A-N.1.1.1 & A-N.1.1.2

► Reporting Category, Assessment Anchor, Anchor Descriptor

The Number System

Demonstrate an understanding of rational and irrational numbers.

Apply concepts of rational and irrational numbers.

► Eligible Content

A-N.1.1.1 Determine whether a number is rational or irrational. For rational numbers, show that the decimal expansion terminates or repeats (limit repeating decimals to thousandths).

A-N.1.1.2 Convert a terminating or repeating decimal into a rational number (limit repeating decimals to thousandths).

► Instructional Supports

Rational and irrational

- Remind students that the decimal representations of rational numbers either terminate or have repeating digits, and recognize that the decimal representations of irrational numbers are non-repeating and nonterminating and so cannot be expressed as fractions. For example, the fraction $\frac{5}{9}$ can be expressed as a decimal by computing 5 ÷ 9, which gives $0.\overline{5}$. The irrational number $\sqrt{3}$ written as a decimal is 1.7320 . . . with no repeating pattern or termination, and so it cannot be expressed as a fraction unless it is rounded.

- Ask students to identify the irrational numbers in a given set of numbers. For example, give students the numbers $\sqrt{81}$, $-\frac{8}{3}$, $-\sqrt{81}$, $\sqrt{13}$, $\frac{8}{3}$, and $-\sqrt{13}$. Students should identify $\sqrt{13}$ and $-\sqrt{13}$ as irrational numbers.

- Ask students to identify all categories (natural, whole, integer, rational, and irrational) that a given number belongs to. For example, give students the numbers $\sqrt{71}$ and 87. Students can identify $\sqrt{71}$ as irrational and 87 as natural, whole, integer, and rational.

- Give students a repeating decimal and a nonterminating, non-repeating decimal and ask them to explain which number is rational and which number is irrational and why. For example, give students the numbers 1.3714714714 . . . and 3.6272772777 . . . . Students can recognize that “714” repeats in the first number, and thus, the number can be written as a ratio of two integers (i.e., $\frac{4,567}{3,330}$) and is therefore rational. Students should explain that the second number does not have a repeating set of digits and cannot be written as a ratio of two integers, and it is therefore irrational.

Fractions and repeating decimals

- Ask students to determine whether a given number’s decimal expansion is terminating, nonterminating and repeating, or nonterminating and non-repeating. For example, give students the numbers $\frac{14}{3}$, $\frac{5}{8}$, and $\sqrt{39}$. Students can identify that $\frac{14}{3}$ has a decimal expansion that is nonterminating and repeating, $\frac{5}{8}$ has a decimal expansion that is terminating, and $\sqrt{39}$ has a decimal expansion that is nonterminating and non-repeating.
- Ask students to explore patterns in fraction forms of repeating decimals. For example, ask students to find the fraction forms of $0.\overline{1}$, $0.\overline{4}$, and $0.\overline{8}$ and help them recognize that these have the fraction equivalents of $\frac{1}{9}$, $\frac{4}{9}$, and $\frac{8}{9}$, respectively. Then, ask students to find the fraction forms of $0.\overline{24}$, $0.\overline{53}$, and $0.\overline{78}$ and help them recognize that these have the fraction equivalents of $\frac{24}{99}$, $\frac{53}{99}$, and $\frac{78}{99}$, respectively. Ask students to make and test conjectures about longer repeating decimals.
- Ask students to convert a repeating decimal with all digits in the decimal part of the repeating pattern to a fraction. For example, give students the number $3.\overline{73}$ and ask them to convert it to a fraction.
    - Students can decompose the given number into $3 + 0.\overline{73}$. Since $0.\overline{73}$ can be written as the fraction $\frac{73}{99}$, their expression is equivalent to $3 + \frac{73}{99}$, $3\frac{73}{99}$, or the improper fraction $\frac{370}{99}$.
    - Alternatively, since there are 2 digits in the repeating portion of the decimal, students can start by multiplying $3.\overline{73}$ by 100 to get $373.\overline{73}$. Students can then subtract $3.\overline{73}$ from $373.\overline{73}$ to get a difference of 370. This is equivalent to 99 times the given number

because they are subtracting the original number from 100 times the original number. Students can then write the expression 370 ÷ 99 as equivalent to the given number, which can be written as the fraction $\frac{370}{99}$ or the mixed number $3\frac{73}{99}$.

- Ask students to convert a repeating decimal, containing digits that are not part of the repeated pattern, to a fraction. For example, ask students to convert $0.5\overline{47}$ to a fraction.
  - Students can decompose the given number into $0.5 + 0.0\overline{47}$. This is equal to $\frac{5}{10} + \frac{1}{10}(0.\overline{47})$, which can be rewritten as $\frac{5}{10} + \frac{1}{10}(\frac{47}{99})$. Students can then simplify this expression to the fraction $\frac{542}{990}$.
  - Alternatively, since there are 2 digits that repeat, students can start by multiplying the given number by 100 to get $54.\overline{74}$. They can then subtract $0.5\overline{47}$ from $54.\overline{74}$ to get 54.2. This difference is equivalent to 99 times the given number because they are subtracting the original number from 100 times the original number. Students can therefore write the expression 54.2 ÷ 99 as equivalent to the given number. Students can multiply both the dividend and divisor by 10 to generate the equivalent expression 542 ÷ 990, which can be written as the fraction $\frac{542}{990}$.

► Identifying and Addressing Common Misconceptions

- Students may associate a radical symbol with irrational numbers. Ask students to identify the irrational numbers in the set $-\frac{3}{4}$, $\sqrt{28}$, $\frac{7}{6}$, $\sqrt{16}$, and $\sqrt{47}$. Students who include $\sqrt{16}$ in the set of irrational numbers may have this misconception. Ask students to estimate the values of the radicals and help them notice that $\sqrt{16}$ has a value of 4.
- Students may confuse “non-repeating” with “nonterminating” in the definition of an irrational number and therefore identify any repeating decimal as an irrational number. Ask students to identify whether $5.\overline{373}$ is rational or irrational. Students who classify the number as irrational may have this misconception. Help students convert the decimal to a fraction and then classify the number again.

- Students may truncate a repeating decimal when converting from a repeating decimal to a fraction. Ask students to convert $0.\overline{53}$ to a fraction. Students who convert the decimal to the fraction $\frac{53}{100}$ may have this misconception. Ask students to reverse the conversion back to decimal form and compare the result to the given decimal.
- Students may think that a number with a repeating pattern rather than a repeating sequence is rational. Ask students to identify whether 10.4040040004 . . . is rational or irrational. Students who classify the number as rational may have this misconception. Ask students to write the decimal using repeating bar notation or ask students to attempt to convert the decimal to a fraction. Discuss with them why they are not able to complete the conversion and help them understand the difference between having a repeating decimal and a repeating pattern in a decimal.

► Enrichment

- Ask students to identify real-world scenarios in which irrational numbers appear. Students could identify the relationship between the radius and circumference of a circle.
- Ask students to identify real-word scenarios in which fractions and negative numbers would make sense but irrational numbers would not make sense. Students could identify money in a bank account.
- Give students two numbers and ask them to determine if the sum of the numbers is rational or irrational. For example, give students the expressions $3.5 + \pi$ and $\sqrt{2} + -\sqrt{2}$. Students can determine that the first is irrational since its decimal form does not repeat or terminate and the second is rational since the two addends are opposites and sum to 0.

► Key Terms

approximate, decimal, decimal expansion, fraction, irrational, rational, repeating, terminating, nonterminating

A-N.1.1.1 & A-N.1.1.2

► Reporting Category, Assessment Anchor, Anchor Descriptor

The Number System

Demonstrate an understanding of rational and irrational numbers.

Apply concepts of rational and irrational numbers.

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► Enrichment

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► Key Terms

approximate, decimal, decimal expansion, fraction, irrational, rational, repeating, terminating, nonterminating

A-N.1.1.3, A-N.1.1.4, & A-N.1.1.5

► Reporting Category, Assessment Anchor, Anchor Descriptor

The Number System

Demonstrate an understanding of rational and irrational numbers.

Apply concepts of rational and irrational numbers.

► Eligible Content

A-N.1.1.3 Estimate the value of irrational numbers without a calculator (limit whole number radicand to less than 144).

A-N.1.1.4 Use rational approximations of irrational numbers to compare and order irrational numbers.

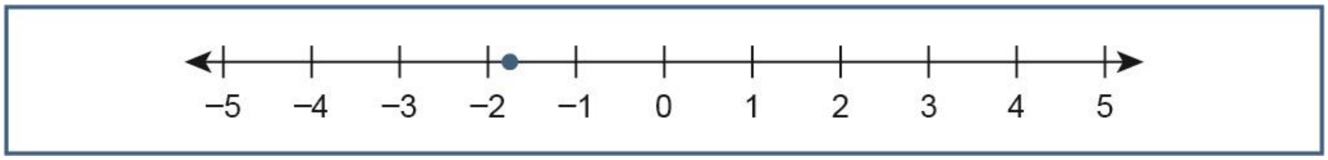
A-N.1.1.5 Locate/identify rational and irrational numbers at their approximate locations on a number line.

► Instructional Supports

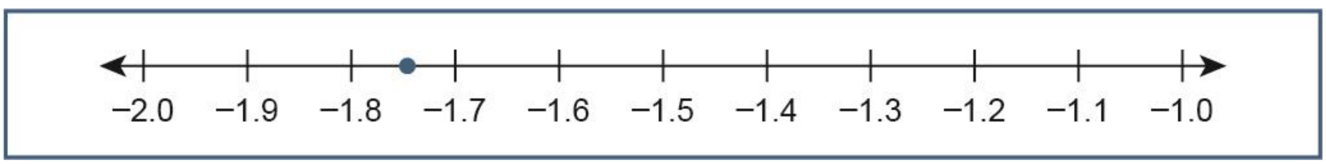
- Ask students to find an irrational number between two given consecutive whole numbers. For example, give students the numbers 7 and 8. Students can use n to represent an irrational number. To indicate that n is between the two given numbers, students can write the inequality $7 < n < 8$. Students can then write the inequality $7^2 < n^2 < 8^2$, which can also be written as $49 < n^2 < 64$. Students can use this inequality to determine some values of n^2 that satisfy the inequality, such as 50, 57, and 61. They can then use these values to determine some positive values of n that are between 7 and 8: $\sqrt{50}$, $\sqrt{57}$, and $\sqrt{61}$.
- Ask students to approximate a given irrational number to the nearest tenth. For example, give students the number $\sqrt{35}$. Students can first determine that the value of $\sqrt{35}$ must be between 5 and 6 because $5^2 < 35 < 6^2$. Thus, the ones digit must be 5. Students should notice that the tenths place can be any digit from 1 to 9. Students can use trial and error to compare 35 with 5.1^2, 5.2^2, 5.3^2, and so on up to 5.9^2. Since 35 is much closer to 36 than to 25, students may decide to start by comparing with 5.8^2 or 5.9^2 rather than going in order. After making comparisons, students

can write the inequality $5.9^2 < 35 < 6^2$, which is equivalent to $34.81 < 35 < 36$. Since 35 is closer to 34.81 than it is to 36, students can conclude that 5.9 approximates $\sqrt{35}$ to the tenths place.

- Ask students to plot an approximation of a given irrational number on a number line. For example, give students the expression $6 - \sqrt{60}$ and ask them to plot a point on the number line at the approximate value. Students should first find that the value of $\sqrt{60}$ is between 7 and 8 because $7^2 < 60 < 8^2$. So the value of $6 - \sqrt{60}$ is between $6 - 7$ and $6 - 8$. Therefore, students should determine that the point should be plotted between –1 and –2 on the number line.



Further, considering the tenths place, students should find that the value of $\sqrt{60}$ is between 7.7 and 7.8 because $7.7^2 < 60 < 7.8^2$. Therefore, students may conclude that the value of $6 - \sqrt{60}$ is between $6 - 7.7$ and $6 - 7.8$, so the point should be plotted between –1.7 and –1.8 on the number line.



Students should note that the value of 60 is about the same distance from 7.7^2 as it is from 7.8^2 but slightly closer to 7.7^2. Therefore, the value of $6 - \sqrt{60}$ is about halfway between $6 - 7.7$ and $6 - 7.8$ (between –1.7 and –1.8) but slightly closer to –1.7.

- Ask students to place a given list of rational and irrational numbers in order from least to greatest. For example, give students the numbers $8\frac{9}{10}$, 3π, $\frac{37}{4}$, and $\sqrt{79}$. Students can determine that the rational numbers in the set have decimal equivalents of 8.9 for $8\frac{9}{10}$ and 9.25 for $\frac{37}{4}$. Further, the irrational numbers in the set are approximately 9.42 for 3π and 8.89 for $\sqrt{79}$. Students can then place the numbers in order as $\sqrt{79}$, $8\frac{9}{10}$, $\frac{37}{4}$, and 3π.

- Ask students to solve a given real-world problem involving irrational numbers by using rational approximations. For example, give students the problem "The area of a square garden is

108 square meters. What is the approximate length of one side of the garden?" Students should explain that one side of the garden has a length of $\sqrt{108}$ meters, which can be approximated as 10.4 meters because 108 is between 10.3^2 and 10.4^2 but is closer to 10.4^2.

► Identifying and Addressing Common Misconceptions

- Students may make an incorrect comparison between an irrational number and a rational approximation of the irrational number due to utilizing too few digits in their rational approximation. Ask students to compare the numbers 4π and the rational approximation 12.5. Students who compare the numbers as $4\pi < 12.5$ may have this misconception. Ask students to write out the irrational number to one more decimal point than the rational approximation before any computations and then compare the calculated values.
- Students may estimate an irrational square root by estimating the value as half of the radicand. Ask students to estimate $\sqrt{19}$. Students who estimate 9.5 may have this misconception. Help students explain the meaning of the square root. Then ask students to square the estimation of the square root and compare it to the original radicand.
- When plotting a point on a number line to approximate a given irrational number, some students may always place a point at the center of the correct interval without considering whether it is closer to one end of the interval or the other. Ask students to plot a point representing an estimate of $\sqrt{34}$. Students who place the point halfway between 5 and 6 may have this misconception. Ask students to square the number halfway between the ends of the interval and compare the square to the radicand. Students can square 5.5 to find 30.25. Because $\sqrt{30.25}$ is 5.5, $\sqrt{34}$ must be greater than 5.5, so their point should be closer to 6 than to 5.

► Enrichment

- Ask students to find an irrational number between two given rational numbers. For example, give students the rational numbers 9.988 and 9.989. A possible solution is 9.988525225222 . . . .
- Ask students to investigate operations with irrational numbers without requiring estimation with rational numbers. For example, ask students to find a number that can be multiplied by 5π to produce a product that is a rational number. Students could identify 0 or $\frac{1}{\pi}$ as possible values.

- Ask students to investigate the values of numbers using subsequently better approximations. For example, ask students to estimate the value of the expression 2.5 • π, first using 3 as an approximation for π, then 3.1, 3.14, 3.142, and 3.1416. Ask students to make observations about the changes in values between each estimation. Students can calculate values of 7.5, 7.75, 7.85, 7.855, and 7.854. Students can observe that each increase in accuracy has less effect on the value than the previous one.

► Key Terms

approximate, estimate, irrational, number line, rational

A-N.1.1.3, A-N.1.1.4, & A-N.1.1.5

► Reporting Category, Assessment Anchor, Anchor Descriptor

The Number System

Demonstrate an understanding of rational and irrational numbers.

Apply concepts of rational and irrational numbers.

► Eligible Content

A-N.1.1.3 Estimate the value of irrational numbers without a calculator (limit whole number radicand to less than 144).

A-N.1.1.4 Use rational approximations of irrational numbers to compare and order irrational numbers.

A-N.1.1.5 Locate/identify rational and irrational numbers at their approximate locations on a number line.

► Instructional Supports

- Ask students to find an irrational number between two given consecutive whole numbers. For example, give students the numbers 7 and 8. Students can use n to represent an irrational number. To indicate that n is between the two given numbers, students can write the inequality $7 < n < 8$. Students can then write the inequality $7^2 < n^2 < 8^2$, which can also be written as $49 < n^2 < 64$. Students can use this inequality to determine some values of n^2 that satisfy the inequality, such as 50, 57, and 61. They can then use these values to determine some positive values of n that are between 7 and 8: $\sqrt{50}$, $\sqrt{57}$, and $\sqrt{61}$.
- Ask students to approximate a given irrational number to the nearest tenth. For example, give students the number $\sqrt{35}$. Students can first determine that the value of $\sqrt{35}$ must be between 5 and 6 because $5^2 < 35 < 6^2$. Thus, the ones digit must be 5. Students should notice that the tenths place can be any digit from 1 to 9. Students can use trial and error to compare 35 with 5.1^2, 5.2^2, 5.3^2, and so on up to 5.9^2. Since 35 is much closer to 36 than to 25, students may decide to start by comparing with 5.8^2 or 5.9^2 rather than going in order. After making comparisons, students

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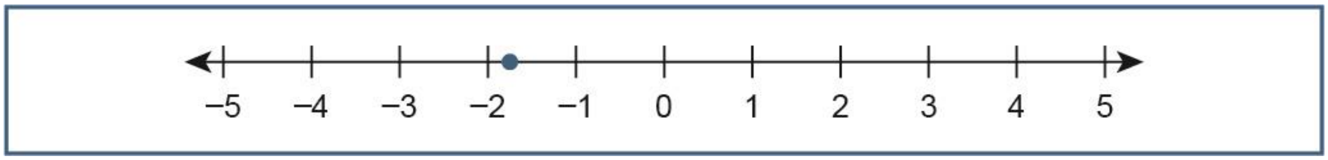
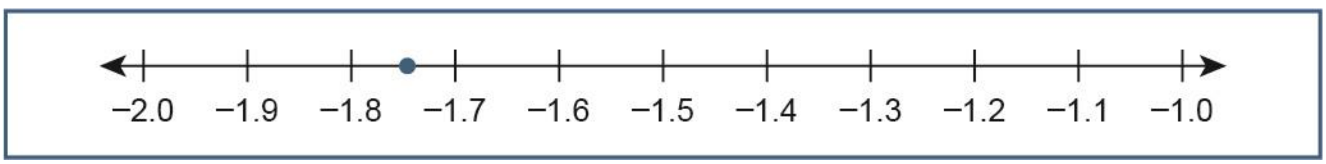


Further, considering the tenths place, students should find that the value of $\sqrt{60}$ is between 7.7 and 7.8 because $7.7^2 < 60 < 7.8^2$. Therefore, students may conclude that the value of $6 - \sqrt{60}$ is between $6 - 7.7$ and $6 - 7.8$, so the point should be plotted between –1.7 and –1.8 on the number line.



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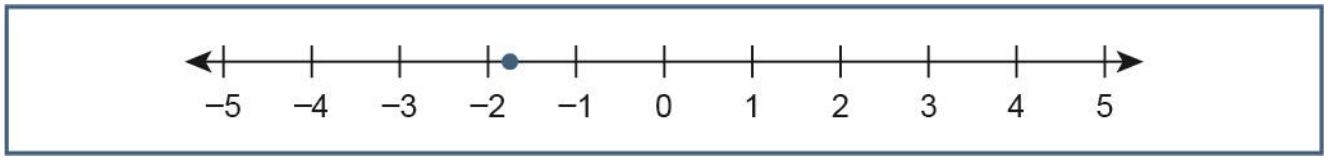
► Instructional Supports

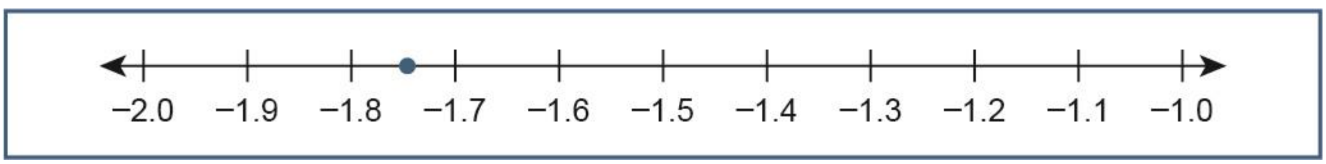
- Ask students to find an irrational number between two given consecutive whole numbers. For example, give students the numbers 7 and 8. Students can use n to represent an irrational number. To indicate that n is between the two given numbers, students can write the inequality $7 < n < 8$. Students can then write the inequality $7^2 < n^2 < 8^2$, which can also be written as $49 < n^2 < 64$. Students can use this inequality to determine some values of n^2 that satisfy the inequality, such as 50, 57, and 61. They can then use these values to determine some positive values of n that are between 7 and 8: $\sqrt{50}$, $\sqrt{57}$, and $\sqrt{61}$.
- Ask students to approximate a given irrational number to the nearest tenth. For example, give students the number $\sqrt{35}$. Students can first determine that the value of $\sqrt{35}$ must be between 5 and 6 because $5^2 < 35 < 6^2$. Thus, the ones digit must be 5. Students should notice that the tenths place can be any digit from 1 to 9. Students can use trial and error to compare 35 with 5.1^2, 5.2^2, 5.3^2, and so on up to 5.9^2. Since 35 is much closer to 36 than to 25, students may decide to start by comparing with 5.8^2 or 5.9^2 rather than going in order. After making comparisons, students

can write the inequality $5.9^2 < 35 < 6^2$, which is equivalent to $34.81 < 35 < 36$. Since 35 is closer to 34.81 than it is to 36, students can conclude that 5.9 approximates $\sqrt{35}$ to the tenths place.

- Ask students to plot an approximation of a given irrational number on a number line. For example, give students the expression $6 - \sqrt{60}$ and ask them to plot a point on the number line at the approximate value. Students should first find that the value of $\sqrt{60}$ is between 7 and 8 because $7^2 < 60 < 8^2$. So the value of $6 - \sqrt{60}$ is between $6 - 7$ and $6 - 8$. Therefore, students should determine that the point should be plotted between –1 and –2 on the number line.



Further, considering the tenths place, students should find that the value of $\sqrt{60}$ is between 7.7 and 7.8 because $7.7^2 < 60 < 7.8^2$. Therefore, students may conclude that the value of $6 - \sqrt{60}$ is between $6 - 7.7$ and $6 - 7.8$, so the point should be plotted between –1.7 and –1.8 on the number line.



Students should note that the value of 60 is about the same distance from 7.7^2 as it is from 7.8^2 but slightly closer to 7.7^2. Therefore, the value of $6 - \sqrt{60}$ is about halfway between $6 - 7.7$ and $6 - 7.8$ (between –1.7 and –1.8) but slightly closer to –1.7.

- Ask students to place a given list of rational and irrational numbers in order from least to greatest. For example, give students the numbers $8\frac{9}{10}$, 3π, $\frac{37}{4}$, and $\sqrt{79}$. Students can determine that the rational numbers in the set have decimal equivalents of 8.9 for $8\frac{9}{10}$ and 9.25 for $\frac{37}{4}$. Further, the irrational numbers in the set are approximately 9.42 for 3π and 8.89 for $\sqrt{79}$. Students can then place the numbers in order as $\sqrt{79}$, $8\frac{9}{10}$, $\frac{37}{4}$, and 3π.

- Ask students to solve a given real-world problem involving irrational numbers by using rational approximations. For example, give students the problem "The area of a square garden is

108 square meters. What is the approximate length of one side of the garden?" Students should explain that one side of the garden has a length of $\sqrt{108}$ meters, which can be approximated as 10.4 meters because 108 is between 10.3^2 and 10.4^2 but is closer to 10.4^2.

▶ Identifying and Addressing Common Misconceptions

- Students may make an incorrect comparison between an irrational number and a rational approximation of the irrational number due to utilizing too few digits in their rational approximation. Ask students to compare the numbers 4π and the rational approximation 12.5. Students who compare the numbers as $4\pi < 12.5$ may have this misconception. Ask students to write out the irrational number to one more decimal point than the rational approximation before any computations and then compare the calculated values.
- Students may estimate an irrational square root by estimating the value as half of the radicand. Ask students to estimate $\sqrt{19}$. Students who estimate 9.5 may have this misconception. Help students explain the meaning of the square root. Then ask students to square the estimation of the square root and compare it to the original radicand.
- When plotting a point on a number line to approximate a given irrational number, some students may always place a point at the center of the correct interval without considering whether it is closer to one end of the interval or the other. Ask students to plot a point representing an estimate of $\sqrt{34}$. Students who place the point halfway between 5 and 6 may have this misconception. Ask students to square the number halfway between the ends of the interval and compare the square to the radicand. Students can square 5.5 to find 30.25. Because $\sqrt{30.25}$ is 5.5, $\sqrt{34}$ must be greater than 5.5, so their point should be closer to 6 than to 5.

▶ Enrichment

- Ask students to find an irrational number between two given rational numbers. For example, give students the rational numbers 9.988 and 9.989. A possible solution is 9.988525225222 . . . .
- Ask students to investigate operations with irrational numbers without requiring estimation with rational numbers. For example, ask students to find a number that can be multiplied by 5π to produce a product that is a rational number. Students could identify 0 or $\frac{1}{\pi}$ as possible values.

- Ask students to investigate the values of numbers using subsequently better approximations. For example, ask students to estimate the value of the expression 2.5 • π, first using 3 as an approximation for π, then 3.1, 3.14, 3.142, and 3.1416. Ask students to make observations about the changes in values between each estimation. Students can calculate values of 7.5, 7.75, 7.85, 7.855, and 7.854. Students can observe that each increase in accuracy has less effect on the value than the previous one.

► Key Terms

approximate, estimate, irrational, number line, rational

B-E.1.1.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Demonstrate an understanding of expressions and equations with radicals and integer exponents.

Represent and use expressions and equations to solve problems involving radicals and integer exponents.

► Eligible Content

B-E.1.1.1 Apply one or more properties of integer exponents to generate equivalent numerical expressions without a calculator (with final answers expressed in exponential form with positive exponents). Properties will be provided.

► Instructional Supports

Non-positive exponents

- Ask students to investigate the value of an exponential expression with a power of 0. For example, ask students to look for patterns in powers of 4, including the value of 4^0. Give them a table to complete to help them find and discuss patterns.

Powers of 4	Value
4^3	64
4^2	16
4^1	4
4^0	1

  Students should determine that as the power of 4 increases by 1, the value of the expression is multiplied by 4. As the power of 4 decreases by 1, the value of the expression is divided by 4. Students may then conclude that $4^0 = 1$ because 4 divided by 4 is 1. Help students extend this to conclude that $a^0 = 1$ for any nonzero number a.

- Ask students to generate a rule that can be used for exponential expressions written with a negative exponent, a^{-n}. For example, give students the expressions shown in the left column of the table and ask them to write the equivalent forms shown in the other column of the table.

Powers of 3	Value
3^3	27
3^2	9
3^1	3
3^0	1
3^{-1}	$\frac{1}{3}$ or $\frac{1}{3^1}$
3^{-2}	$\frac{1}{9}$ or $\frac{1}{3^2}$
3^{-3}	$\frac{1}{27}$ or $\frac{1}{3^3}$

Have students discuss how as the exponent decreases, the values are divided by 3 each time. Students should determine that an exponential expression of the form a^{-n} is equivalent to $\frac{1}{a^n}$, as shown in the table.

Products of powers

- Ask students to decompose an expression that contains the product of two exponential expressions with a common base into an expression with no exponential terms. For example, ask students to rewrite the expression $7^5 \cdot 7^3$ as $(7 \cdot 7 \cdot 7 \cdot 7 \cdot 7) \cdot (7 \cdot 7 \cdot 7)$, which is equivalent to $7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7$ by the associative property. Then, ask students to rewrite the expression with a single base and exponent, 7^8.

- Ask students to create a general rule that can be used to simplify the product of two exponential expressions with a common base, $a^m a^n$. For example, have students complete a table such as the following and have them discuss the patterns they notice.

Product of Same Bases	Decomposed Form	Single Base
$3^3 \cdot 3^2$	$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$	3^5
$8^1 \cdot 8^5$	$8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8$	8^6
$(-6)^4 \cdot (-6)^4$	$(-6) \cdot (-6) \cdot (-6) \cdot (-6) \cdot (-6) \cdot (-6) \cdot (-6) \cdot (-6)$	$(-6)^8$
$(-9)^4 \cdot (-9)^3$	$(-9) \cdot (-9) \cdot (-9) \cdot (-9) \cdot (-9) \cdot (-9) \cdot (-9)$	$(-9)^7$

Based on the results of the table, students should generate the rule $a^m a^n = a^{m+n}$ and explain their thinking.

- Ask students to find and rewrite products of exponential expressions with common bases as expressions using only the base and one exponent. For example, give students the expression $x^6 \cdot x^2 \cdot x^{-3}$. Students should rewrite the expression as x^5 and explain how they found their answer.

- Ask students to use the exponent addition properties to derive the values of expressions with negative exponents. For example, ask students to use the expression $3^{-1} \cdot 3^3$ to help explain why $3^{-1} = \frac{1}{3}$. The expression $3^{-1} \cdot 3^3$ is equal to 3^2. Help students use the facts that $3^3 = 27$ and $3^2 = 9$ to create the equation $3^{-1} \cdot 27 = 9$. This equation can be simplified into $3^{-1} = \frac{3^2}{3^3} = \frac{9}{27} = \frac{1}{3}$.

Power of a power

- Ask students to expand and then simplify an expression that is a number raised to a power which is then raised to a power. Then, ask students to simplify the expression using a single base and a single exponent. For example, given the expression $(3^3)^4$, write $(3^3) \cdot (3^3) \cdot (3^3) \cdot (3^3)$ and then $(3 \cdot 3 \cdot 3) \cdot (3 \cdot 3 \cdot 3) \cdot (3 \cdot 3 \cdot 3) \cdot (3 \cdot 3 \cdot 3)$. Finally, ask students to write the expression in exponential form with a single base and exponent, 3^{12}.

- Ask students to create a general rule that can be used to simplify an expression that is a number raised to a power which is then raised to a power, $(a^m)^n$. For example, have students complete a table such as the following and have them discuss the patterns they notice.

Exponential Expression Raised to a Power	Decomposed Form	Single Exponent
$(5^3)^4$	$5^3 \cdot 5^3 \cdot 5^3 \cdot 5^3$ or $(5 \cdot 5 \cdot 5) \cdot (5 \cdot 5 \cdot 5) \cdot (5 \cdot 5 \cdot 5) \cdot (5 \cdot 5 \cdot 5)$	5^{12}
$(6^5)^2$	$6^5 \cdot 6^5$ or $(6 \cdot 6 \cdot 6 \cdot 6 \cdot 6) \cdot (6 \cdot 6 \cdot 6 \cdot 6 \cdot 6)$	6^{10}
$[(-2)^3]^2$	$(-2)^3 \cdot (-2)^3$ or $[(-2) \cdot (-2) \cdot (-2)] \cdot [(-2) \cdot (-2) \cdot (-2)]$	$(-2)^6$
$(a^3)^5$	$a^3 \cdot a^3 \cdot a^3 \cdot a^3 \cdot a^3$ or $(a \cdot a \cdot a) \cdot (a \cdot a \cdot a) \cdot (a \cdot a \cdot a) \cdot (a \cdot a \cdot a) \cdot (a \cdot a \cdot a)$	a^{15}

Based on the results of the table, students should generate the rule $(a^m)^n = a^{mn}$ and explain their thinking.

Power of a product

- Ask students to expand and then simplify an exponential expression that contains a product raised to a power. For example, have students rewrite the expression $(4w)^5$ first as $(4 \cdot w)^5$ and then as $(4 \cdot w) \cdot (4 \cdot w) \cdot (4 \cdot w) \cdot (4 \cdot w) \cdot (4 \cdot w)$. Then, applying both the commutative property of multiplication and the associative property of multiplication, students should rewrite the expression as $(4 \cdot 4 \cdot 4 \cdot 4 \cdot 4) \cdot (w \cdot w \cdot w \cdot w \cdot w)$. Finally, ask students to write the expression with each base raised to an exponent, 4^5w^5.
- Ask students to create a general rule for any expression that contains a product raised to a power, $(ab)^n$. For example, have students complete a table such as the following and have them discuss the patterns they notice.

Product Raised to a Power	Decomposed Form	Each Base Raised to Power
$(-8t)^6$	$(-8t) \bullet (-8t) \bullet (-8t) \bullet (-8t) \bullet (-8t) \bullet (-8t)$ or $(-8) \bullet (-8) \bullet (-8) \bullet (-8) \bullet (-8) \bullet (-8) \bullet t \bullet t \bullet t \bullet t \bullet t \bullet t$	$(-8)^6t^6$
$(12y)^4$	$(12y) \bullet (12y) \bullet (12y) \bullet (12y)$ or $12 \bullet 12 \bullet 12 \bullet 12 \bullet y \bullet y \bullet y \bullet y$	12^4y^4
$(9d)^5$	$(9d) \bullet (9d) \bullet (9d) \bullet (9d) \bullet (9d)$ or $9 \bullet 9 \bullet 9 \bullet 9 \bullet 9 \bullet d \bullet d \bullet d \bullet d \bullet d$	9^5d^5
$(-3n)^3$	$(-3n) \bullet (-3n) \bullet (-3n)$ or $-3 \bullet -3 \bullet -3 \bullet n \bullet n \bullet n$	$(-3)^3n^3$

Based on the results of the table, students should generate the rule $(ab)^n = a^nb^n$ and explain their thinking.

- Ask students to find equivalent exponential expressions to those containing products raised to powers. For example, write the expression $(-6p^3)^6$ as $(-6)^6p^{18}$.

Powers of quotients

- Ask students to create a general rule that can be used to simplify the quotient of two exponential expressions with a common base, $\frac{a^m}{a^n}$. For example, given the expressions shown in the left column of the table below, ask students to write the equivalent forms in the other columns and then discuss any patterns they notice.

Quotient of Same Bases	Decomposed Form	Single Base
$\frac{6^8}{6^4}$	$\frac{6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6}{6 \cdot 6 \cdot 6 \cdot 6}$	6^4
$\frac{9^7}{9^5}$	$\frac{9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9}{9 \cdot 9 \cdot 9 \cdot 9 \cdot 9}$	9^2
$\frac{8^3}{8^9}$	$\frac{8 \cdot 8 \cdot 8}{8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8}$	$\frac{1}{8^6} = 8^{-6}$
$\frac{4^2}{4^{12}}$	$\frac{4 \cdot 4}{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4}$	$\frac{1}{4^{10}} = 4^{-10}$

Based on the results of the table, students should generate the rule that for nonzero values of a, $\frac{a^m}{a^n}$ equals a^{m-n}.

► Identifying and Addressing Common Misconceptions

- Some students believe that a number multiplied by a number raised to the zero power yields a result of zero. Ask students to evaluate $3^3 \cdot 3^0$. Some students may give an answer of 0. If $3^0 = 0$, then $3^3 \cdot 3^0 = 3^3 \cdot 0 = 0$ because any number multiplied by $0 = 0$. Remind students that $3^3 \cdot 3^0 = 3^{3+0} = 3^3$. Since $3^3 \cdot 3^0 = 3^3$, 3^0 must equal 1 (if $ab = a$, and $a \neq 0$, then $b = 1$).

- Some students believe that a number cannot be divided by a number raised to the zero power (because dividing by zero is undefined). Ask students to evaluate $\frac{2^3}{2^0}$. Some students may say that this cannot be done because numbers cannot be divided by zero. Remind students that $\frac{2^3}{2^0} = 2^{3-0} = 2^3$. Since $\frac{2^3}{2^0} = 2^3$, 2^0 must equal 1 (if $\frac{a}{b} = a$ and $a \neq 0$, then $b = 1$).

- Some students believe that a negative exponent changes the sign of the base. Ask students to evaluate $2^3 \cdot 2^{-1}$. Some students may give a negative number as an answer. Help students use exponent properties to show that $2^3 \cdot 2^{-1} = 2^{3+(-1)} = 2^2 = 2 \cdot 2 = 4$. Remind them that a positive number multiplied by a negative number could not equal positive 4.

- Some students think that exponents can be "distributed" over a sum or difference. Ask students whether $(3 + 4)^2$ or $3^2 + 4^2$ has a greater value. Some students may say that they are equal. Help them calculate both values by using the order of operations and recognize that the two expressions are not equal.

► Enrichment

- Ask students to investigate the effect of parentheses in exponentials with negative bases. For example, discuss the difference between -5^4 and $(-5)^4$. Then, ask students to compare that to the values of -2^3 and $(-2)^3$. Ask them to experiment with other combinations of even or odd bases and exponents and discuss patterns they notice.
- Ask students to predict the value of $a^{0.5}$ and, in particular, to consider whether it is greater than or less than a. For example, give students the value $2^{0.5}$. The value of 2^0 is 1 and the value of 2^1 is 2. Therefore, the value of $2^{0.5}$ is likely to be between 1 and 2. As a further investigation, look at patterns with exponents to determine whether $2^{0.7}$ is greater than or less than $2^{0.5}$.
- Ask students to determine whether greater exponents will always result in greater values when the base remains constant.
- Ask students to determine the value of an expression with a negative exponent in the denominator of a fraction. For example, ask students to determine the value of $\frac{1}{3^{-4}}$.

► Key Terms

base, equivalent, exponent, integer, inverse, negative, power, radical, reciprocal

B-E.1.1.2

► Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Demonstrate an understanding of expressions and equations with radicals and integer exponents.

Represent and use expressions and equations to solve problems involving radicals and integer exponents.

► Eligible Content

B-E.1.1.2 Use square root and cube root symbols to represent solutions to equations of the form x2 = p and x3 = p, where p is a positive rational number. Evaluate square roots of perfect squares (up to and including 122) and cube roots of perfect cubes (up to and including 53) without a calculator.

► Instructional Supports

Square roots

- Formally introduce students to the radical symbol. Explain to students that if p is a positive number, then $\sqrt{p}$ represents a positive number that when multiplied by itself is equal to p. For example, show students that $\sqrt{9} = 3$ because $3 \cdot 3 = 9$. Explain to students that "$\sqrt{9}$" is read as "the square root of nine." In other words, three is the square root of nine because $3 \cdot 3 = 3^2$ ("three squared") = 9. The "root" of a number is the "base" of the exponential.
- Ask students to explain what a perfect square is, to give an example of a perfect square, and to explain why their example fits their definition. Students should explain that a perfect square is a whole number whose square root is also a whole number. They could give an example of 16 being a perfect square and explain that the square root of 16 is 4 and 4 is a whole number.
- Ask students to explain why $\sqrt{2}$ is irrational. Students should explain that since 2 is not a perfect square, its square root is not a rational number. In other words, $\sqrt{2}$ is irrational because there is no rational value of x such that x^2 is equal to 2. Help students generalize a rule for all non-square

whole numbers. Students should conclude that if p is a non-square whole number, then $\sqrt{p}$ is irrational.

- Ask students to identify fractions that would have rational square roots. For example, $\sqrt{\frac{4}{9}} = \frac{2}{3}$ because 4 and 9 are perfect squares. Explain to students that although fractions are not perfect squares, the square root of a fraction is rational if the numerator and the denominator are both perfect squares.
- Ask students to evaluate radical expressions with rational values. For example, ask students to complete the following table as shown.

Expression	Value
$\sqrt{81}$	9
$\sqrt{49}$	7
$\sqrt{169}$	13
$\sqrt{1}$	1
$\sqrt{\frac{25}{121}}$	$\frac{5}{11}$
$\sqrt{\frac{64}{49}}$	$\frac{8}{7}$
$\sqrt{\frac{1}{100}}$	$\frac{1}{10}$

- Ask students to solve an equation of the form $x^2 = p$ with the understanding that the solution is the number that when multiplied by itself is equal to p. Show students that they can use radical symbols to evaluate equations of the form $x^2 = p$. For example, if $x^2 = 24$, then $x = \sqrt{24}$. Explain to students that since 24 is not a perfect square, $\sqrt{24}$ is irrational, so the radical sign must be used to write the exact value of the number. Explain to students that there is another solution to

$x^2 = 24$. Ask students if they can think of any other number that when multiplied by itself equals 24. Show students that $(-\sqrt{24}) \cdot (-\sqrt{24}) = 24$. Therefore, the equation $x^2 = 24$ has two solutions: $x = \sqrt{24}$ and $x = -\sqrt{24}$. Explain to students that the radical symbol refers only to positive numbers. For example, even though $(-6)^2 = 36$ and $6^2 = 36$, the radical expression $\sqrt{36}$ is equal to 6 but not −6. Point out that $-\sqrt{}$ is used to denote the negative square root of a number, so $-\sqrt{36} = -6$. And if $p^2 = 36$, then $p = \pm\sqrt{36}$. It should be stressed to students that the equation $x^2 = 81$ is not equivalent to $\sqrt{81} = x$. The first equation has two solutions, $x = 9$ and $x = -9$, while the second equation has only one solution, $x = 9$.

- Ask students to find numbers that square to a given target number (i.e., find the square roots), using ± notation. For example, ask students to complete the table shown.

p^2	p
25	± 5
34	$\pm\sqrt{34}$
$\frac{3}{8}$	$\pm\sqrt{\frac{3}{8}}$
4	± 2
99	$\pm\sqrt{99}$
48	$\pm\sqrt{48}$
$\frac{5}{12}$	$\pm\sqrt{\frac{5}{12}}$
$\frac{6}{7}$	$\pm\sqrt{\frac{6}{7}}$

Cube roots

- Formally introduce students to the cube root. Explain to students that the cube root of x is q if $q^3 = x$. The radical symbol $\sqrt[3]{}$ is used to represent the cube root of a number. For example, show students that $\sqrt[3]{64} = 4$ because $4 \cdot 4 \cdot 4 = 64$, or $4^3 = 64$. In words, 4 is the cube root of 64 because 4 cubed equals 64.

- Ask students to explain what a perfect cube is, to give an example, and to explain why their example fits their definition. Students should explain that a perfect cube is an integer whose cube root is also an integer. They could give an example of 64 as a perfect cube and explain that the cube root of 64 is 4 and 4 is an integer. Also, explain to students that although fractions are not perfect cubes, the cube root of a fraction is rational if the numerator and the denominator are both perfect cubes. For example, $\sqrt[3]{\frac{8}{125}} = \frac{2}{5}$ because 8 and 125 are perfect cubes.
- Ask students to evaluate radical expressions involving cube roots. For example, ask students to find the values in the right column of the table shown.

Expression	Value
$\sqrt[3]{27}$	3
$\sqrt[3]{216}$	6
$\sqrt[3]{343}$	7
$\sqrt[3]{1}$	1
$\sqrt[3]{\frac{8}{27}}$	$\frac{2}{3}$
$\sqrt[3]{\frac{64}{125}}$	$\frac{4}{5}$
$\sqrt[3]{\frac{1}{1{,}000}}$	$\frac{1}{10}$

- Ask students to determine the values of q for which $\sqrt[3]{q}$ is irrational. Students should explain that if q is a non-cube integer or q is a fraction where either the numerator or the denominator is a non-cube integer, then $\sqrt[3]{q}$ is irrational. For example, $\sqrt[3]{55}$ is irrational because 55 is a non-cube integer.

- Ask students to evaluate expressions with small perfect cubes. For example, give students the expressions shown in the left column of the table and ask them to find the values shown in the right column.

q^3	q
512	8
29	$\sqrt[3]{29}$
$\frac{4}{9}$	$\sqrt[3]{\frac{4}{9}}$
1,000	10
18	$\sqrt[3]{18}$

▶ Identifying and Addressing Common Misconceptions

- Some students believe that the square root of a number is equal to half the number. Ask students to evaluate $\sqrt{64}$. Some students might give an answer of 32. Remind students that the square root of a number is a solution to the equation $x^2 = p$. Therefore, to solve $\sqrt{64}$, students should write $x^2 = 64$. Ask students to rewrite the equation as $x \bullet x = 64$ and then ask students to substitute their solution into the equation for x and compute their product. The result is $32 \bullet 32 = 1{,}024$, which is not equal to 64. This shows that 32 is not the solution to the equation. Ask students what number when multiplied by itself is equal to 64. Help students find the solution of 8. Also, remind students that since the radical symbol represents only the positive square root, $\sqrt{64} \neq -8$.
- Some students believe that the cube root of a number is found by dividing the number by three. Ask students to evaluate $\sqrt[3]{216}$. Some students might give an answer of 72. Remind students that the cube root of a number is the solution to the equation $x^3 = p$. Therefore, to solve $\sqrt[3]{216}$, students should write $x^3 = 216$. Ask students to rewrite the equation as $x \bullet x \bullet x = 216$ and then ask students to substitute their solution into the equation for x and find their product. The result is $72 \bullet 72 \bullet 72 = 373{,}248$, which is not equal to 216. This shows that 72 is not the solution to the

equation. Ask students what number when multiplied by itself and then multiplied by itself again is equal to 216. If students cannot come up with 6, ask them to use guess and check.

- Some students believe that numbers have both a positive and a negative cube root. Ask students to list all cube roots of 125. Some students might give an answer of 5 and −5. Remind students that the cube root of a number is the solution to the equation $x^3 = p$. So the cube roots of 125 are solutions to the equation $x^3 = 125$. Remind students that $x^3 = x \cdot x \cdot x$ and ask them to substitute −5 for x. Help students use the rules about multiplying negative numbers to conclude that $(-5) \cdot (-5) \cdot (-5) = -125$ and that $\sqrt[3]{125} \neq -5$. The only cube root of 125 is 5.
- Some students confuse radical symbols with exponents. Ask students to solve $x^2 = 9$. Some students might give an answer of 81. Ask students to rewrite the equation using multiplication, without the exponent symbol. Students should write $x \cdot x = 9$. Then, ask students to substitute their solution into the equation for x and compute their product. Students should write $81 \cdot 81 = 6{,}561$, which is not equal to 9. Ask students what number when multiplied by itself is equal to 9. Most students will conclude that the solution is 3. Remind students that −3 is also a solution because multiplying a negative number by another negative number will have a positive solution.

► Enrichment

- Ask students to evaluate $\sqrt{p^2}$, where p is any real number. For example, ask students to evaluate $\sqrt{(-14)^2}$. Many students will say the answer is −14. Remind students that the radical symbol represents only positive square roots. So even though we started with −14 under the radical, there is only one solution, and it is 14. Repeat with other values to create the general rule that, for any real number p, $\sqrt{p^2} = |p|$.
- Ask students to investigate how to multiply two square roots. For example, ask students to investigate the value of $\sqrt{3} \cdot \sqrt{6}$. Students may theorize that the given expression is equivalent to $\sqrt{18}$. Help them demonstrate that this is true by showing that $(\sqrt{3} \cdot \sqrt{6})^2 = 18$.

$$(\sqrt{3} \cdot \sqrt{6})^2 = \sqrt{3} \cdot \sqrt{6} \cdot \sqrt{3} \cdot \sqrt{6} = \sqrt{3} \cdot \sqrt{3} \cdot \sqrt{6} \cdot \sqrt{6} = 3 \cdot 6 = 18$$

Any number that when multiplied by itself results in 18 is, by definition, the square root of 18.

As a further investigation, ask students to show that $\frac{\sqrt{2}}{\sqrt{5}}$ is equivalent to $\sqrt{\frac{2}{5}}$ by showing that $\left(\frac{\sqrt{2}}{\sqrt{5}}\right)^2 = \frac{2}{5}$.

- Ask students to evaluate $\sqrt{\frac{p}{q}}$, where p and q are whole numbers, either p or q is a perfect square, and $q \neq 0$. For example, ask students to evaluate $\sqrt{\frac{3}{16}}$. Some students will note that since 3 is not a perfect square, the number is irrational. Show students that while that is correct, the denominator is a perfect square; therefore, it can be written as 4. In other words, show students that when a fraction is under a radical, it can be rewritten so that the numerator is under a radical and the denominator is under a radical. That is, $\sqrt{\frac{3}{16}}$ can be rewritten as $\frac{\sqrt{3}}{\sqrt{16}}$. Now, $\frac{\sqrt{3}}{\sqrt{16}}$ can be simplified to $\frac{\sqrt{3}}{4}$.

▶ Key Terms

cube root, exponent, irrational, perfect cube, perfect square, power, principal square root, radical, rational, square root

B-E.1.1.3

► Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Demonstrate an understanding of expressions and equations with radicals and integer exponents.

Represent and use expressions and equations to solve problems involving radicals and integer exponents.

► Eligible Content

B-E.1.1.3 Estimate very large or very small quantities by using numbers expressed in the form of a single digit times an integer power of 10, and express how many times larger or smaller one number is than another.

► Instructional Supports

- Ask students to approximate a very large number by using a single digit multiplied by a power of 10. For example, give students the number 7,927,000,000,000. This number rounds to 8,000,000,000,000 when rounded to the nearest trillion, so this approximation can be expressed as 8×10^{12}. Help students see that the first digit of the rounded number is used as the single digit and then is multiplied by 10 twelve times to arrive at the value of 8,000,000,000,000.
- Ask students to approximate a very small number by using a single digit multiplied by a power of 10. For example, give students the number 0.0000073. This number rounds to 0.000007 when rounded to the nearest millionth, so this approximation can be expressed as 7×10^{-6}. Help students see that the last digit of the rounded number is used as the single digit and then divided by 10 six times to arrive at the value of 0.000007. The process of dividing by 10 six times is equivalent to multiplying by 10^{-6} because 10^{-6} is equal to $\frac{1}{10^6}$, or $\frac{1}{10 \times 10 \times 10 \times 10 \times 10 \times 10}$.
- Ask students to make multiplicative comparisons between numbers that are represented as a single digit times an integer power of 10. For example, give students the numbers 4×10^{-8} and 2×10^{-6}. Students can determine that 2×10^{-6} is the greater of the two values because 10^{-6} is greater than 10^{-8}. Students can then compare the single digits to find that 2 is 0.5 times as great

as 4. Then, help students compare the powers of 10 to find that 10^{-6} is 10^2 times as great as 10^{-8}. The product of 0.5 and 10^2 is 50. Therefore, students should find that 2×10^{-6} is 50 times as great as 4×10^{-8}.

► Identifying and Addressing Common Misconceptions

- Students may reverse the relationship between two numbers when comparing numbers with negative exponents. Ask students which is greater, 6×10^{-9} or 6×10^{-6}. Students who identify 6×10^{-9} as being greater than 6×10^{-6} may have this misconception. Ask students to rewrite the numbers as repeated multiplication or division expressions and evaluate the expressions to produce decimal equivalents.

- Students may not include the nonzero digit when writing an approximation for a small number. For example, ask students to use a single digit times a power of 10 to estimate 0.0000032. Students who count 5 zeros and respond with 3×10^{-5} may have this misconception. Ask students to check their work by writing the decimal representation of 3×10^{-5}. Students can think of 3×10^{-5} as 3 divided by 10^5 or 3 divided by 10 five times. Each time 3 is divided by 10, the digit is moved one place to the right. So 3 divided by 10^5 is 0.00003. However, that is not equivalent to the estimated value of 0.000003, in which the 3 was moved to the right 6 places. The correct answer must be 3×10^{-6}.

- Students may use incorrect methods to compare relative sizes of numbers written as single digits multiplied by powers of 10. For example, ask students to determine how many times as great as the smaller number the larger number is in the set 6×10^5 and 3×10^7. Some students may recognize that 6 is 2 times as large as 3 and 10^7 is 100 times as great as 10^5 and respond with 200. Remind students to be consistent with comparing each part of the number. Compare 10^7 with 10^5 and 3 with 6. For example, 10^7 is 100 times as great as 10^5, and 3 is $\frac{1}{2}$ as great as 6. Therefore, 3×10^7 is 100 times $\frac{1}{2}$ as great as 6×10^5, which is 50.

► Enrichment

- Ask students to explore connections between powers of 10 and prefixes like mega-, giga-, tera-, etc.
- Ask students to investigate the ideas of “very large” and “very small.” For example, ask students to find a real-world scenario involving “very large” distances. Students could give examples of the distance across the Pacific Ocean, the distance to the moon, or the distance between two stars.
- Ask students to explore how increasing or decreasing the power to which the 10 is raised changes the value represented. For example, give students the number 4×10^6 and ask how the value changes when the 6 in the exponent is reduced to a 5. Students can observe the changed value is one-tenth the original value.

► Key Terms

estimate, exponent, integer, power, significant digit

B-E.1.1.4

▶ Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Demonstrate an understanding of expressions and equations with radicals and integer exponents.

Represent and use expressions and equations to solve problems involving radicals and integer exponents.

▶ Eligible Content

B-E.1.1.4 Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Express answers in scientific notation and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading). Interpret scientific notation that has been generated by technology (e.g., interpret 4.7EE9 displayed on a calculator as 4.7 × 109).

▶ Instructional Supports

Scientific notation

- Ask students to explain why extremely large quantities or extremely small quantities lend themselves to the use of scientific notation. Some examples are shown.
    - The distance between New York City and Boston measured in miles does not warrant the use of scientific notation, because the distance can be expressed with a relatively low number of digits. Conversely, the distance from the sun to Saturn measured in miles does warrant the use of scientific notation because the number of digits is relatively high.
    - The width of a human hair measured in millimeters warrants the use of scientific notation because the standard form of the number would use a relatively high number of digits. Conversely, the width of a coin measured in millimeters would not warrant the use of scientific notation, because the standard form of the number contains a relatively low number of digits.

- Ask students to find the sum or difference of two numbers expressed in scientific notation. For example, give students $(6.7 \times 10^6) - (4.4 \times 10^5)$. Ask students to write both terms with the same power of 10. The subtrahend, 4.4×10^5, is equivalent to 0.44×10^6. This results in the expression $(6.7 \times 10^6) - (0.44 \times 10^6)$. Ask students to use the distributive property to rewrite the equation as $(6.7 - 0.44) \times 10^6$ and then subtract to get a difference of 6.26×10^6.

- Ask students to find the product of two numbers given in scientific notation by applying the commutative and associative properties. For example, give students the product $(8.8 \times 10^3)(1.5 \times 10^8)$. The commutative and associative properties can be used to rewrite the expression as $(8.8 \times 1.5)(10^3 \times 10^8)$. The multiplication within each set of parentheses can then be performed to arrive at 13.2×10^{11}. This number can be expressed in scientific notation by dividing the first factor by 10 and multiplying the second factor by 10 to get 1.32×10^{12}.

- Ask students to find the quotient of two numbers given in scientific notation. For example, give students the quotient $(3.8 \times 10^4) \div (7.6 \times 10^{11})$. The commutative and associative properties along with the inverse relationship between multiplication and division can be used to rewrite the expression as $(3.8 \div 7.6)(10^4 \div 10^{11})$. The division within each set of parentheses can then be performed to arrive at 0.5×10^{-7}. This number can be expressed in scientific notation by multiplying the first factor by 10 and dividing the second factor by 10 to get 5×10^{-8}.

- Ask students to use a calculator to perform a calculation that results in an output on the calculator expressed in scientific notation and interpret the result displayed by the calculator. For example, give students the problem $(7.4 \times 10^4) \div (4 \times 10^{15})$. Most scientific calculators will display the quotient as either 1.85E–11 or 1.85×10^{-11}. On calculators that display the former result, the “E” indicates scientific notation and replaces the multiplication by the base of 10.

Units of measurement

- Ask students to choose an appropriate unit of measurement for a given context and ask them to explain why the unit is appropriate. For example, give students the context of the area of the United States. Students could choose units of square miles and explain that since the area is very large, the larger unit of square miles is more appropriate than square feet.

- Ask students to analyze the units used in a given context. For example, give students the context “The weight of a ship is measured in pounds.” Students can analyze the situation as shown.

Units of pounds are not an appropriate unit in this case because the weight of a ship would be a large number when measured in pounds. The unit of tons would be more appropriate in this situation.

► Identifying and Addressing Common Misconceptions

- Some students may add or subtract the exponents and the bases when performing the corresponding operation. Ask students to evaluate $(3.5 \times 10^8) - (7.3 \times 10^5)$. Students who respond with -3.8×10^3 may have this misconception. Ask students whether 1×10^2 is less than, greater than, or equal to 1×10^1. Help students recognize that because 1×10^2 is greater, subtracting the two quantities should result in a number greater than 0. However, their way of calculating $(1 \times 10^2) - (1 \times 10^1)$ would result in 0×10^1, which is equal to 0. Make sure students understand that when two positive values are expressed in scientific notation, the value with the larger exponent is greater, regardless of the constant multiplier; therefore, the difference of the original expression $(3.5 \times 10^8) - (7.3 \times 10^5)$ should also result in a number greater than 0.

- Some students may multiply or divide the exponents and bases when performing the associated operation. Ask students to evaluate $(8.6 \times 10^8) \div (4.3 \times 10^2)$. Students who respond with 2×10^4 may have this misconception. Ask students to rewrite the expression as $\frac{8.6 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10}{4.3 \times 10 \times 10}$. Help them use their understanding of fractions to cancel out two 10s from both the numerator and denominator of the fraction and then to rewrite the result in scientific notation as 2×10^6.

- Some students may incorrectly interpret the output of a calculator by not including the 10. Ask students to interpret the output of a calculator for the product $(3.1 \times 10^5)(3 \times 10^4)$. Students who interpret the output 9.3E9 as 9.3^9 may have this misconception. Ask students to evaluate the expression using the commutative and associative properties and compare the result to the interpretation of the calculator output.

- Some students may incorrectly rewrite a number in scientific notation. Show students that $(4 \times 10^3)(7 \times 10^6) = 28 \times 10^9$ and ask them to rewrite the product in scientific notation. Students who divide both the numeral and the exponential by 10 to get a result of 2.8×10^8 may have this misconception. Ask students to identify pairs of numbers that multiply to 48. Help them identify relationships between factor pairs (such as 16 × 3 and 4 × 12) in which one number in each factor pair has been multiplied by a number and the other number has been divided by that same number (in this case, 4). Help students recognize that multiplying one factor of a product and dividing the other factor of a product preserves the value of the product.

► Enrichment

- Ask students to solve equations in which the variable is an exponent of a number written in scientific notation. For example, ask students to solve for n in the equation $(1.8 \times 10^n)(6.5 \times 10^4) = 1.17 \times 10^{10}$. Students could solve by rewriting the equation as $(1.8 \times 6.5)(10^n \times 10^4) = 1.17 \times 10^{10}$. Students could then perform the first multiplication to get $11.7 \times (10^n \times 10^4) = 1.17 \times 10^{10}$. Students could then recognize 11.7 as equivalent to 1.17×10^1, and thus rewrite the equation as $(1.17 \times 10^1)(10^n \times 10^4) = 1.17 \times 10^{10}$. Using the associative property, this can be rewritten as $1.17(10^1 \times 10^n \times 10^4) = 1.17 \times 10^{10}$. Students can then recognize that $4 + 1 + n$ must be equal to 10, and therefore $n = 5$.
- Ask students to explore the relationships between the prefixes kilo- and milli- used for units of measure and the expression of measurements in scientific notation. For example, ask students to convert the measurement 8.3×10^{12} kilometers to meters and then to millimeters. Ask students to make observations about how the numbers change in each conversion and relate the change to the conversion factor between the units. Students can determine that the given value is equivalent to 8.3×10^{15} meters and 8.3×10^{18} millimeters. Students can observe that the change from kilometers to meters and the change from meters to millimeters each increase the exponent by 3 and that 10^3 is the same as the conversion factor of 1,000 between the measurements.
- Ask students to investigate raising a number expressed in scientific notation to a power. For example, ask students to evaluate the expression $(2.1 \times 10^5)^3$. Students can rewrite this expression as $(2.1 \times 10^5)(2.1 \times 10^5)(2.1 \times 10^5)$, which can then be rewritten as $(2.1 \times 2.1 \times 2.1)(10^5 \times 10^5 \times 10^5)$. Students can then evaluate the operations within each set of parentheses to get 9.261×10^{15}.

► Key Terms

decimal, exponent, power, radical, scientific notation, significant digit, unit

B-E.2.1.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Understand the connections between proportional relationships, lines, and linear equations.

Analyze and describe linear relationships between two variables, using slope.

► Eligible Content

B-E.2.1.1 Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways.

► Instructional Supports

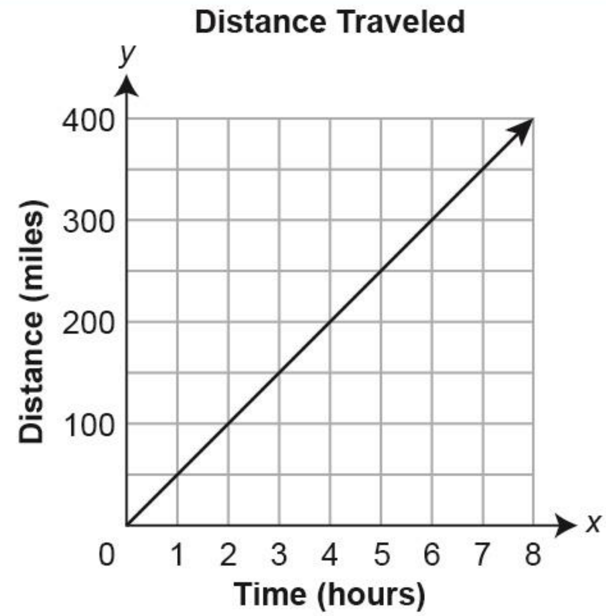
- Ask students to determine the unit rate, or slope, of a proportional relationship given as a table, graph, equation, diagram, or verbal description and explain its meaning. For example, give students the following table and ask them to find the constant of proportionality and explain their thinking.

Number of Oranges	Cost (dollars)
3	3.75
6	7.50
9	11.25
12	15

A possible student response is shown.

The constant of proportionality can be found by first determining that when the number of oranges increases by 3, the cost increases by $3.75. Then divide the price increase by the increase in number to get $1.25.

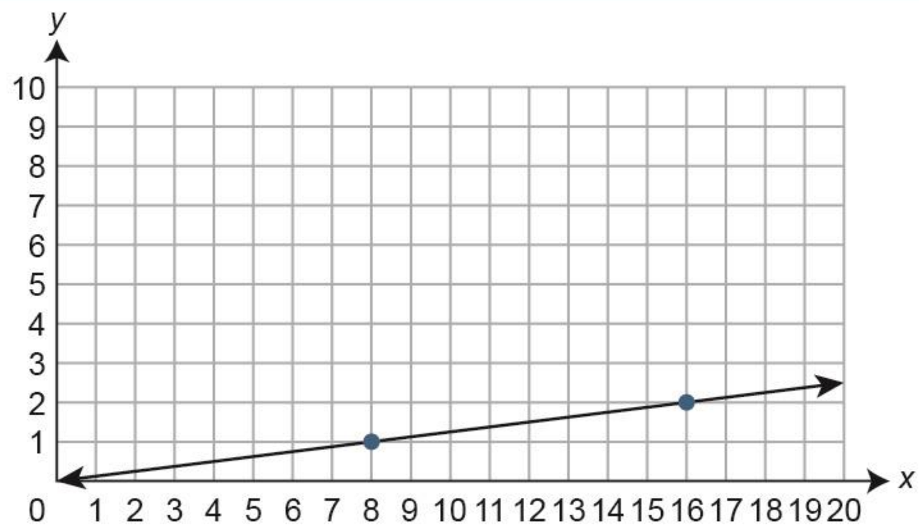
- Give students a proportional relationship represented using a table, graph, equation, diagram, or verbal description and ask them to represent it in a different way. Some examples are shown.
    - Give students the graph shown and ask them to create a table.



The constant of proportionality can be determined to be 50 miles per hour using any point along the line, such as (3, 150). The ratio of the values is 50 miles per hour. This information can be used to make a table as shown.

Time (hours)	**Distance (miles)**
0	0
1	50
2	100
3	150
4	200
5	250

- Give students the equation $y = \frac{1}{8}x$ and ask them to create a graph. A graph can be constructed by selecting two values of x and multiplying them by $\frac{1}{8}$, the unit rate, to find the corresponding values of y. Students using the values $x = 8$ and $x = 16$ should find the corresponding values of $y = 1$ and $y = 2$. Students then create the graph by plotting the two points and connecting them with a line.

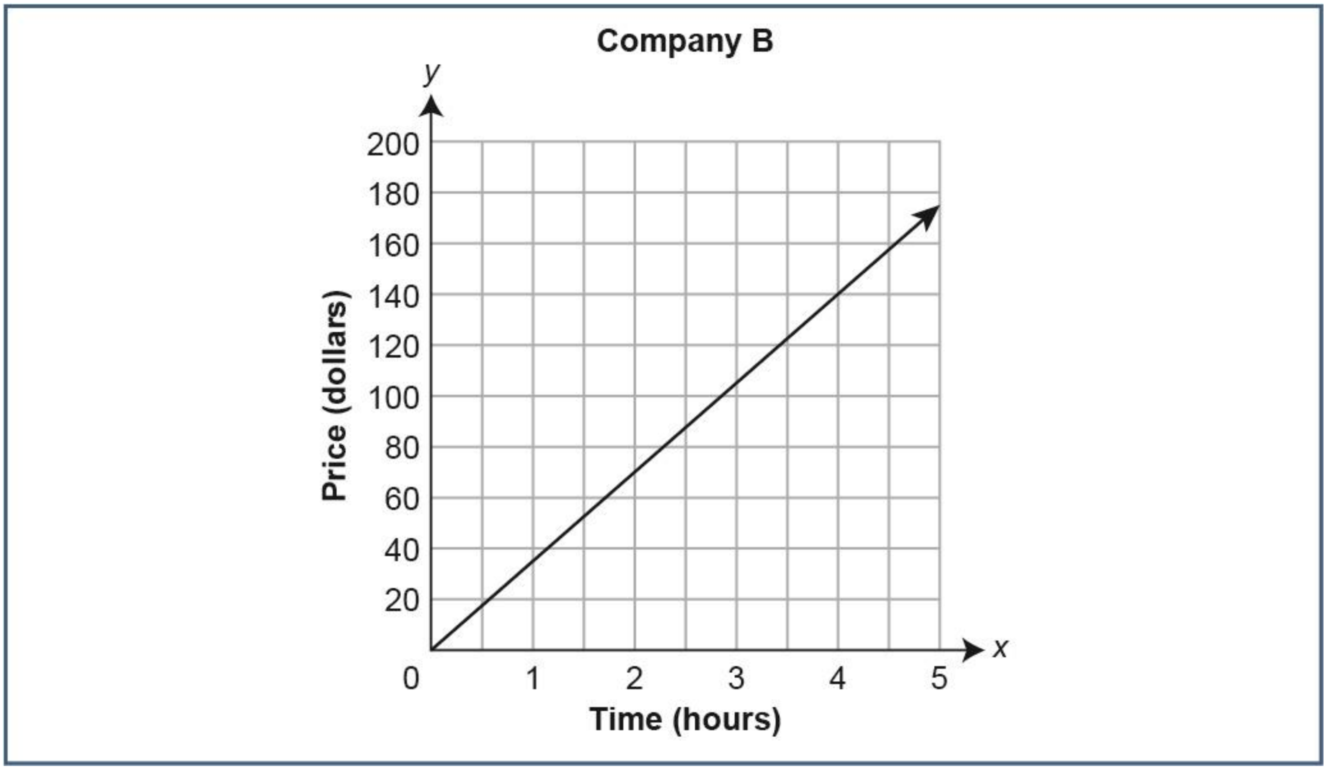


Students can confirm the values by noting that the slope of the line through these points is equal to the unit rate.

- Ask students to compare the slopes of two proportional relationships represented in different ways. For example, give students the table and graph shown.

Company A

Time (hours)	Price (dollars)
1	40
2	80
3	120
4	160



A possible student response is shown.

Company A charges $40 per hour, and company B charges $35 per hour. Company A has a price that is $5 per hour more than the price of company B.

► Identifying and Addressing Common Misconceptions

- Some students may not consider the changes in x-values when comparing proportional relationships given in different ways. Give students the equation $y = 0.2x$ and the following table.

x	y
0	0
2	0.4
4	0.8
6	1.2

Ask students to compare the rates of change of the two relationships. Students who identify the rate of change of the second relationship as twice that of the first may have this misconception. Give students an x-value from the table and ask them to give the corresponding y-value from the equation. Help students recognize that the values are the same in both cases. Then ask students to graph the relationships on a coordinate plane and then answer the question using the graph.

► Enrichment

- Ask students to investigate how changing the unit of measurement affects the unit rate.
- Ask students to investigate the difference between two proportional relationships. For example, give students a proportional relationship representing the distance run by person A over time and a proportional relationship representing the distance run by person B over time. Help students recognize that the difference between the two proportional relationships is also a proportional relationship and that this difference can be interpreted as the size of the distance gap between the two people running.
- Ask students to use unit rates of proportional relationships given in different ways to solve problems. For example, give students an equation showing the growth of a plant and a graph showing the growth of a different plant. Ask students to determine how much taller one plant will be than the other plant after 4 days and to explain how their answer was determined.

► Key Terms

constant of proportionality, coordinate plane, equation, graph, input, output, proportional, slope, table, unit rate, variable

B-E.2.1.2 & B-E.2.1.3

► Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Understand the connections between proportional relationships, lines, and linear equations.

Analyze and describe linear relationships between two variables, using slope.

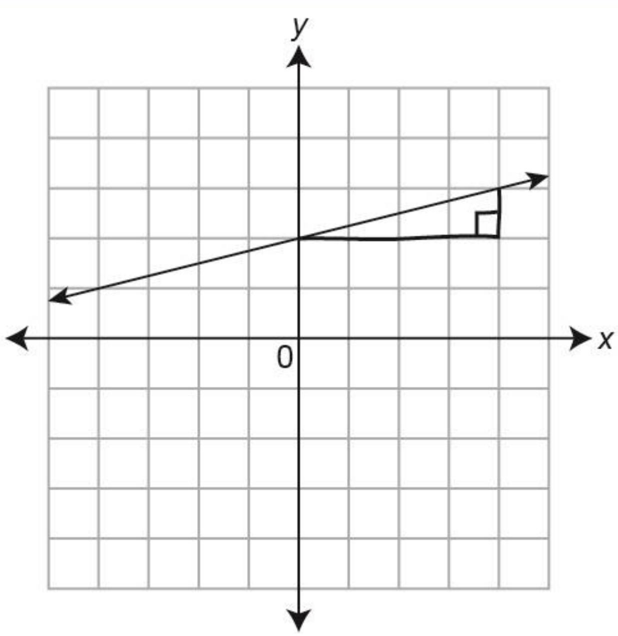
► Eligible Content

B-E.2.1.2 Use similar right triangles to show and explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane.

B-E.2.1.3 Derive the equation y = mx for a line through the origin and the equation y = mx + b for a line intercepting the vertical axis at b.

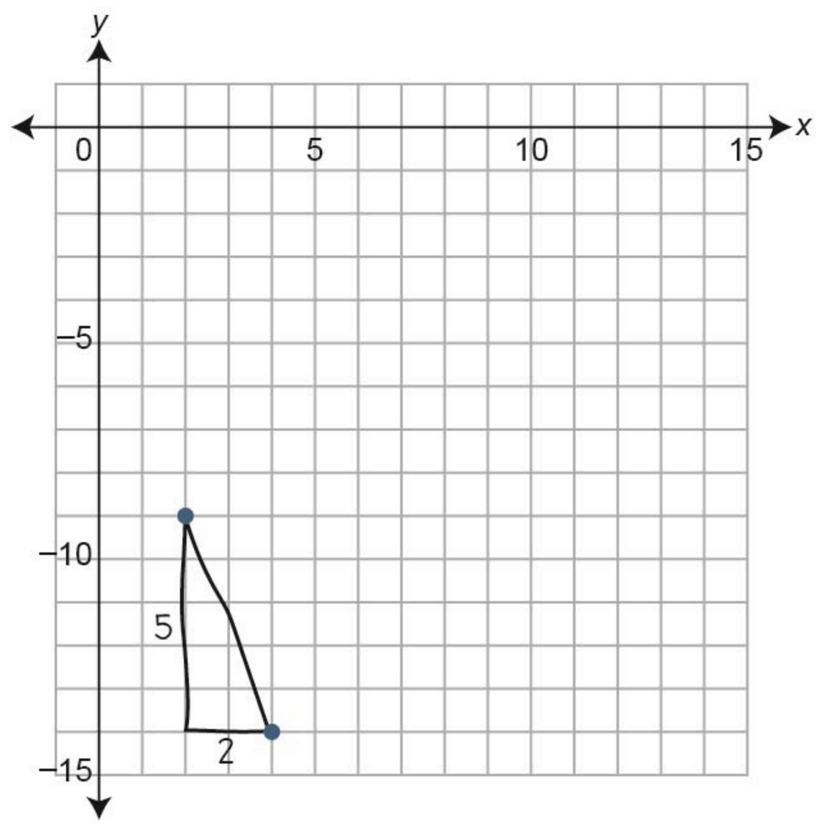
► Instructional Supports

- Give students a graph of a line and ask students to find the slope of the line. For example, give students the line shown.



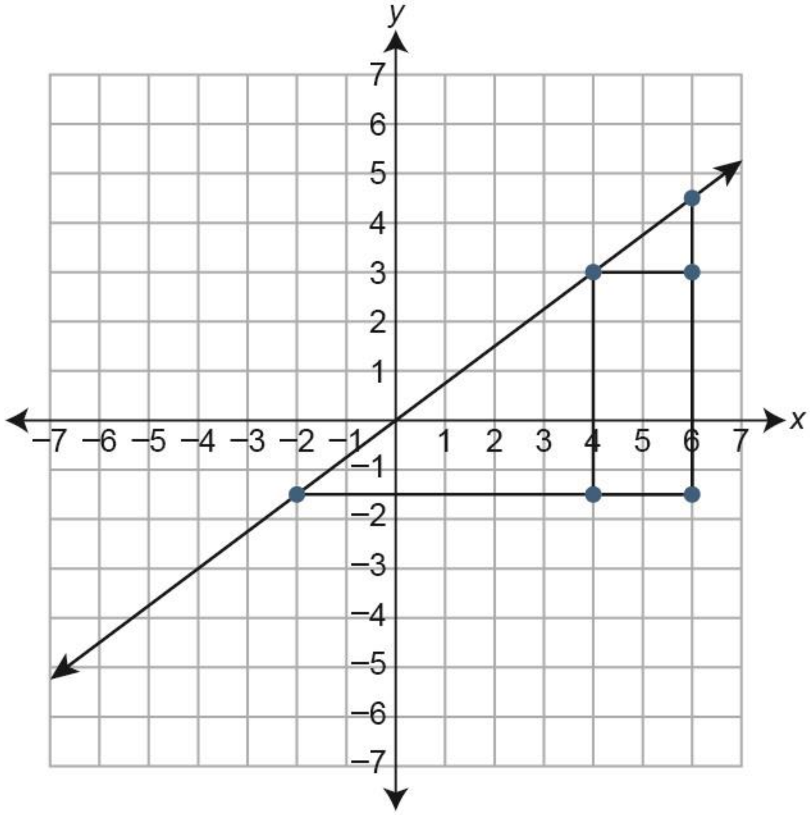
Students can create a right triangle by drawing a horizontal line segment from (0, 2) to (4, 2) and a vertical line segment from (4, 3) to (4, 2). Students can then find the ratio of the triangle height to its width, which gives the slope of the line as $\frac{1}{4}$.

- Ask students to find the slope of a line when given two points on the line. For example, give students the points (2, –9) and (4, –14). Students can create a right triangle with a width equal to the difference in the *x*-values of the two points and a height equal to the difference of the *y*-values of the two points.



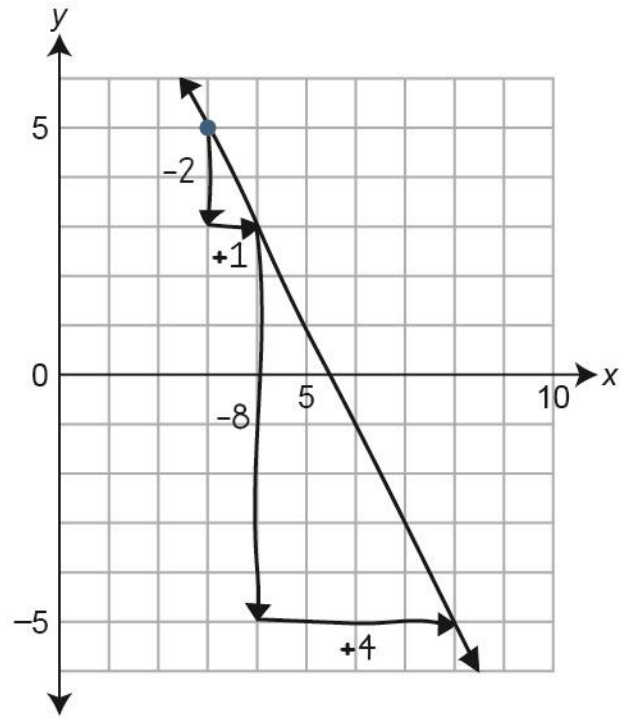
In this case, the height is 5 and the width is 2. Students can then find the ratio of the triangle height to its width. In this case, the *y*-values decrease as the *x*-values increase, so the sign on the 5 is negative. This gives a ratio of $\frac{-5}{2}$, which is the slope of the graph.

- Ask students to use similar triangles to show that the slope of a non-vertical line is a constant. For example, tell students that a line passes through the points (–2, –1.5), (4, 3), and (6, 4.5). The figure shown has that line as well as three right triangles of varying sizes, all using the line as the hypotenuse.



The acute angles of the triangles are congruent because they are corresponding angles by using the corresponding legs of the triangles as the parallel lines. Because the triangles all have the same angle measures, the three triangles are all similar. Therefore, the ratios of the lengths of the vertical legs to the horizontal legs of the right triangles are all the same: $\frac{1.5}{2} = \frac{4.5}{6} = \frac{6}{8}$. The line has the same slope no matter which pair of points is used to calculate the slope.

- Give students the slope of a line and a point on the line and ask them to explain how to use similar right triangles to identify additional points on the line. For example, give students the point (3, 5) and a slope of –2. A possible student response is shown.



I drew a vertical line segment from (3, 5) down 2 units and then a horizontal line segment 1 unit to the right to the point (4, 3) since the slope is −2. Then I drew a vertical line segment 8 units down from (4, 3) and a horizontal line segment 4 units to the right to (8, −5) because $\frac{-8}{4} = -2$.

- Ask students to derive the equation of a line through the origin when given one other point on the line. For example, give students the point (9, 6) on a line through the origin. The equation $y = \frac{2}{3}x$ can be derived for this situation since a straight line through the origin represents a proportional relationship. The constant of proportionality is $\frac{6}{9} = \frac{2}{3}$.
- Ask students to determine whether the line defined by two given points passes through the origin. For example, give students the points (6, 2) and (10, 7). It can be determined that the ratio represented by the first point is $\frac{2}{6}$ and by the second point is $\frac{7}{10}$. Since the ratios are not equal, the

relationship represented by the two points is not proportional and the line does not pass through the origin.

- Ask students to show the relationship between the equations $y = mx$ and $y = mx + b$ for specified values of m and b. For example, give students the equations $y = 0.5x$ and $y = 0.5x + 4$. Ask students to create tables to represent the equations.

$y = 0.5x$

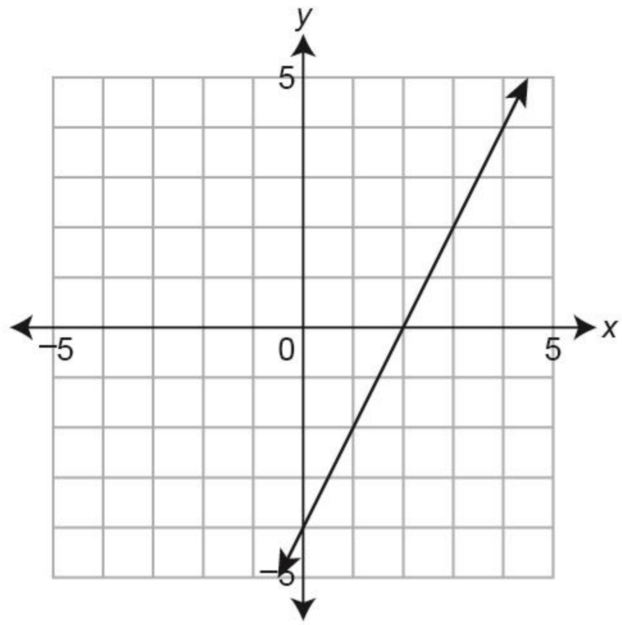
x	*y*
0	0
2	1
4	2
6	3

$y = 0.5x + 4$

x	*y*
0	4
2	5
4	6
6	7

The points in the second equation have y-values that are 4 greater than those of the first equation for the same x-values. Also, the change in y-values is the same for each equation when the change in x-values is the same. Ask students to plot the points and show that each point on the graph of the second equation is 4 units higher than a point on the graph of the first equation. Note that because the first graph passes through the origin, the second graph crosses the y-axis at 4.

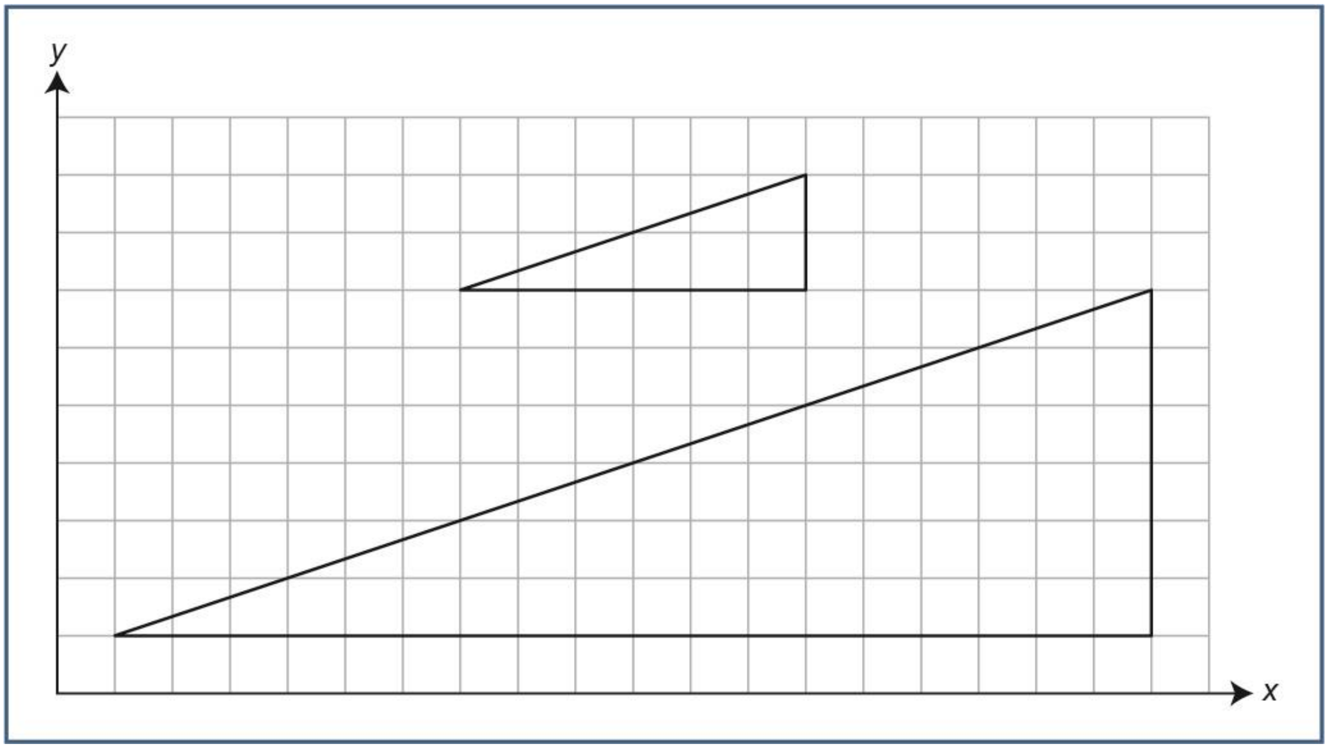
- Ask students to create an equation in the form $y = mx + b$ when given a graph of a non-vertical line. For example, give students the graph shown.



The equation $y = 2x + -4$ can be written for the given line. The value of b is determined by finding the point where the line crosses the y-axis, (0, –4). The value of m is determined by finding the change in y-values between two consecutive integer x-values, such as (1, –2) and (2, 0), where the change in y-value is 2.

► Identifying and Addressing Common Misconceptions

- Students may associate a larger triangle with a steeper slope. For example, give students the following graphic and ask students to compare the slopes of the slanted line segments.



Students who identify the longer line segment as having a steeper slope may have this misconception. Ask students to cut out a right triangle with a width of 6 and a height of 2, lay it on the grid, and find its slope. Then ask them to rotate the triangle so that it has a width of 2 and a height of 6, lay it on the grid, and find its slope. Students should recognize that a large slope and a small slope can both be shown using the same triangle.

- Students may perform operations in an incorrect order when calculating the slope for an equation. For example, ask students to write an equation for a line passing through the points (2, 8) and (3, 6). Students who respond with $y = 2x + 4$ may have this misconception. Ask students to substitute the x-values into the equation and calculate the associated y-values. Have them compare the resulting values to those originally given. Also, have the students plot the two points and their calculated y-intercept on a coordinate grid to see that their y-intercept does not fall on the same line as the two given points. Remind students that subtraction is not commutative so they must be consistent when finding the difference between two coordinates. If the difference in y-coordinates is determined using $8 - 6$, then the difference in x-coordinates must be determined using $2 - 3$ because 2 is the x-coordinate paired with 8.

- Students may not consider the shift of a line's y-intercept from the origin. For example, ask students to write an equation for the line passing through the points (4, 7) and (5, 9). Students who write $y = 2x$ may have this misconception. Ask students to explain the significance of the 2 in the equation. Then ask them to draw multiple different lines with a slope of 2. Help students recognize that a single number in an equation is not sufficient to describe a specific line with a slope of 2. Then ask students to use the slope to determine that other points on the line include (3, 5), (2, 3), (1, 1), and (0, –1). Help students recognize how the y-intercept can be used to specify a unique line of slope 2.

► Enrichment

- Ask students to explore vertical shifts of a given linear relationship. For example, give students the equation $y = 3x + 6$. Ask them to graph a line representing the equation. Then ask students to subtract 2 from the y-value of each point on the line and graph a line representing these new points. Ask students to represent the new line with an equation and note how the equation is different from the given equation. Students can determine that the equation of the new line is $y = 3x + 4$, which is found by taking the given equation and subtracting 2 from the b term.
- Ask students to explore the nonlinear relationship between slopes and angles. Ask students to draw lines with slopes of 0, 1, 2, 3, and 4. Students should notice that there is a large angle between the lines with slopes of 0 and 1 but that there are gradually decreasing angles between the lines of subsequent slopes.
- Ask students to explore the slopes of parallel lines using similar triangles. For example, ask students to use triangles to compare the slopes of the linear equations $y = 3.5x$ and $y = 3.5x + 6$. Students can observe that right triangles drawn to have the hypotenuse along each of the lines are similar.

► Key Terms

axis, coordinate plane, graph, horizontal, hypotenuse, input, intercept, origin, output, right triangle, similar, slope, slope triangle, unit rate, vertical, y-intercept

B-E.2.1.2 & B-E.2.1.3

► Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Understand the connections between proportional relationships, lines, and linear equations.

Analyze and describe linear relationships between two variables, using slope.

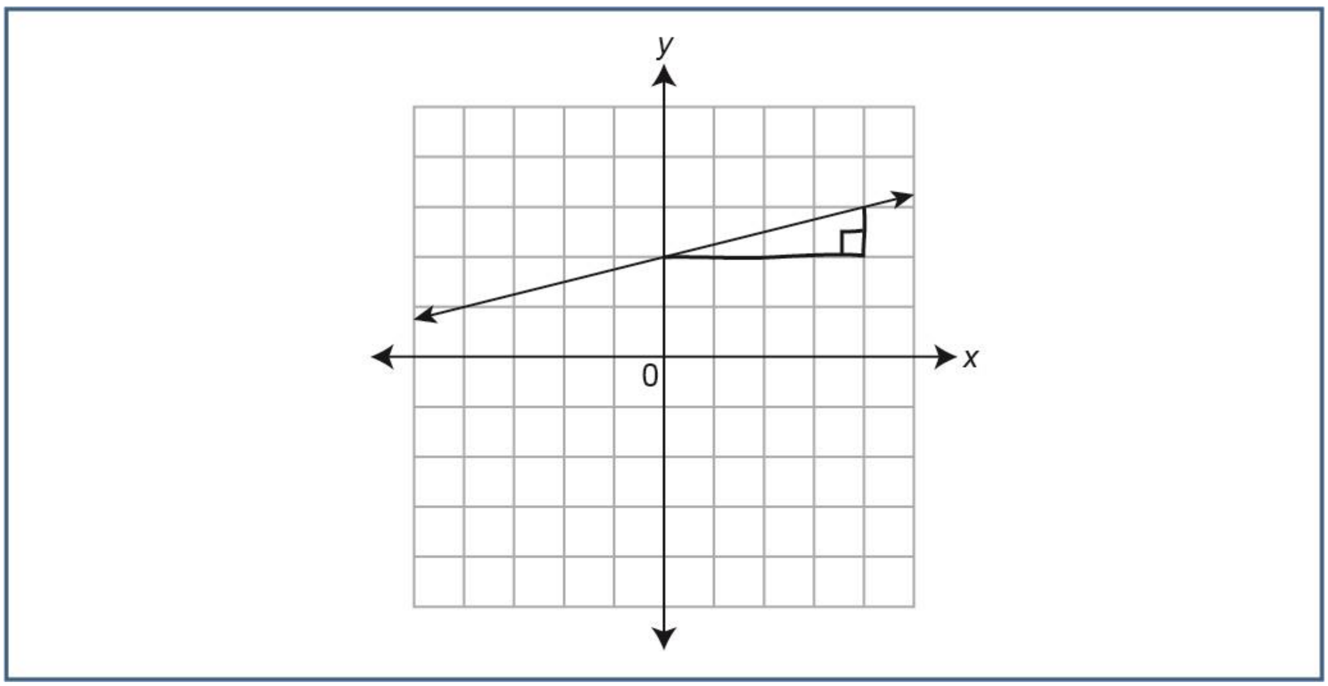
► Eligible Content

B-E.2.1.2 Use similar right triangles to show and explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane.

B-E.2.1.3 Derive the equation y = mx for a line through the origin and the equation y = mx + b for a line intercepting the vertical axis at b.

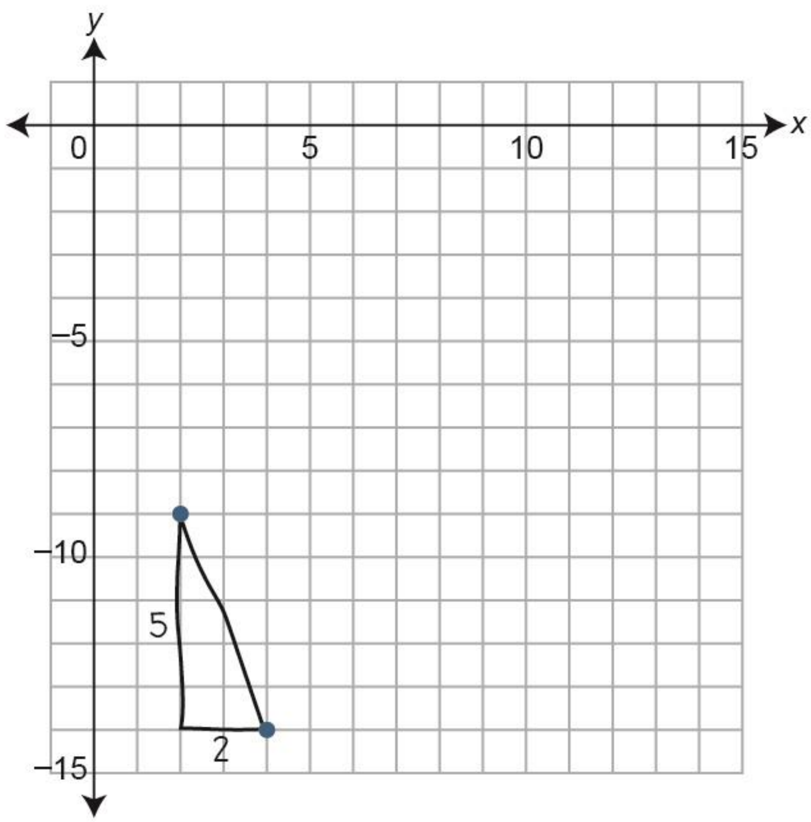
► Instructional Supports

- Give students a graph of a line and ask students to find the slope of the line. For example, give students the line shown.



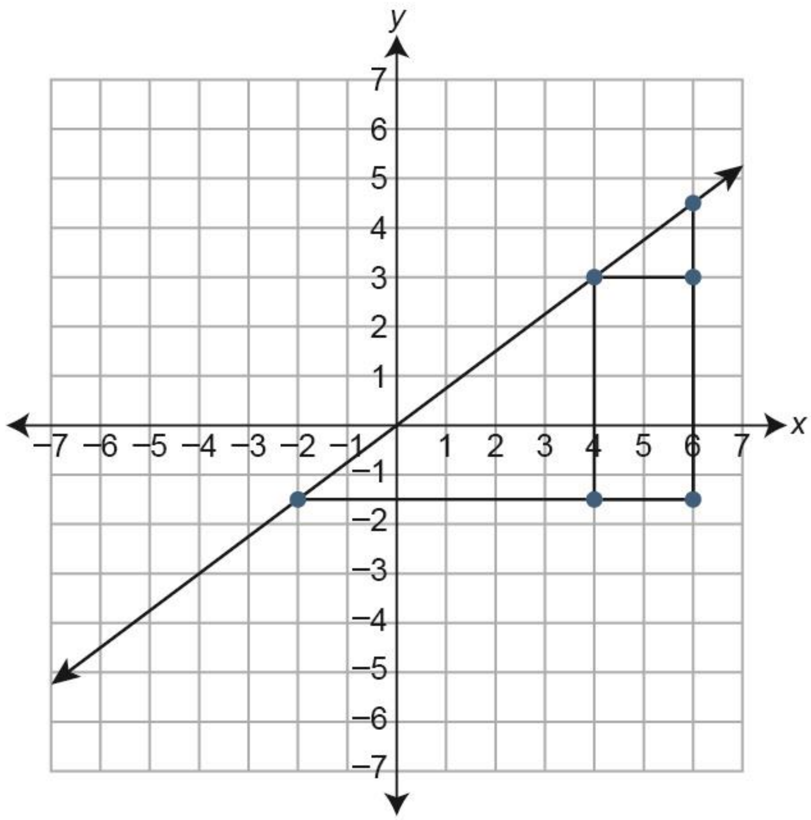
Students can create a right triangle by drawing a horizontal line segment from (0, 2) to (4, 2) and a vertical line segment from (4, 3) to (4, 2). Students can then find the ratio of the triangle height to its width, which gives the slope of the line as $\frac{1}{4}$.

- Ask students to find the slope of a line when given two points on the line. For example, give students the points (2, –9) and (4, –14). Students can create a right triangle with a width equal to the difference in the *x*-values of the two points and a height equal to the difference of the *y*-values of the two points.



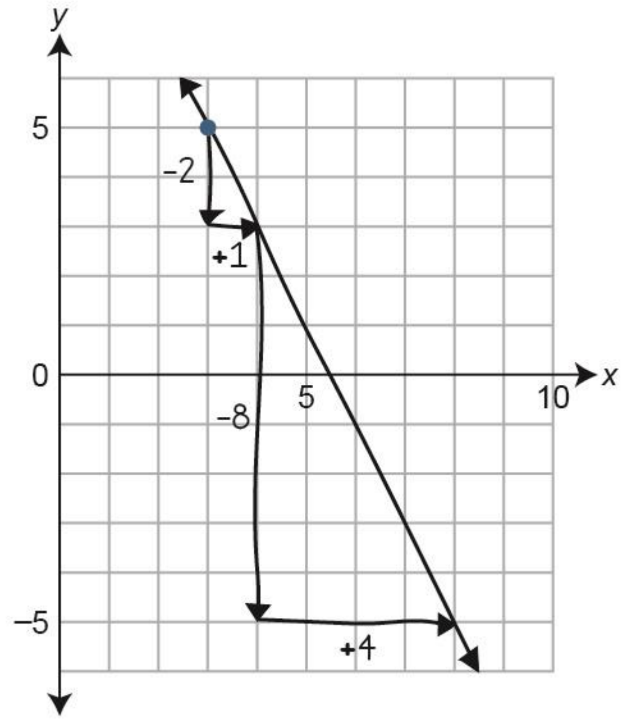
In this case, the height is 5 and the width is 2. Students can then find the ratio of the triangle height to its width. In this case, the *y*-values decrease as the *x*-values increase, so the sign on the 5 is negative. This gives a ratio of $\frac{-5}{2}$, which is the slope of the graph.

- Ask students to use similar triangles to show that the slope of a non-vertical line is a constant. For example, tell students that a line passes through the points (–2, –1.5), (4, 3), and (6, 4.5). The figure shown has that line as well as three right triangles of varying sizes, all using the line as the hypotenuse.



The acute angles of the triangles are congruent because they are corresponding angles by using the corresponding legs of the triangles as the parallel lines. Because the triangles all have the same angle measures, the three triangles are all similar. Therefore, the ratios of the lengths of the vertical legs to the horizontal legs of the right triangles are all the same: $\frac{1.5}{2} = \frac{4.5}{6} = \frac{6}{8}$. The line has the same slope no matter which pair of points is used to calculate the slope.

- Give students the slope of a line and a point on the line and ask them to explain how to use similar right triangles to identify additional points on the line. For example, give students the point (3, 5) and a slope of –2. A possible student response is shown.



I drew a vertical line segment from (3, 5) down 2 units and then a horizontal line segment 1 unit to the right to the point (4, 3) since the slope is −2. Then I drew a vertical line segment 8 units down from (4, 3) and a horizontal line segment 4 units to the right to (8, −5) because $\frac{-8}{4} = -2$.

- Ask students to derive the equation of a line through the origin when given one other point on the line. For example, give students the point (9, 6) on a line through the origin. The equation $y = \frac{2}{3}x$ can be derived for this situation since a straight line through the origin represents a proportional relationship. The constant of proportionality is $\frac{6}{9} = \frac{2}{3}$.
- Ask students to determine whether the line defined by two given points passes through the origin. For example, give students the points (6, 2) and (10, 7). It can be determined that the ratio represented by the first point is $\frac{2}{6}$ and by the second point is $\frac{7}{10}$. Since the ratios are not equal, the

relationship represented by the two points is not proportional and the line does not pass through the origin.

- Ask students to show the relationship between the equations $y = mx$ and $y = mx + b$ for specified values of m and b. For example, give students the equations $y = 0.5x$ and $y = 0.5x + 4$. Ask students to create tables to represent the equations.

$y = 0.5x$

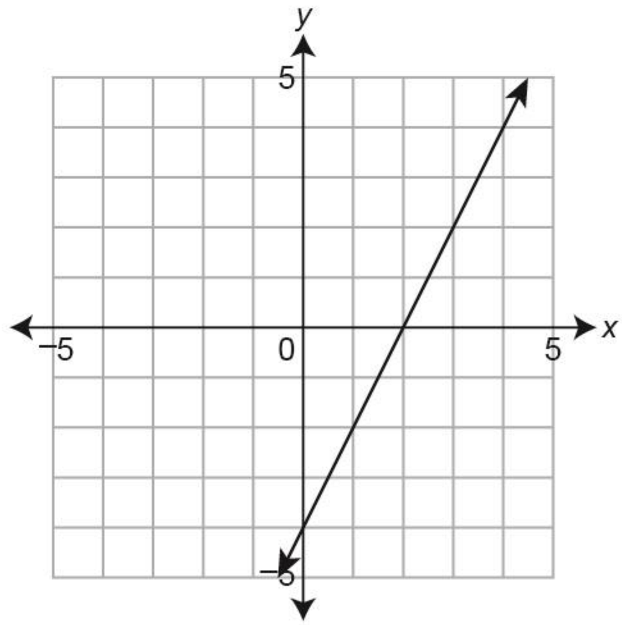
x	*y*
0	0
2	1
4	2
6	3

$y = 0.5x + 4$

x	*y*
0	4
2	5
4	6
6	7

The points in the second equation have y-values that are 4 greater than those of the first equation for the same x-values. Also, the change in y-values is the same for each equation when the change in x-values is the same. Ask students to plot the points and show that each point on the graph of the second equation is 4 units higher than a point on the graph of the first equation. Note that because the first graph passes through the origin, the second graph crosses the y-axis at 4.

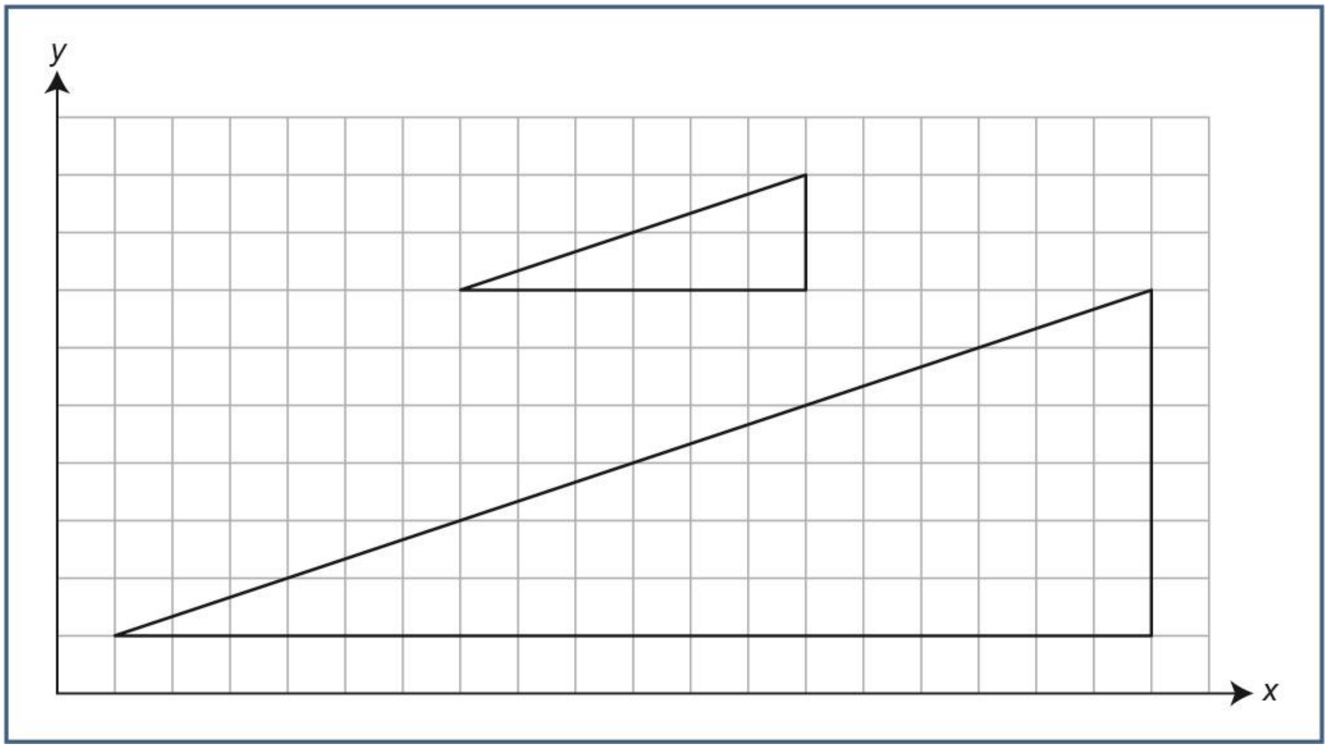
- Ask students to create an equation in the form $y = mx + b$ when given a graph of a non-vertical line. For example, give students the graph shown.



The equation $y = 2x + -4$ can be written for the given line. The value of b is determined by finding the point where the line crosses the y-axis, (0, –4). The value of m is determined by finding the change in y-values between two consecutive integer x-values, such as (1, –2) and (2, 0), where the change in y-value is 2.

► Identifying and Addressing Common Misconceptions

- Students may associate a larger triangle with a steeper slope. For example, give students the following graphic and ask students to compare the slopes of the slanted line segments.



Students who identify the longer line segment as having a steeper slope may have this misconception. Ask students to cut out a right triangle with a width of 6 and a height of 2, lay it on the grid, and find its slope. Then ask them to rotate the triangle so that it has a width of 2 and a height of 6, lay it on the grid, and find its slope. Students should recognize that a large slope and a small slope can both be shown using the same triangle.

- Students may perform operations in an incorrect order when calculating the slope for an equation. For example, ask students to write an equation for a line passing through the points (2, 8) and (3, 6). Students who respond with $y = 2x + 4$ may have this misconception. Ask students to substitute the x-values into the equation and calculate the associated y-values. Have them compare the resulting values to those originally given. Also, have the students plot the two points and their calculated y-intercept on a coordinate grid to see that their y-intercept does not fall on the same line as the two given points. Remind students that subtraction is not commutative so they must be consistent when finding the difference between two coordinates. If the difference in y-coordinates is determined using 8 – 6, then the difference in x-coordinates must be determined using 2 – 3 because 2 is the x-coordinate paired with 8.

- Students may not consider the shift of a line's y-intercept from the origin. For example, ask students to write an equation for the line passing through the points (4, 7) and (5, 9). Students who write $y = 2x$ may have this misconception. Ask students to explain the significance of the 2 in the equation. Then ask them to draw multiple different lines with a slope of 2. Help students recognize that a single number in an equation is not sufficient to describe a specific line with a slope of 2. Then ask students to use the slope to determine that other points on the line include (3, 5), (2, 3), (1, 1), and (0, –1). Help students recognize how the y-intercept can be used to specify a unique line of slope 2.

► Enrichment

- Ask students to explore vertical shifts of a given linear relationship. For example, give students the equation $y = 3x + 6$. Ask them to graph a line representing the equation. Then ask students to subtract 2 from the y-value of each point on the line and graph a line representing these new points. Ask students to represent the new line with an equation and note how the equation is different from the given equation. Students can determine that the equation of the new line is $y = 3x + 4$, which is found by taking the given equation and subtracting 2 from the b term.
- Ask students to explore the nonlinear relationship between slopes and angles. Ask students to draw lines with slopes of 0, 1, 2, 3, and 4. Students should notice that there is a large angle between the lines with slopes of 0 and 1 but that there are gradually decreasing angles between the lines of subsequent slopes.
- Ask students to explore the slopes of parallel lines using similar triangles. For example, ask students to use triangles to compare the slopes of the linear equations $y = 3.5x$ and $y = 3.5x + 6$. Students can observe that right triangles drawn to have the hypotenuse along each of the lines are similar.

► Key Terms

axis, coordinate plane, graph, horizontal, hypotenuse, input, intercept, origin, output, right triangle, similar, slope, slope triangle, unit rate, vertical, y-intercept

B-E.3.1.1

▶ Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Analyze and solve linear equations and pairs of simultaneous linear equations.

Write, solve, graph, and interpret linear equations in one or two variables, using various methods.

▶ Eligible Content

B-E.3.1.1 Write and identify linear equations in one variable with one solution, infinitely many solutions, or no solutions. Show which of these possibilities is the case by successively transforming the given equation into simpler forms, until an equivalent equation of the form x = a, a = a, or a = b results (where a and b are different numbers).

▶ Instructional Supports

- Ask students to solve an equation of the form $ax + b = c$, where a, b, and c are all rational, justifying each step in the process. For example, give students the equation $\frac{1}{4}x - \frac{3}{4} = 2$.

$\frac{1}{4}x - \frac{3}{4} = 2$	given
$\frac{1}{4}x - \frac{3}{4} + \frac{3}{4} = 2 + \frac{3}{4}$	addition property of equality
$\frac{1}{4}x + 0 = \frac{11}{4}$	additive inverse; add $2 + \frac{3}{4}$
$\frac{1}{4}x = \frac{11}{4}$	additive identity
$4 \bullet \frac{1}{4}x = \frac{11}{4} \bullet 4$	multiplication property of equality
$1x = 11$	multiplicative inverse; multiply $\frac{11}{4} \bullet 4$
$x = 11$	multiplicative identity

- Ask students to identify errors made in a solution process for an equation in the form $ax + b = c$. For example, give students the work shown and ask them to identify and correct the errors in the work.

$$\frac{1}{4}x - 5 = 7$$

$$\frac{1}{4}x = 12$$

$$x = 3$$

Students can determine that the error was made in going from the second equation to the third. Both sides of the equation should have been multiplied by 4. A possible correction is shown.

$$\begin{array}{rcl}
\frac{1}{4}x - 5 & = & 7 \\
+5 & & +5 \\
\hline
\frac{1}{4}x & = & 12 \\
\cdot 4 & & \cdot 4 \\
\hline
x & = & 48
\end{array}$$

- Ask students to explain what it means for an equation to have no solution or infinitely many solutions. For example, give students the equation $7 + a = 2 + a$. There is no value of a for which 7 can be added to a and 2 can be added to a and the result is the same number. Also, give students the equation $4 + 2 + x = x + 6$. For any value of x, the left and right sides of the equation are equal; therefore, the equation has infinitely many solutions.
- Ask students to solve equations having no solution or infinitely many solutions. For example, give students the following equations.
    - $4(a + 3) – a = 3(a + 5) – 5$

$$4(a + 3) - a \stackrel{?}{=} 3(a + 5) - 5$$

$$4a + 12 - a \stackrel{?}{=} 3a + 15 - 5$$

$$3a + 12 \stackrel{?}{=} 3a + 10$$

$$3a + 12 \neq 3a + 10$$

no solution

Students can conclude that the equation has no solution because adding 12 to $3a$ and 10 to $3a$ will never yield the same value.

- $3(b + 8) = 2(b + 12) + b$

$$3(b + 8) \stackrel{?}{=} 2(b + 12) + b$$
$$3b + 24 \stackrel{?}{=} 2b + 24 + b$$
$$3b + 24 \stackrel{?}{=} 3b + 24$$
$$3b + 24 = 3b + 24$$

infinitely many solutions

Students can conclude the equation has infinitely many solutions since any value for b will make the equation true.

► Identifying and Addressing Common Misconceptions

- Some students may interpret any equation in which all variables can be eliminated during the process of successive transformations as having no solutions. For example, ask students to identify the number of solutions for $\frac{1}{2}(6x + 4x) = 5x$. Students who identify the equation as having no solution may have this misconception. Ask students to try evaluating each side of the given equation for some values of x.
- Some students may confuse the value of x with the number of solutions. For example, ask students to identify the number of solutions to the equation $4(x + 4) = 16$. Students who solve the equation to get $x = 0$ and then identify the equation as having 0 solutions may have this misconception. Ask students to explain what it means for a number to be a solution of the equation. Then ask students to substitute the value $x = 0$ into the equation.
- Students may identify an equation showing two different numbers equal to each other as indicating two solutions. For example, ask students how many solutions there are for the equation $2x + 4 = 2(3 + x)$. Students who identify the equation as having two solutions may have

this misconception. Ask students to substitute the values they identified as solutions into the original equation and then check to see if the two sides are equivalent.

▶ Enrichment

- Ask students to identify real-world situations that can be represented by equations having no solution or infinitely many solutions. For example, students could identify that two vehicles traveling in the same direction and traveling at the same speed on the same road but starting from different points could be represented by an equation with no solution. Also, two plants growing at the same rate and starting at the same height could be represented by an equation with infinitely many solutions.
- Ask students to interpret a situation in which an equation has one solution, but the solution is not meaningful in the context. For example, ask students to examine the situation "Two students play 5 rounds of a game. One student scores 20 points per round and receives a bonus of 60 points, and the other student scores 30 points per round and does not receive a bonus. After how many rounds will both players have the same score?" Students could write the equation $20r + 60 = 30r$ to describe the situation. They can then determine that the equation has one solution, $r = 6$. When interpreted within the given context, however, there is no solution because a solution of 6 rounds is impossible if the game lasts only 5 rounds.
- Ask students to determine values for a constant in an equation that result in the equation having no solution or infinitely many solutions. For example, give students the equation $5(x + b) = x + 4(x + 10) + 5$ and ask them to determine the values of b for which the equation has no solution and the values of b for which the equation has infinitely solutions. Students can determine that when $b = 9$, the equation has infinitely many solutions, and for all other values of b, the equation has no solution.

▶ Key Terms

constant, equation, equivalent, infinite, linear, solution, variable

B-E.3.1.2

▶ Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Analyze and solve linear equations and pairs of simultaneous linear equations.

Write, solve, graph, and interpret linear equations in one or two variables, using various methods.

▶ Eligible Content

B-E.3.1.2 Solve linear equations that have rational number coefficients, including equations whose solutions require expanding expressions using the distributive property and collecting like terms.

▶ Instructional Supports

- Ask students to solve equations in which there are variables on both sides of the equation. For example, give students the equation $\frac{1}{8}x + 8 = \frac{3}{8}x - 13$. Students can start by subtracting $\frac{1}{8}x$ from each side and combining like terms.

$$\begin{array}{rcl} \frac{1}{8}x + 8 & = & \frac{3}{8}x - 13 \\ -\frac{1}{8}x \quad & & -\frac{1}{8}x \\ \hline 8 & = & \frac{2}{8}x - 13 \end{array}$$

The fraction $\frac{2}{8}$ can be rewritten as $\frac{1}{4}$. Students can then add 13 to each side of the equation and combine like terms to begin isolating the variable. Students can then finish isolating the variable by multiplying both sides of the equation by 4 to arrive at a solution of $x = 84$.

$$
\begin{aligned}
8 &= \frac{1}{4}x - 13 \\
+13 & \qquad\quad +13 \\
\hline
21 &= \frac{1}{4}x \\
\cdot 4 & \quad \cdot 4 \\
\hline
84 &= x
\end{aligned}
$$

Students can verify the solution to the equation by inserting 84 into the original equation for x and then following the order of operations to evaluate each side of the equation.

$$\frac{1}{8}(84) + 8 = \frac{3}{8}(84) - 13$$

$$\frac{21}{2} + 8 = \frac{63}{2} - 13$$

$$\frac{21}{2} + \frac{16}{2} = \frac{63}{2} - \frac{26}{2}$$

$$\frac{37}{2} = \frac{37}{2}$$

Students are thus able to show that 84 is the correct solution to the given equation.

- Ask students to solve equations where the distributive property can be used to generate an equivalent expression. For example, give students the equation $\frac{9}{5} - \frac{3}{5}(x - \frac{1}{2}) = \frac{1}{3}x - \frac{1}{10} + \frac{1}{6}x$. Students can generate an equivalent expression for $\frac{9}{5} - \frac{3}{5}(x - \frac{1}{2})$ by using the distributive property and combining like terms.

$$\frac{9}{5} - \frac{3}{5}(x - \frac{1}{2})$$
$$\frac{9}{5} - \frac{3}{5}x + \frac{3}{10}$$
$$\frac{9}{5} + \frac{3}{10} - \frac{3}{5}x$$
$$\frac{18}{10} + \frac{3}{10} - \frac{3}{5}x$$
$$\frac{21}{10} - \frac{3}{5}x$$

Students can generate an equivalent expression for $\frac{1}{3}x - \frac{1}{10} + \frac{1}{6}x$ by combining like terms.

$$\frac{1}{3}x - \frac{1}{10} + \frac{1}{6}x$$
$$\frac{1}{3}x + \frac{1}{6}x - \frac{1}{10}$$
$$\frac{2}{6}x + \frac{1}{6}x - \frac{1}{10}$$
$$\frac{3}{6}x - \frac{1}{10}$$

Students can recognize that $\frac{3}{6} = \frac{1}{2}$ and use the equivalent expressions generated to create the equivalent equation $\frac{21}{10} - \frac{3}{5}x = \frac{1}{2}x - \frac{1}{10}$. Students can then solve by adding $\frac{3}{5}x$ to each side and then adding $\frac{1}{10}$ to each side of the equation. Finally, students can multiply each side of the equation by $\frac{10}{11}$ to isolate the variable.

$$
\begin{aligned}
\frac{21}{10} - \frac{3}{5}x &= \frac{1}{2}x - \frac{1}{10} \\
+\frac{3}{5}x \quad & \quad +\frac{3}{5}x \\
\hline
\frac{21}{10} &= \frac{11}{10}x - \frac{1}{10} \\
+\frac{1}{10} \quad & \quad +\frac{1}{10} \\
\hline
\frac{22}{10} &= \frac{11}{10}x \\
\cdot\frac{10}{11} \quad & \quad \cdot\frac{10}{11} \\
\hline
2 &= x
\end{aligned}
$$

Students can verify the solution by inserting 2 into the original equation for x and evaluating each side to check that both sides are equivalent.

$$
\begin{aligned}
\frac{9}{5} - \frac{3}{5}\left(2 - \frac{1}{2}\right) &= \frac{1}{3}(2) - \frac{1}{10} + \frac{1}{6}(2) \\
\frac{9}{5} - \frac{3}{5}\left(\frac{3}{2}\right) &= \frac{1}{3}(2) - \frac{1}{10} + \frac{1}{6}(2) \\
\frac{9}{5} - \frac{9}{10} &= \frac{2}{3} - \frac{1}{10} + \frac{1}{3} \\
\frac{9}{10} &= \frac{9}{10}
\end{aligned}
$$

► Identifying and Addressing Common Misconceptions

- Some students may not use the inverse operation when trying to move all variables to one side of the equation and all constants to the other side of the equation. Ask students to solve the equation $8 - \frac{3}{4}x = \frac{1}{6}x - \frac{7}{6}$. Students who find a solution of $-\frac{82}{7}$ may have this misconception. Give students the correct answer of $x = 10$. Ask them to start with the given equation and substitute 10 for x in each step of their work to find the step at which the first error was made. They will know their process includes an error when the left and right sides do not have the same value when $x = 10$.

- Some students may distribute a coefficient only to the first term in a set of parentheses. Ask students to solve the equation $\frac{2}{3}(3x - 7) = x + 4$. Students who solve the equation as $x = 11$ may have this misconception. Ask students to substitute their solution into the given equation and realize that their answer is incorrect. Then, to help students determine where they made a mistake, ask students to substitute their incorrect answer into each step of their solution process. Students can identify that their incorrect solution creates a true equation beginning with the step where they distributed incorrectly. Therefore, the distribution step is where they had an error.
- Some students may attempt to remove the coefficient of a variable without applying the factor to the constants in the equation. Ask students to solve the equation $\frac{1}{5}x + 4 = x - 4$. Students who multiply both sides of the equation by 5 and get $x + 4 = 5x - 4$ may have this misconception. Help students recognize that when multiplying both sides of an equation by a number, the distributive property is being applied. Some students may find it helpful to put each side of the equation in parentheses.

► Enrichment

- Ask students to solve for a coefficient in an equation when given information about the solution of the equation. For example, ask students to find the value of a in the equation $a(6x - 8) = 3x - 4$, given that the equation has infinitely many solutions. Students can determine that the value of a must be $\frac{1}{2}$.
- Ask students to work with situations where the quantity within a set of parentheses can be a factor for both sides of an equation. For example, give students the equation $x(x - 3) = 5x - 15$. Students can factor a 5 out of the expression on the right side, resulting in $x(x - 3) = 5(x - 3)$. Therefore, $x = 5$ will be a solution because $5(5 - 3) = 5(5 - 3)$. Further, if $x = 3$, then both sides are equal to 0 because $3(3 - 3) = 5(3 - 3)$, or $3(0) = 5(0)$. Therefore, $x = 3$ is another solution to the equation.
- Ask students to investigate the effects of multiplying an equation by a constant. For example, give students the equation $\frac{3}{4}x - 6 = -x + 8$. Ask students to solve the equation for x. Then ask students to multiply each side of the equation by 4 and solve the resulting equation. Then ask

students to multiply both sides of the given equation by $\frac{1}{2}$ and solve the resulting equation. Students can find that in each case the solution is $x = 8$.

► Key Terms

coefficient, constant, distributive property, equation, equivalent, expression, like terms, linear, reciprocal, solution, unknown quantity, variable

B-E.3.1.3

► Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Analyze and solve linear equations and pairs of simultaneous linear equations.

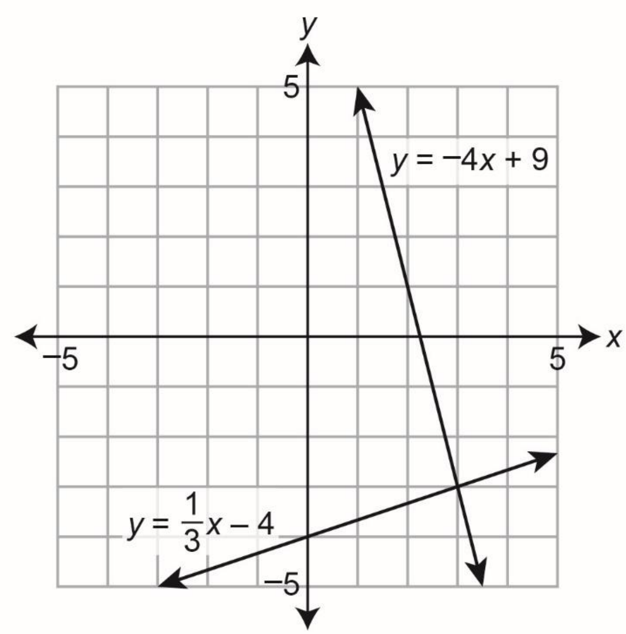
Write, solve, graph, and interpret linear equations in one or two variables, using various methods.

► Eligible Content

B-E.3.1.3 Interpret solutions to a system of two linear equations in two variables as points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.

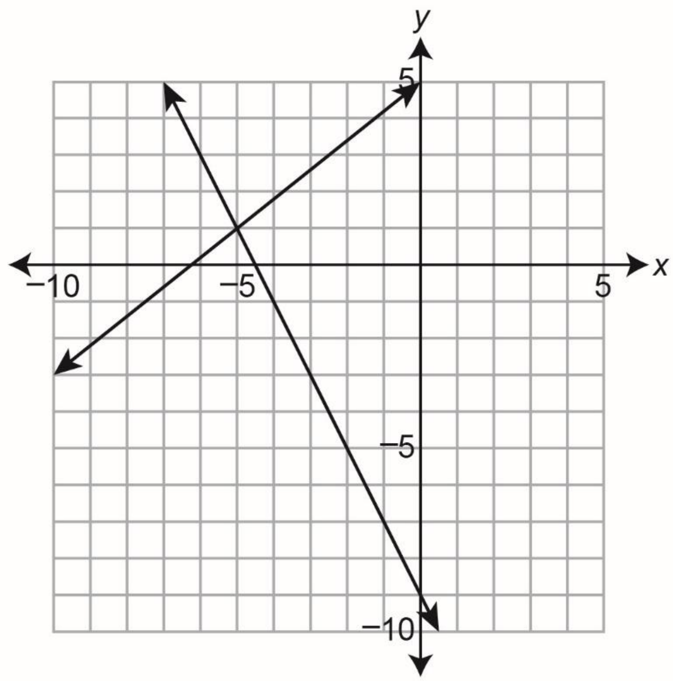
► Instructional Supports

- Ask students to solve a given system of equations by graphing the lines representing each equation and finding the point of intersection. For example, give students the system of equations $y = -4x + 9$ and $y = \frac{1}{3}x - 4$. Students can graph the lines as shown.



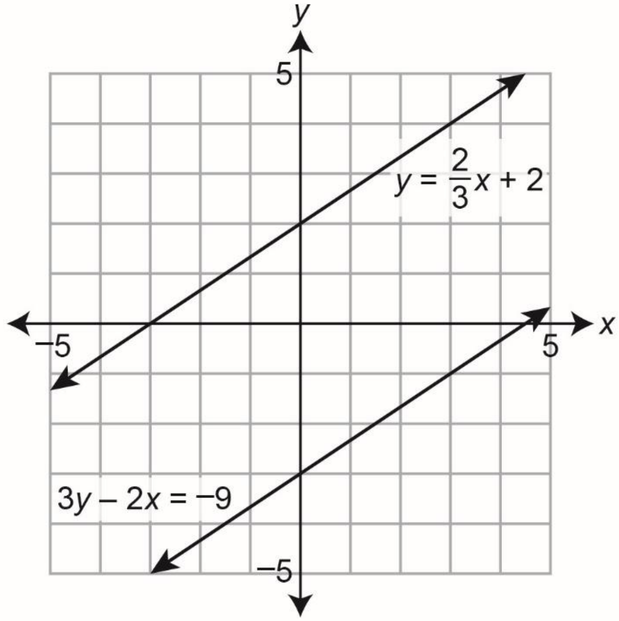
Students can determine the solution is (3, –3) since that is the point of intersection of the two lines. The solution can be verified by substituting 3 in for x in each equation and solving to find a y-value of –3 for each equation.

- Ask students to create the system of equations represented by a graph. For example, give students the graph shown.



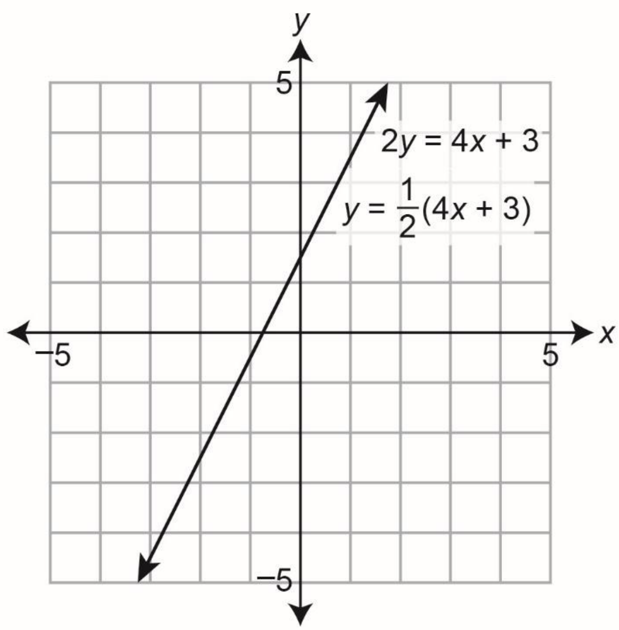
Students can determine that the system of equations is $y = -2x - 9$ and $y = \frac{4}{5}x + 5$. Students can find these equations by determining the slopes and the y-intercepts of the lines and substituting those values in for m and b, respectively, in the equation $y = mx + b$.

- Ask students to solve systems of equations that have no solution or infinitely many solutions.
    - Give students the system of equations $y = \frac{2}{3}x + 2$ and $3y - 2x = -9$. Students can graph the system as shown.



Students can determine that the system has no solution because the lines are parallel and thus have no points of intersection.

- Give students the system of equations $2y = 4x + 3$ and $y = \frac{1}{2}(4x + 3)$. Students can graph the lines as shown.



Students can conclude the system of equations has infinitely many solutions since the two lines coincide. Every point on the graph is a point that is common to both equations.

- Ask students to rewrite equations in slope-intercept form, $y = mx + b$, to determine the number of solutions to a system of equations. When both equations are in slope-intercept form, students can compare the values of the slopes, m, and y-intercepts, b. When the slopes are the same and the y-intercepts are different, the system has no solution; when the slopes are the same and the y-intercepts are the same, the system has infinitely many solutions. For example, give students the system of equations $y – 1 = 3(x + 3)$ and $3x – y = 1$. Students can rewrite the equations as $y = 3x + 10$ and $y = 3x – 1$. Since the slopes are the same and the y-intercepts are different, students can conclude the system has no solution.

► Identifying and Addressing Common Misconceptions

- Some students may interpret x- or y-intercepts as solutions to a system of equations. Ask students to identify the solutions to the system of equations $y = –6x + 3$ and $y = 3x – 6$. Students who identify (0, 3), (0, –6), (0.5, 0), or (2, 0) as solutions may have this misconception. Ask students to explain the connection between the equation of a function and the graph of a function. Students should recognize that an ordered pair lies on a graph if and only if that ordered pair is a solution of the equation. For each of the intercepts, ask students whether the ordered pair lies on both lines and whether it is a solution of both equations. Help students recognize that the intercepts lie only on one line and, therefore, fit only one equation.
- Some students may interpret coinciding lines as having no solution. Ask students to find the solutions of the system of equations $y = \frac{3}{4}x – \frac{1}{2}$ and $4y = 3x – 2$. Students who claim the system has no solution may have this misconception. Ask students to test points on the graphed line, such as (4, 2.5) and (8, 5.5), in both equations. Students should find that those points make both equations true.

▶ Enrichment

- Ask students to explore systems of three equations in two variables. For example, give students the system of equations $y = -\frac{1}{5}x + 4$, $y - 3 = \frac{1}{10}(x - 5)$, and $y = 2x - 7$. Students can graph the three equations and find the lines all intersect at (5, 3); thus, the solution to the system is (5, 3). If there had not been a single point at which all three lines intersected, the system would have had no solution.
- Ask students to solve an equation in one variable by graphing a system of equations in two variables. For example, give students the equation $4 = 2(x + 3) + 2$. Students can graph the equations $y = 4$ and $y = 2(x + 3) + 2$. The x-value of the point of intersection is the solution to the given equation.
- Ask students to consider a graph showing a linear function and a nonlinear function and ask students to identify the solutions to the system of equations represented. For example, give students a graph of the functions $y = 2x - 4$ and $y = x^2 - 4$. Students can identify that this system has two points of intersection, and thus two solutions, at (0, –4) and (2, 0).

▶ Key Terms

coordinate, graph, intersection, linear equation, simultaneous linear equations, solution, system of equations, variable

B-E.3.1.4

► Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Analyze and solve linear equations and pairs of simultaneous linear equations.

Write, solve, graph, and interpret linear equations in one or two variables, using various methods.

► Eligible Content

B-E.3.1.4 Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection.

► Instructional Supports

- Ask students to use the substitution method to algebraically solve a system of equations. For example, give students the system of equations $y = 2x - 3$ and $2y - 6x = -8$. Since one of the equations is solved for y, the substitution method is a good choice. The expression $2x - 3$ can be substituted into the second equation for y.

$$2(2x - 3) - 6x = -8$$

The 2 can be distributed on the left side of the equation and the like terms can be combined.

$$\begin{aligned} 4x - 6 - 6x &= -8 \\ -2x - 6 &= -8 \end{aligned}$$

The equation can be solved for x by adding 6 to each side and then dividing each side by –2.

$$\begin{array}{r} -2x - 6 = -8 \\ \underline{\quad +6 \quad +6} \\ \dfrac{-2x}{-2} = \dfrac{-2}{-2} \\ x = 1 \end{array}$$

The x-value can then be substituted into the first equation to calculate the y-value.

$$y = 2(1) - 3$$
$$y = 2 - 3$$
$$y = -1$$

Students can check their solution by substituting 1 for x and –1 for y in the original equations. The x- and y-values can then be used to write the solution to the system as (1, –1).

- Ask students to use the elimination method to algebraically solve a system of equations. For example, give students the system of equations $3x - 2y = 8$ and $9x + 5y = 13$. Since the coefficients of the x-values have a low least common multiple, the elimination method is a good choice. Multiplying the first equation by 3 makes the coefficients of x in each equation the same, and the second equation can then be subtracted from the new first equation to eliminate the x-variable.

$$\begin{array}{rcl} 9x - 6y & = & 24 \\ -(9x + 5y) & & -(13) \\ \hline \dfrac{-11y}{-11} & = & \dfrac{11}{-11} \\ y & = & -1 \end{array}$$

The value of y can then be substituted into either equation to calculate the x-value.

$$\begin{array}{lcr} 3x - 2(-1) & = & 8 \\ 3x \quad + 2 & = & 8 \\ \quad - 2 & & - 2 \\ \hline \dfrac{3x}{3} & = & \dfrac{6}{3} \\ x & = & 2 \end{array}$$

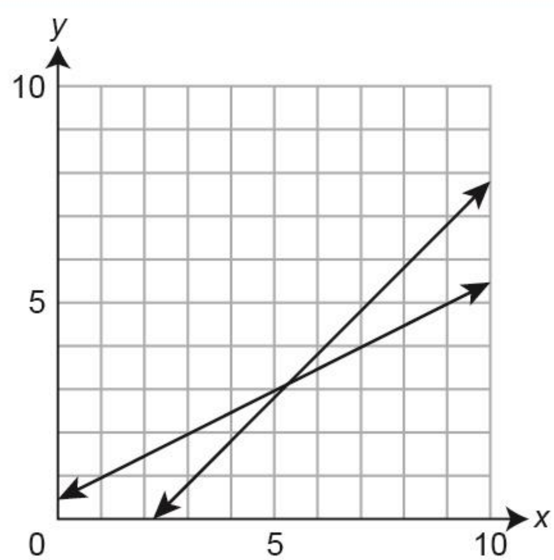
Students can check their solution by substituting 2 for x and –1 for y in the original equations. The x- and y-values can then be used to write the solution to the system as (2, –1).

- Ask students to solve a system of equations that has no solution or an infinite number of solutions. For example, give students the system of equations $-3x + y = 10$ and $3x - y = -6$. Students can solve the system of equations as shown.

$$\begin{array}{r} -3x + y = 10 \\ +(3x - y) \quad +(-6) \\ \hline 0 = 4 \end{array}$$

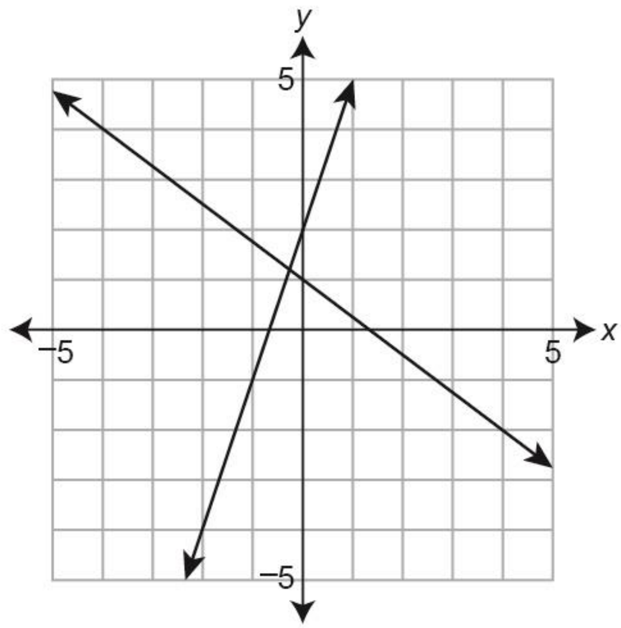
The equation $0 = 4$ cannot be true; therefore, the system of equations has no solution and would be represented as parallel lines on a graph.

- Ask students to use a graph of a system of equations to estimate the solution to the system.



Students can estimate the solution to the given system as approximately (5, 3) by estimating the coordinates of the intersection point of the lines.

- Ask students to test the reasonableness of an estimated solution to a system of equations by substituting the solution into both equations. For example, give students a graph of the system of equations $3x + 4y = 4$ and $6x - 2y = -4$.



Students can estimate the solution to be (0, 1). This estimate can be tested by inserting the values into the given equations for x and y. The first equation has equal values on both sides and the second is relatively close, so the estimate is reasonable.

$$\begin{array}{r} 3(0) + 4(1) = 4 \\ 0 + 4 = 4 \\ \hline 4 = 4 \end{array} \qquad \begin{array}{r} 6(0) - 2(1) = -4 \\ 0 - 2 = -4 \\ \hline -2 = -4 \end{array}$$

▶ Identifying and Addressing Common Misconceptions

- Some students may use an incorrect operation when solving using the elimination method. Give students the system of equations $2x + 4y = 12$ and $-6x + 4y = 4$ and ask them to find the solution using the elimination method. Students who add the equations and find a solution of $x = -4$ and $y = 5$ or $y = -5$ may have this misconception. Ask students to test the solution in the equation that was not used to find the y-variable. Ask students to explain what happens to each pair of terms when the equations are added. Help them recognize that adding $4y$ and $4y$ results in $8y$, which does not successfully eliminate the y-variable.

- Some students may not distribute a coefficient to all terms when using the substitution method. Ask students to solve the system of equations $x + 2y = 18$ and $y = 2x - 1$. Students who find a solution of $(\frac{19}{5}, \frac{33}{5})$ may have this misconception. Ask students to explain the distributive property or to give an example of it. Then ask students to explain each step they took to solve the equation. Help them recognize that the first 2 in the equation $x + 2(2x - 1) = 18$ should be distributed to both terms inside the parentheses.

- Some students may think that any system of equations in which both variables get eliminated has no solution. Ask students to solve the system of equations $4x + 6y = 10$ and $8x + 12y = 20$. Students who give a response of "no solution" may have this misconception. Ask students to choose any x-value, solve for the corresponding y-value using the first equation from the system, and substitute both values into the second equation. Have the students repeat this process for different values of x. The students should see that no matter what choice they make for the value of x in the first equation, the solution will also be true for the second equation.

► Enrichment

- Ask students to find another equation that shares the same solution as a given system of equations. For example, give students the system of equations $y - 2x = 2$ and $y + 4x = 8$, and ask students to find an equation with the same solution as the system. Students could come up with the equations $y = 3x + 1$ or any equation which fits the form $y - 4 = m(x - 1)$.

- Ask students to solve a simple system of three equations in three variables by applying the substitution method twice. For example, give students the system of equations $6x - y + 2z = 12$, $y = 3x + 2z$, and $z = x + 8$. Students can start by substituting $x + 8$ for z in the first two equations and then use the new second equation to substitute for y in the first equation. The first equation can then be solved to find $x = 4$. The value of x can be substituted into the third equation to find $z = 12$, and the values of x and z can then be substituted into the second equation to find $y = 36$.

- Ask students to use a given solution to a system of equations to find missing coefficients in a system of equations. For example, give students the system of equations $-ax - y = -25$ and $ax + by = 13$ and a solution of $(7, -3)$. Students can use the first equation to determine the value of a is 4, and then they can use the second equation to determine the value of b is 5.

► Key Terms

algebraic, elimination, equivalent, graph, intersection, linear equation, parallel lines, solution, substitution, system of equations, variable

B-E.3.1.5

► Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Analyze and solve linear equations and pairs of simultaneous linear equations.

Write, solve, graph, and interpret linear equations in one or two variables, using various methods.

► Eligible Content

B-E.3.1.5 Solve real-world and mathematical problems leading to two linear equations in two variables.

► Instructional Supports

- Ask students to write a system of equations to represent a given situation. For example, give students the situation "Small and large tiles are used to make a design. Each small tile covers 2 square inches, and each large tile covers 8 square inches. A total of 78 tiles are used to make the design, and the total area covered by the design is 222 square inches. How many small tiles (x) and large tiles (y) are used in the design if none of the tiles overlap?" Students can write the equation $2x + 8y = 222$ to represent the area of the design. Students can also write the equation $x + y = 78$ to represent the number of tiles used in the design.
- Ask students to solve a real-world problem using a system of linear equations. For example, give students the situation "The heights of two plants are measured over time. One plant has a height of 1.2 inches and grows at a rate of 0.3 inches per day. The other plant has a height of 8.4 inches and grows at a rate of 0.1 inches per day. How many days will it take for both plants to be the same height, and what is the height at that time?" Students can write the equation $h = 0.3t + 1.2$ to represent the height of the first plant and $h = 0.1t + 8.4$ to represent the height of the second plant. The substitution method can then be used to create the equation $0.3t + 1.2 = 0.1t + 8.4$. Students subtract $0.1t$ and 1.2 from each side of the equation to get $0.2t = 7.2$. They then divide each side of the equation by 0.2 to get the solution of $t = 36$ days. The value of t can then be substituted into the first equation to find the value of h for the solution. Students then determine that the plants will each be 12 inches tall on day 36.

► Identifying and Addressing Common Misconceptions

- Some students may incorrectly assign values when writing equations. Ask students to write a system of equations to represent the situation "Two students are reading the same book. One student has read 36 pages and will read 10 pages per day. Another student has read 76 pages and will read 6 pages per day." Some students may try to write one equation for the number of pages and one equation for the rates, resulting in the system of equations $y = 36x + 76$ and $y = 10x + 6$. Ask students to explain what each of their variables represents. Help them analyze the meaning of their variables in each equation and recognize that the meaning of the variables is not the same in their two equations. One of the equations is about number of pages read and the other is about pages read per day.

- Some students may not recognize a situation as requiring a system of equations. Ask students to use equations to represent the situation "A theater sells 200 tickets for the performance of a play. Regular tickets cost \$8, and student tickets cost \$6. The number of regular tickets sold is x, and the number of student tickets sold is y. The total amount of ticket sales is \$1,432." Students who write only the equation $8x + 6y = 1{,}432$ may have this misconception. Ask students to try solving for the values of x and y. Then have them check to see if their solution fits all the details included in the problem. Help them recognize that their equation does not account for the fact that 200 tickets are sold.

► Enrichment

- Ask students to interpret a system of equations in the context of a given situation. For example, give students the system of equations $x + y = 7$ and $3x + 4y = 25$ and the context of filling large containers and small containers with a number of gallons of water. Students can determine that each small container holds 3 gallons, each large container holds 4 gallons, and a total of 25 gallons are placed in the containers.

- Ask students to interpret situations that lead to systems of equations with no solution. For example, ask students to use a system of equations to solve the problem "One person has walked 0.7 miles and continues walking at a rate of 2.7 miles per hour. Another person has walked 0.9 miles and continues walking at a rate of 1.35 miles each half hour. When will both people have walked the same distance?" Students can write the system of equations $y = 2.7x + 0.7$ and $y = 2 \cdot 1.35x + 0.9$, which has no solution. Students should recognize that this means there will never come a time when both people have walked the same distance.

- Ask students to interpret a system of equations for which the solution does not work in the context. For example, ask students to solve the problem "Packages are being loaded on a truck. Each small package has a volume of 1.5 cubic feet, and each large package has a volume of 3 cubic feet. The person packing the truck wants to pack the truck with 22 packages that have a total volume of 72 cubic feet." Solving a system of equations gives a solution of (–4, 26); however, this would imply a negative number of boxes, which is not possible. Therefore, it is impossible for the person packing the truck to meet the given criteria.
- Ask students to find the equation of a line containing two given points by solving a system of equations to determine the slope and y-intercept of the line containing both of the given points. For example, to find the equation of a line through (5, 8) and (2, 11), students create the equations $8 = 5m + b$ and $11 = 2m + b$ and solve for m and b.

► Key Terms

elimination, equivalent, graph, intersection, linear equation, substitution, system of equations, variable

B-F.1.1.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

Functions

Analyze and interpret functions.

Define, evaluate, and compare functions displayed algebraically, graphically, numerically in tables, or by verbal descriptions.

► Eligible Content

B-F.1.1.1 Determine whether a relation is a function.

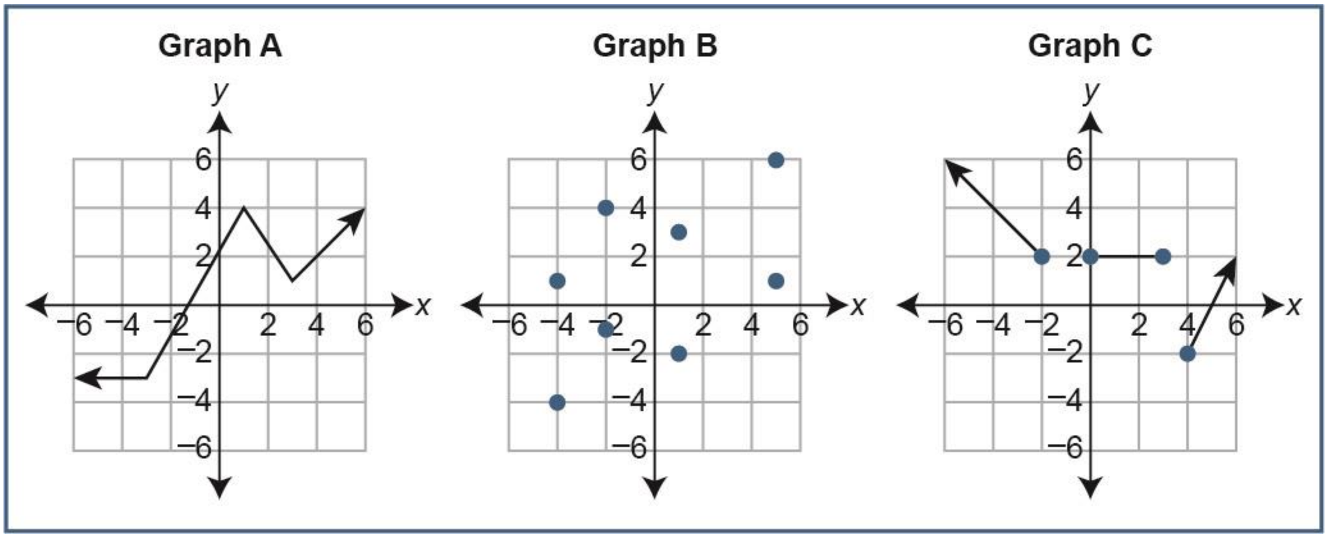
► Instructional Supports

- Ask students to determine whether a list of a set of inputs and outputs represents a function. For example, give students the list of points (1, 6), (4, –2), (–5, 1), (2, –2), (0, –8). The list of points represents a function since each input value has exactly one output value.
- Ask students to determine whether a table of inputs and outputs represents a function. For example, give students the following table.

Input (x)	**Output (y)**
–4	–2
3	–1
7	2
0	4
3	5
–1	8

  The table does not represent a function because the input $x = 3$ has multiple output values of $y = -1$ and $y = 5$.

- Ask students to determine whether a graphical representation of a set of inputs and outputs represents a function. For example, give students the following graphs.



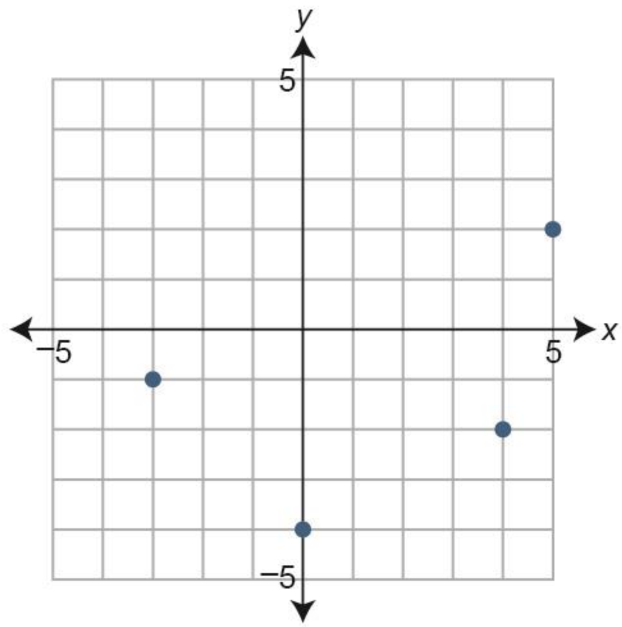
Students can identify that graph A is a function because each input has exactly one output. Graph B is not a function because at least one input has more than one output—in this case, each input has two outputs. Graph C is a function because each input has exactly one output. It does not matter that it has “missing” input values.

- Ask students to identify whether a given equation represents a function. For example, give students the equation $x + y = 10$ and inform them that the input of the function is represented by the variable x. Students can identify that the equation represents a function because for every possible value of x, there is only one value of y that completes the equation.

- Ask students to graph a function on a coordinate grid. For example, give students the following function.

Input	Output
5	2
0	–4
4	–2
–3	–1

Input values of a function are generally interpreted as x-values, and output values are generally interpreted as y-values. Therefore, the function can be shown graphically as a set of points plotted at (5, 2), (0, –4), (4, –2), and (–3, –1).



- Give students a relation with missing values and ask students to determine all the possible values that can be used to make the relation a function. For example, give students the following relation.

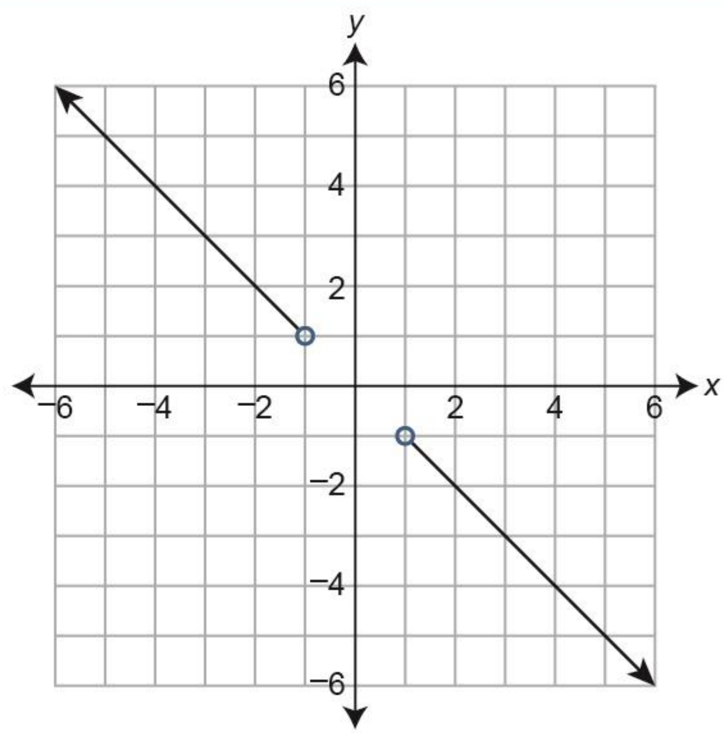
Input	Output
–2	8
0	3
m	12
5	0
9	*n*

Help students recognize that *m* could be any number except –2, 0, 5, and 9 because those input values are already shown in the table and have output values that are not 12. There are no restrictions on the value of *n* because the input value of 9 is only represented one time in the table (unless *m* is 9; in that case, *n* must be 12).

► Identifying and Addressing Common Misconceptions

- Some students may confuse which value is the input and which value is the output of a relation. Give students a relation in some representation and ask them to identify the input and output values. Some students may not know or may reverse the inputs and outputs. Help students recognize that in all representations, the input typically represents the *x*-variable (the independent variable) and is listed first while the output typically represents the *y*-variable (the dependent variable) and is listed second. On a vertically oriented table the column on the left should be interpreted as the input, and in a horizontally oriented table the top row should be interpreted as the input.

- Some students may think that because a graph is "missing" input values that the relation is not a function. Give students the following graph and ask them to explain why it is or is not a function.



Some students may think that it is not a function because it has gaps. Give students the following relation.

Input (x)	**Output (y)**
0	2
1	4
2	6
3	8

Many students would recognize the relation shown in the table as a function because the gaps are not as obvious as in a graph. Help students recognize that a function shown as a table has gaps but is still a function. It does not matter if the relation has missing input values, it only matters that each given input value has exactly one output value.

- Some students may think that a list of points or a table with duplicate inputs and outputs is not a function. Give students the following table and ask them to explain why it is or is not a function.

Input (x)	**Output (y)**
–5	8
3	–6
12	10
3	–6
6	4
0	0
3	–6
2	9

Some students may think that it is not a function because 3 is a repeated input. Ask students to create a graph of the relation and ask them to identify whether the graph represents a function. Help students understand that even though the input 3 is in the table multiple times, the output is always –6. Therefore, the input 3 has only one output and the relation represented in the table is a function.

► Enrichment

- Ask students to explain how to change a relation that is not a function so that it could represent a function.
- Give students a function and ask them to identify whether it would still be a function if the input and output values were reversed. This can be used to introduce the concept of an inverse function.
- Ask students to determine whether a given real-world situation would be represented by a function.

- Ask students to explore whether various operations affect whether a relation is a function. Some examples are shown.
    - Adding 3 to all input values does not change whether a relation is a function.
    - Subtracting 8 from all output values does not change whether a relation is a function.
    - Multiplying all input values by the same number does not change whether a relation is a function unless multiplying by 0.
    - Squaring all input and output values may cause a function to become a non-function.

► Key Terms

function, graph, input, ordered pairs, output, pattern, proportional relationship, rule, table

B-F.1.1.2

► Reporting Category, Assessment Anchor, Anchor Descriptor

Functions

Analyze and interpret functions.

Define, evaluate, and compare functions displayed algebraically, graphically, numerically in tables, or by verbal descriptions.

► Eligible Content

B-F.1.1.2 Compare properties of two functions each represented in a different way (i.e., algebraically, graphically, numerically in tables, or by verbal descriptions).

► Instructional Supports

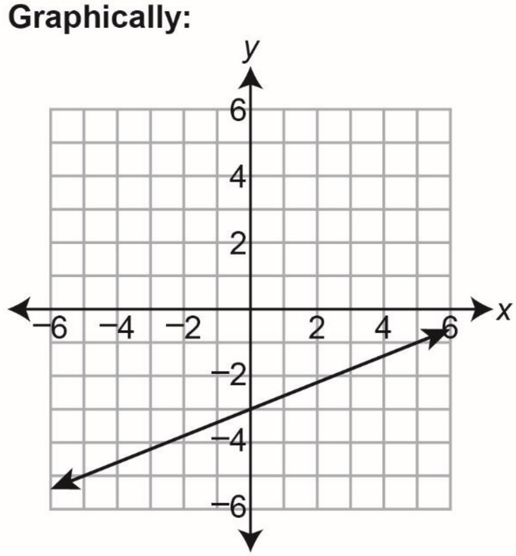
- Give students a function and ask them to represent the function in different ways. For example, give students the verbal description of a linear function with a slope of $\frac{2}{5}$ and a *y*-intercept at –3. Ask students to represent the function algebraically, graphically, or numerically in a table.

Algebraically:

$y = \frac{2}{5}x - 3$

Verbal Description:
a linear function with a slope of $\frac{2}{5}$ and a *y*-intercept at –3

Graphically:

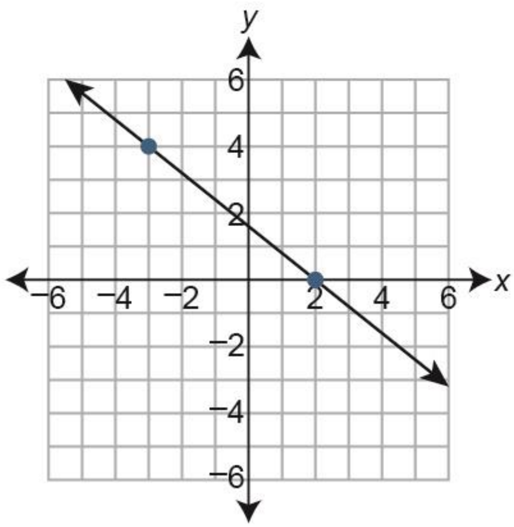


Numerically in a Table:

x	*y*
–15	–9
–10	–7
–5	–5
0	–3
5	–1
10	1
15	3

- Ask students to compare the slopes of two functions represented in different ways.
  - Give students two functions: one represented graphically and the other as an equation.

Function A



Function B

$y = \frac{3}{5}x + 2$

The graph of function A has a slope of $\frac{-4}{5}$ because the change in *y*-value between the two points is –4 and the change in *x*-value is 5. The equation of function B has a slope of $\frac{3}{5}$, which is easily identifiable since the equation is written in slope-intercept form. As $\frac{3}{5} > \frac{-4}{5}$, the value of the slope of function B is greater than the value of the slope of function A.

  - Give students two linear functions with one represented in a table and the other as a verbal description of a line through two points.

Function M

x	y
−6	8
1	5
8	2
15	−1
22	−4

Function N

a line through points (2, 3) and (−8, 0)

The table representing function M has a slope of $\frac{-3}{7}$ since the change in y-values in the table is –3 and the change in x-values is 7. The equation representing function N has a slope of $\frac{3}{10}$ since the change in the y-values is −3 and the change in x-values is –10. As $\frac{3}{10} > \frac{-3}{7}$, the value of the slope of function N is greater than the value of the slope of function M.

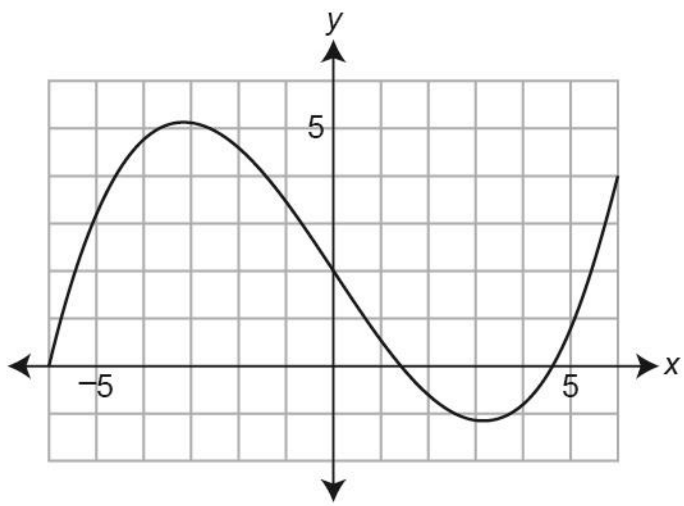
- Ask students to compare the y-intercepts of two functions represented in different ways.
  - Give students two functions with one represented as an equation and the other as a verbal description. Function Q is represented by the equation $5x - 2y = 10$. Function R is represented by a line that passes through the points (2, –6) and (8, 12). The equation given for function Q can be rewritten in slope-intercept form as $y = \frac{5}{2}x - 5$, meaning function Q has a y-intercept of –5. An equation for function R can be determined by calculating the slope between the two given points and then using the slope to find the y-intercept. Students calculate the slope as $\frac{12-(-6)}{8-2} = 3$ and then substitute the slope and the coordinates (8, 12) into the equation $y = mx + b$ to get $12 = 3(8) + b$. They then solve this equation for b to determine the y-intercept of function R is –12. Since –5 is closer to 0 than –12, the y-intercept of function Q is closer to the origin.

- Give students two functions with one represented as a table and the other as a graph.

Function E

x	y
−2	1
−1	3
0	6
1	−2
2	3
3	1

Function F



Students identify the y-intercept of function E as 6 because it contains the ordered pair (0, 6), which is a point on the y-axis. Students identify the y-intercept of function F as 2 because the graph crosses the y-axis at 2. Since 2 is closer to 0 than 6 is, the y-intercept of function F is closer to the origin.

- Ask students to compare the x-intercepts or the number of x-intercepts of two functions represented in different ways. For example, give students the following functions.

Function G

x	y
−4	2
−3	0
−2	0
−1	−2
0	−1
1	1
2	0
3	4

Function H

$2x + y = 6$

Students identify the x-intercepts of function G as –3, –2, and 2 because the ordered pairs (–3, 0), (–2, 0), and (2, 0) represent points on the x-axis. Students may find the x-intercepts of function H by substituting $y = 0$ into the equation and solving for x. Students write the equation $2x + 0 = 6$ and solve to get an x-intercept of 3. Students conclude that function G has more x-intercepts than function H does, and each x-intercept of function G is less than the x-intercept of function H.

► Identifying and Addressing Common Misconceptions

- Some students may think that the coefficient of x always represents the slope of a function. Ask students to determine the slope of the function $3x + y = 6$. Some students may think the slope is 3. Ask students to rewrite the equation as $y = -3x + 6$, and then help students recognize that both equations represent the exact same function. Ask students to graph the function and use the graph to show that the coefficient of x only indicates the slope of the function when the equation is given in slope-intercept form.
- Some students may confuse the y-intercept and the rate of change in a function. Ask students to identify the slope of a function given by the equation $y = 2x + 3$. Some students may give a response of 3. Ask students to create a table or graph from the equation to examine the slope in those representations and confirm that the slope is 2.

► Enrichment

- Ask students to compare more than two functions that are each represented in a different way.
- Ask students to compare properties of nonlinear functions that are represented as equations.
- Give students a function and ask students to create a different function with given characteristics. For example, give students a function shown as a graph and ask students to create a table for a function with a y-intercept that is farther from the origin and a slope that has a lesser value.

► Key Terms

function, graph, initial value, input, output, rate of change, slope, table, verbal description

B-F.1.1.3

► Reporting Category, Assessment Anchor, Anchor Descriptor

Functions

Analyze and interpret functions.

Define, evaluate, and compare functions displayed algebraically, graphically, numerically in tables, or by verbal descriptions.

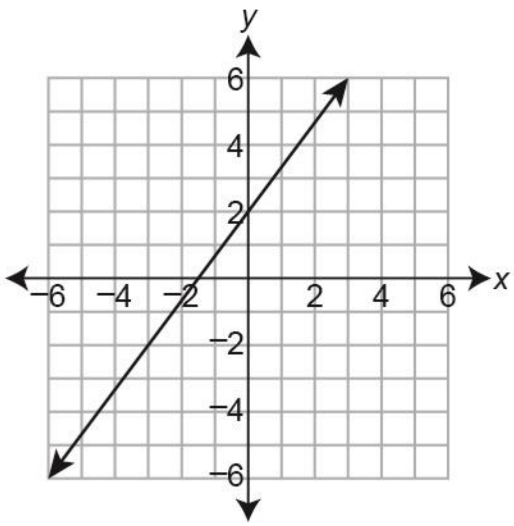
► Eligible Content

B-F.1.1.3 Interpret the equation y = mx + b as defining a linear function whose graph is a straight line; give examples of functions that are not linear.

► Instructional Supports

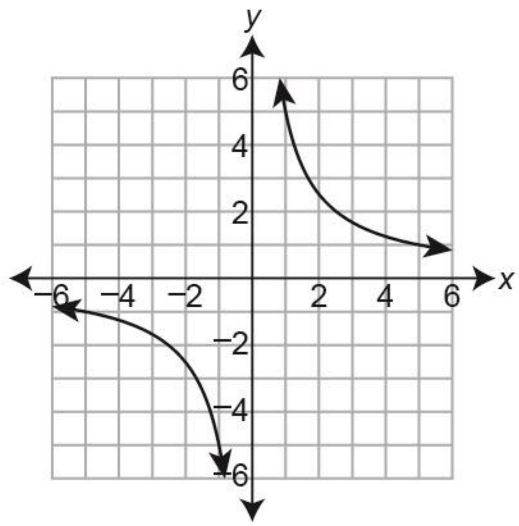
- Ask students to interpret a function written in the form $y = mx + b$ and to describe how the function looks when graphed on a coordinate plane. For example, ask students to interpret the function $y = 3x – 4$ and to explain what the graph of the function looks like. A student response may include that the function has a slope of 3 and a y-intercept at –4. Further, when graphed, the function is represented by a straight line on the coordinate plane. Students may confirm this by selecting several x-values, substituting the x-values into the function to find the corresponding y-values, plotting the points on the coordinate plane, and connecting them with a straightedge to show the line. Have students check that the slope between any two points on the line is 3 and that the line crosses the y-axis at (0, –4).
- Ask students to determine whether a function is linear or not linear by graphing the equation on a coordinate plane. For example, give students the functions $y = \frac{4}{3}x + 2$ and $y = \frac{5}{x}$. Ask students to select several x-values, substitute them into the function to find the corresponding y-values, and plot the points on the coordinate plane. Then ask students to connect the points to see whether the function is linear or not linear. The graphs of the functions are shown.

Function Y



$y = \frac{4}{3}x + 2$

Function Z



$y = \frac{5}{x}$

Function Y is linear, and function Z is not linear.

- Ask students to determine whether a function is linear or not linear by analyzing the slopes between multiple sets of points. For example, ask students to create a table of input and output values for the functions $y = –5x + 6$ and $y = x^2 + 1$. A possible response is shown.

Function C

$y = -5x + 6$

x	y	slope
0	6	
1	1	−5
2	−4	−5
3	−9	−5
4	−14	−5

Function D

$y = x^2 + 1$

x	y	slope
−3	10	
−1	2	−4
1	2	0
3	10	4
5	26	8

Once a table of values has been created for each function, ask students to calculate the slope between consecutive pairs of points. Students should recognize that when the slope between each set of points is the same, the function is linear. Conversely, when the slope between each set of points is different, the function is not linear.

- Ask students to explain which part of a function equation determines whether it is linear or not linear. With linear functions, variables have exponents of 1, they are not multiplied with other variables, and no variable is a divisor. For example, $y = 5$ and $y = \frac{4}{7}x + 2$ are both linear. On the other hand, the equations $y = 3x^3 - 4$ and $y = \frac{-5}{x}$ are both nonlinear.

▶ Identifying and Addressing Common Misconceptions

- Some students may think that any equation that has exponents is nonlinear. Ask students whether the equation $y = 4x + 2^3$ is linear. Some students may respond that it is nonlinear because it has an exponent other than 1. Ask students to create a table of values and graph the function. Help students recognize that the equation is equivalent to $y = 4x + 8$, which is linear, and that exponents with constant terms do not affect whether a function is linear or nonlinear.
- Some students may not recognize equations of horizontal lines as being linear. Ask students whether $y = 9$ is linear. Some students may not recognize it as having the form $y = mx + b$ due to the lack of an x-term. Ask students if $y = 9$ is equivalent to $y = 0x + 9$. Then ask them to create a graph and note that the graph is a straight line. From their response, students may extrapolate that equations given in this way are linear.

▶ Enrichment

- Ask students to evaluate piecewise functions and determine whether some, all, or no pieces of the functions are linear.
- Ask students to examine differences and second differences in tables of linear, quadratic, and cubic functions.
- Ask students to explore sums, differences, and products of functions. For example, is the sum of two linear functions always linear? Is the product of a linear and a nonlinear function always linear, always nonlinear, or a mix of both?

► Key Terms

function, graph, input, linear, linear function, nonlinear, nonlinear function, output, rate of change, side length

B-F.2.1.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

Functions

Use functions to model relationships between quantities.

Represent or interpret functional relationships between quantities using tables, graphs, and descriptions.

► Eligible Content

B-F.2.1.1 Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models and in terms of its graph or a table of values.

► Instructional Supports

Creating equations for linear functions

- Ask students to recognize how and when to use the three main forms of equations for a linear function. The three main forms of equations for linear functions are the following:
    - standard form: $\mathrm{A}x + \mathrm{B}y = \mathrm{C}$, where A, B, and C are integers.
    - slope-intercept form: $y = mx + b$, where m is the slope and b is the y-intercept.
    - point-slope form: $y - y_1 = m(x - x_1)$, where m is the slope and (x_1, y_1) is a point on the graph of the function.

  (Note: Another way of writing standard form is $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$, and another way of writing point-slope form is $y = m(x - x_1) + y_1$.)

  Any linear function can be represented in any of the forms, but typically one of the forms is easiest to use when given different constraints. When a function is given in standard form, the axis intercepts are easily identified with the x-intercept located at $\frac{\mathrm{C}}{\mathrm{A}}$ and the y-intercept located at

$\frac{C}{B}$. When a function is given in slope-intercept form, the slope and the y-intercept are easily identified. When a function is given in point-slope form, the slope and one point on the line are easily identified.

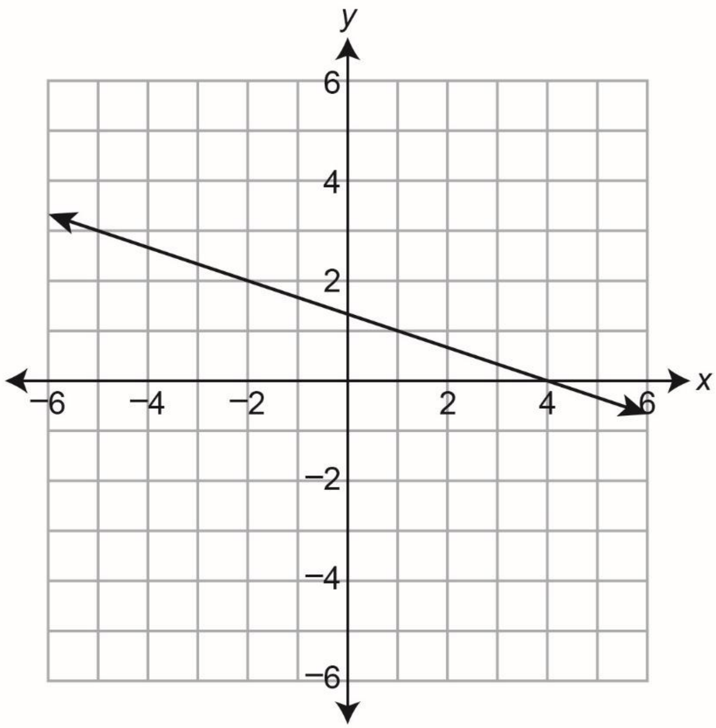
(Note: When using the standard form of $Ax + By + C = 0$, the intercepts are at $-\frac{C}{A}$ and $-\frac{B}{A}$.)

- Ask students to write an equation for a line containing two given points. For example, ask students to write an equation for the line containing the points (4, –2) and (9, 5). In this case, the point-slope form of the function may be the easiest to find. Students calculate the slope, m, to be $\frac{5-(-2)}{9-4} = \frac{7}{5}$. Then, students can substitute either of the given coordinate pairs into the point-slope form of a function to write $y + 2 = \frac{7}{5}(x - 4)$ or $y - 5 = \frac{7}{5}(x - 9)$. Ask students to show that these two resulting functions are equivalent by writing both in slope-intercept form as $y = \frac{7}{5}x - \frac{38}{5}$ or in standard form as $7x - 5y = 38$.
- Ask students to write an equation for a line given a point and the y-intercept. For example, ask students to write an equation for a line with a y-intercept at 12 that passes through the point (–4, 5). In this case, the slope-intercept form of the function may be the easiest to find. Students find the slope, m, by substituting all the known values into the slope-intercept form of a function to get $5 = -4m + 12$; then they solve to get $m = 1.75$. Therefore, the slope is 1.75 and the function can be written in slope-intercept form as $y = 1.75x + 12$.
- Ask students to write an equation for a line containing points given in a table. For example, ask students to write an equation for a line fitting the table shown.

x	y
–8	4
–2	5
4	6
10	7
16	8

Students may determine that the point-slope form of the function is the easiest to find. Students can use any two points in the table to find the slope of $\frac{1}{6}$. Though only two points are needed, students should do this for multiple sets of points to verify that the slope is consistent throughout the table. Students then choose any point in the table and create an equation equivalent to $y - 4 = \frac{1}{6}(x + 8)$.

- Ask students to write an equation for a line when given the graph of the line. For example, ask students to find the equation of the line shown.



Students first identify two points of the line, typically where the line crosses an axis or where the line crosses intersecting gridlines of the coordinate plane. In this case, the easiest points are at (4, 0), (1, 1), (–2, 2), and (–5, 3). Students use any two of these points to find the slope of $\frac{-1}{3}$; then they may substitute the point (4, 0) into the point-slope form of the function to get $y = \frac{-1}{3}(x - 4)$ as the equation for the graphed line. Other equivalent equations are also possible.

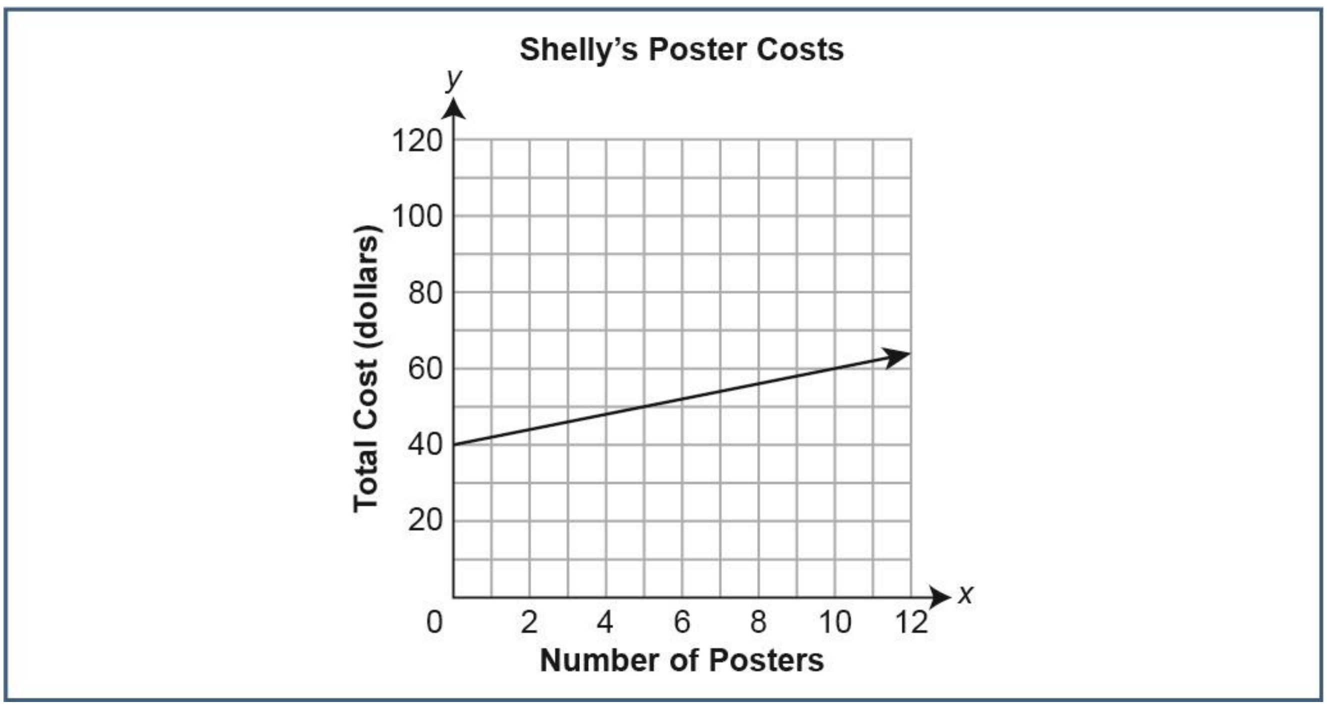
Creating tables and graphs for linear functions

- Ask students to create a table to represent multiple values in a real-world problem. For example, ask students to create a table of values for a function that could represent the height of a building when each floor, x, is 12 feet tall. Students determine that the input variable of the function is the number of floors of the building and the output variable of the function is the height of the building.

Number of Floors	Height of Building (feet)
1	12
4	48
6	72
9	108
13	156

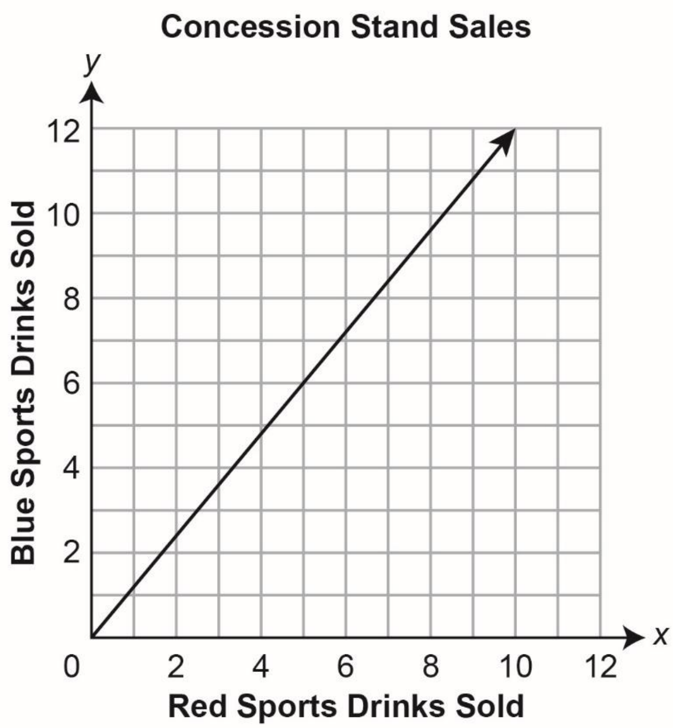
Students create the table by choosing several different possible numbers of floors and multiplying that number by 12 to find the height of a building with that many floors.

- Ask students to create a graph to represent a real-world problem. For example, ask students to graph a line on a coordinate plane to represent the total cost, y, in dollars, Shelly spends to develop x posters after an initial cost of \$40 and a cost of \$2 per poster. When Shelly has sold 0 posters, she has still spent \$40, so the graph has a y-intercept at (0, 40). Since it costs Shelly \$2 for each poster, the line has a slope of 2. Therefore, the points (1, 42), (2, 44), etc. are also on the graph.



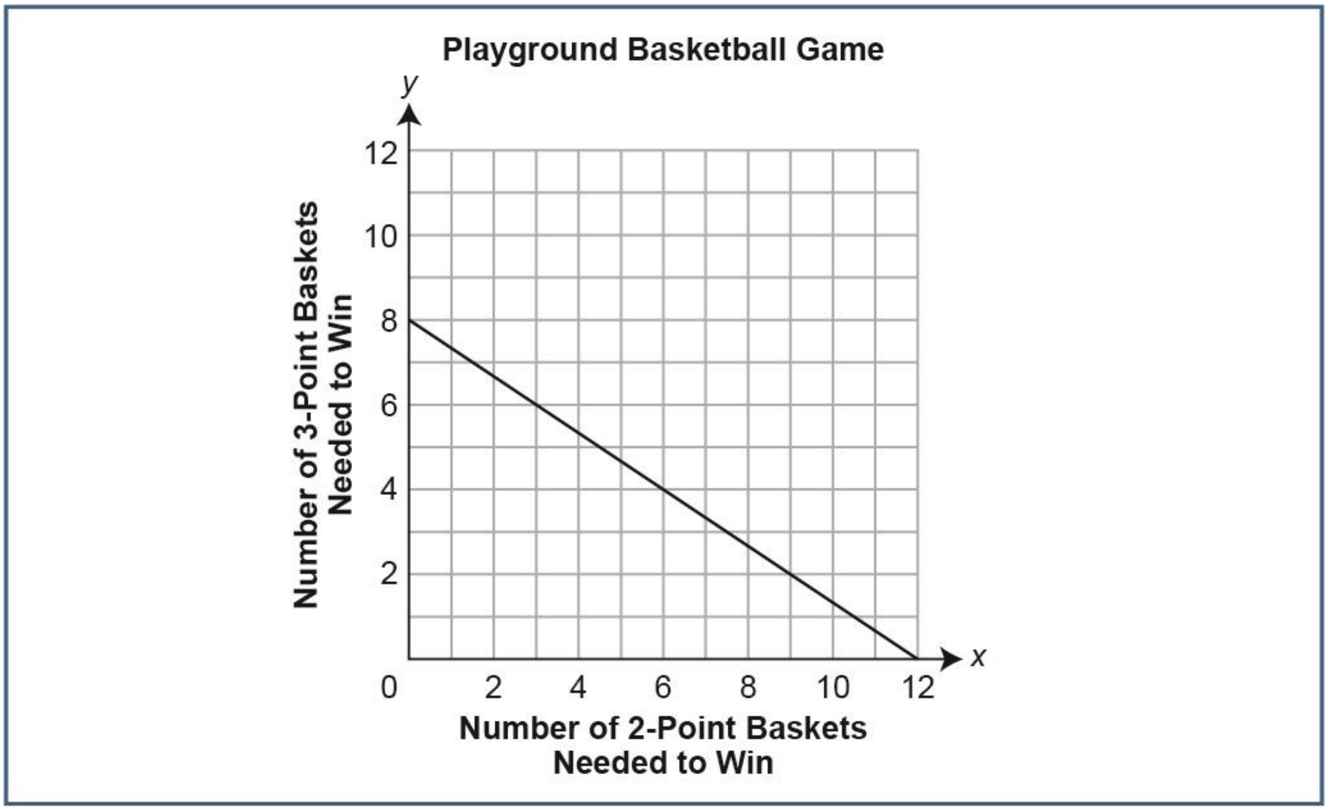
Interpreting features of functions

- Ask students to interpret the rate of change in a real-world problem represented by a line. For example, ask students to explain the meaning of the slope of the line shown.



Students determine that the slope of the line is $\frac{6}{5}$. The *y*-axis label is Blue Sports Drinks Sold, and the *x*-axis label is Red Sports Drinks Sold. Since slope is the change in *y* over the change in *x*, the slope of this line means that for every 6 blue sports drinks sold, 5 red sports drinks are sold.

- Ask students to interpret the *x*-intercept and *y*-intercept in a real-world problem represented by a line. For example, give students the following problem: Julian plays a game of basketball where the first team to 24 points wins. The number of baskets of each type that are needed to win are shown in the graph. What is the meaning of the *x*-intercept and of the *y*-intercept?



Since the y-axis label is Number of 3-Point Baskets Needed to Win and the x-axis label is Number of 2-Point Baskets Needed to Win, the intercepts indicate that Julian's team needs 8 three-point baskets to win if they score 0 two-point baskets and they need 12 two-point baskets to win if they score 0 three-point baskets.

► Identifying and Addressing Common Misconceptions

- Some students may think there is only one equation for a given line. Ask students whether the equations $y = 5x + 6$, $y - 11 = 5(x - 1)$, and $5x - y = -6$ represent the same line. Some students may think all three equations must represent different lines. Ask students to rewrite the three equations in the same form or to create tables for all three equations.

► Enrichment

- Ask students to find equations of lines that are parallel to a given line.
- Ask students to find the equations of line segments that represent the sides of figures, or, given the equations, ask students to identify the figure created by the lines of the equations.

► Key Terms

dependent variable, function, independent variable, initial value, input, linear, output, rate of change, *y*-intercept

B-F.2.1.2

► Reporting Category, Assessment Anchor, Anchor Descriptor

Functions

Use functions to model relationships between quantities.

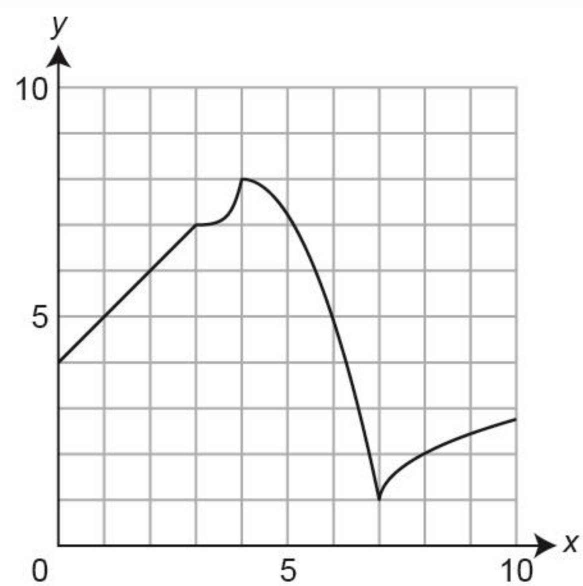
Represent or interpret functional relationships between quantities using tables, graphs, and descriptions.

► Eligible Content

B-F.2.1.2 Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch or determine a graph that exhibits the qualitative features of a function that has been described verbally.

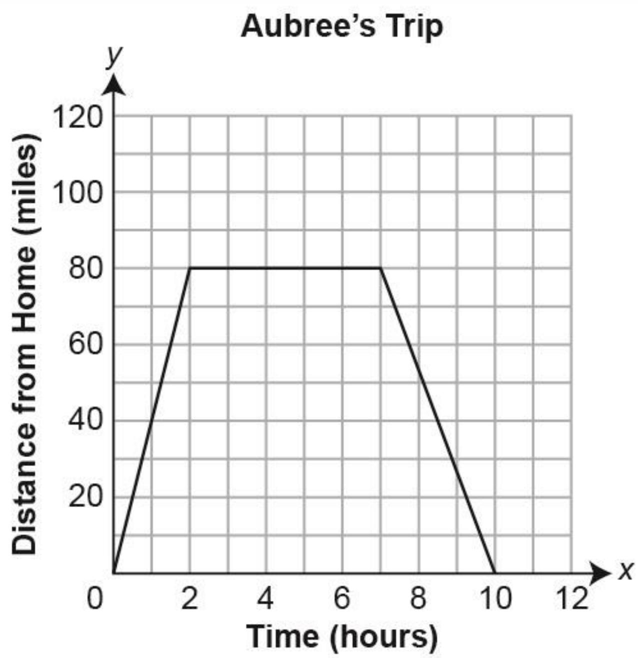
► Instructional Supports

- Ask students to describe various features of the graph of a function. For example, give students the following graph and ask them to identify where the graph has a maximum, where it has a minimum, and where the graph is linear.



The graph has a maximum when $x = 4$ and a minimum when $x = 7$. The graph is linear when x is between 0 and 3.

- Ask students to create a story that matches a given graph. For example, ask students to create a scenario that could be represented by the graph shown.

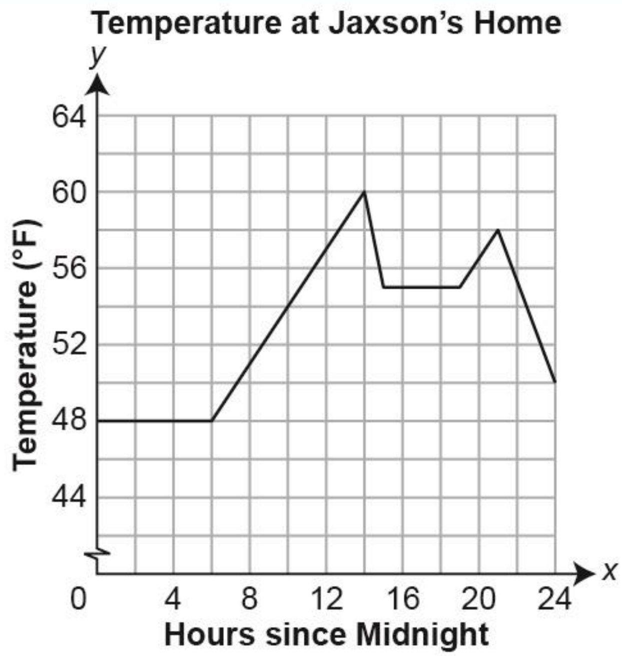


A possible response is shown.

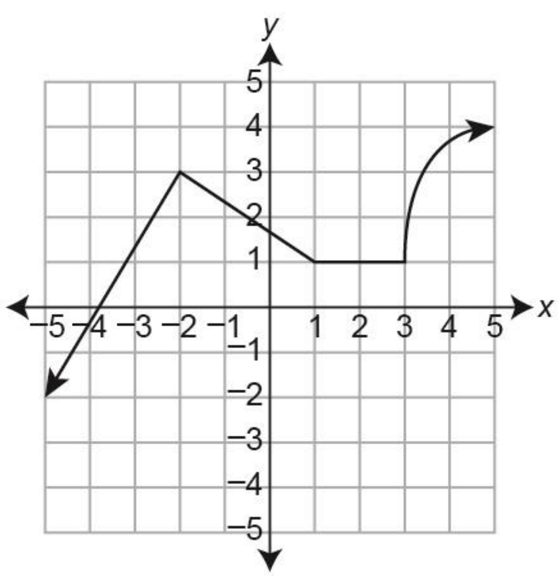
Aubree drove 80 miles to her aunt's home. The drive took her 2 hours. She then stayed and visited her aunt for 5 hours before traveling home. Aubree took a different route home that took 3 hours.

- Ask students to create a graph of a function based on a description of the function. Some examples and possible student responses are shown.
    - Jaxson records the temperature, in degrees Fahrenheit (°F), outside his home one day in May. At midnight, the temperature was 48°. The temperature remained constant until 6 a.m., when it started to steadily increase, and ultimately increased by a total of 12° until 2 p.m. Between 2 p.m. and 3 p.m., the temperature dropped 5° and then remained

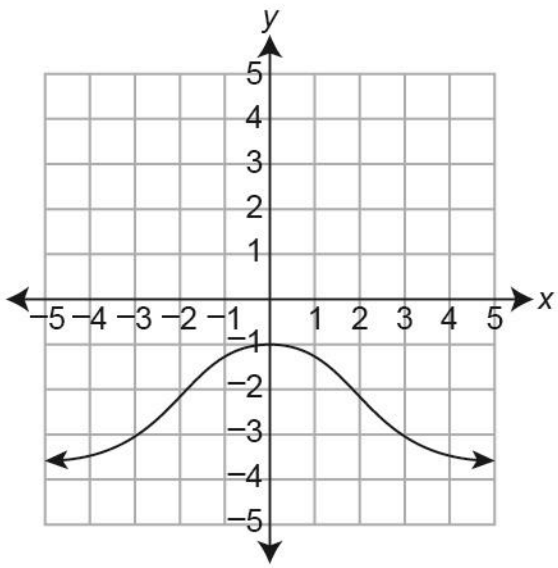
constant until 7 p.m. The temperature then rose 3° until 9 p.m. and then steadily dropped by a total of 8° until midnight.



- A function is increasing at a constant rate for input values less than –2, is decreasing at a constant rate for input values between –2 and 1, and is constant for input values between 1 and 3. When the input is greater than 3, the function increases but is not linear.

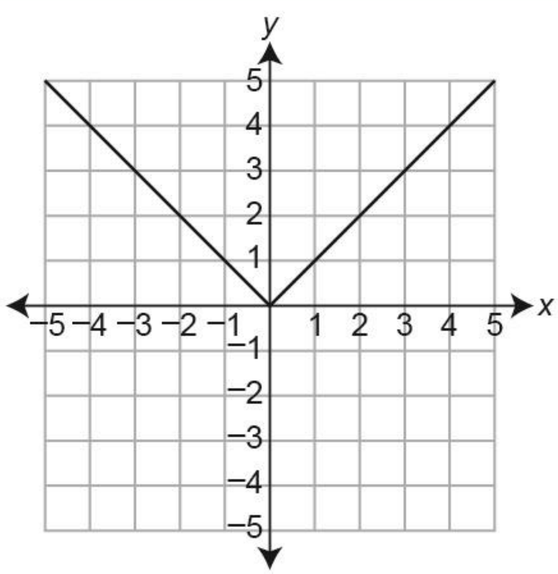


- A function is symmetric and has the y-axis as a line of symmetry. For positive inputs, the graph is always decreasing and always negative.



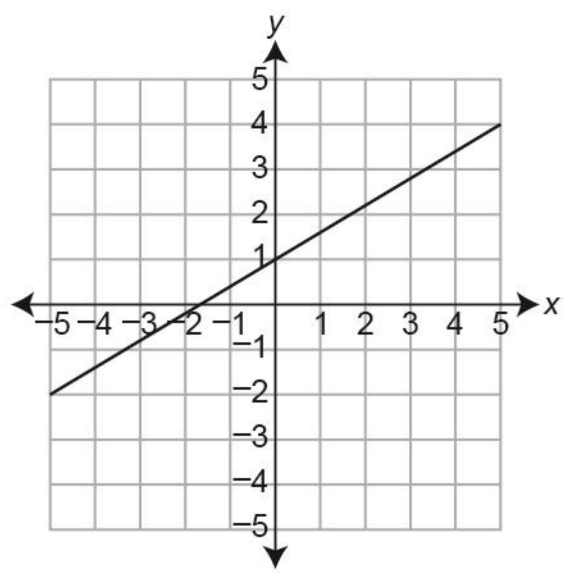
► Identifying and Addressing Common Misconceptions

- Some students may describe whether a function is increasing or decreasing relative to the y-axis rather than from left to right. Ask students whether the following function is increasing.



Some students may think that it is increasing away from the y-axis for all values other than $x = 0$. Ask students whether each half of the graph is increasing from left to right. Then ask students whether each half of the graph is increasing from right to left. Acknowledge that it is possible to think of whether a function is increasing or decreasing in either direction, but the agreed-upon convention is to define it from left to right.

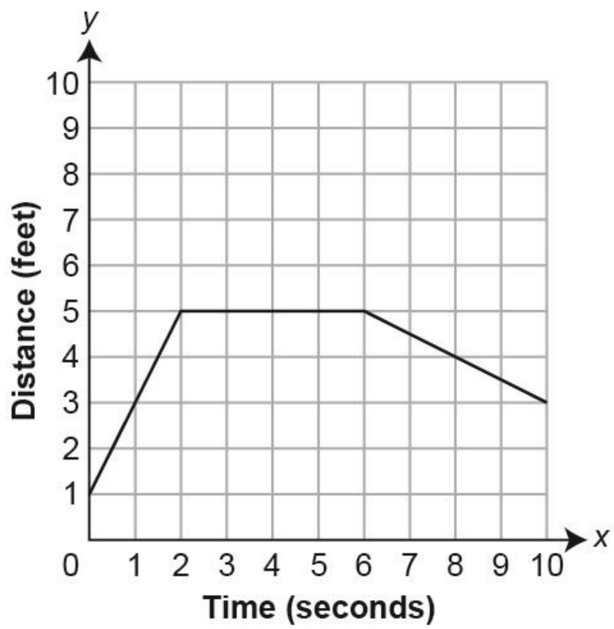
- Some students may confuse a constant function with one that has a constant slope. Ask students whether the following function is a constant function.



Some students may respond that it is. Ask students to explain what they mean when they say that it is constant. Then ask them to identify the output for various inputs. Remind students that saying a function is constant refers to the output values always being the same, not to the slope or the rate of change.

► Enrichment

- Ask students to describe where a graph is concave up, concave down, or neither.
- Ask students to identify whether a graph shows a function that is discrete, continuous, or neither.
- Give students a graph in the first quadrant and ask them to walk closer to or farther from a wall in a way that matches the graph. For example, give students the following graph.



Students start 1 foot away from the wall and walk 4 feet away from the wall over the course of 2 seconds. Students stand in place for 4 seconds. Then students very slowly walk 2 feet towards the wall over the course of 4 seconds.

► Key Terms

constant speed, function, input, interval, linear, nonlinear, output, qualitative, rate of change, sketch

C-G.1.1.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

Geometry

Demonstrate an understanding of geometric transformations.

Apply properties of geometric transformations to verify congruence or similarity.

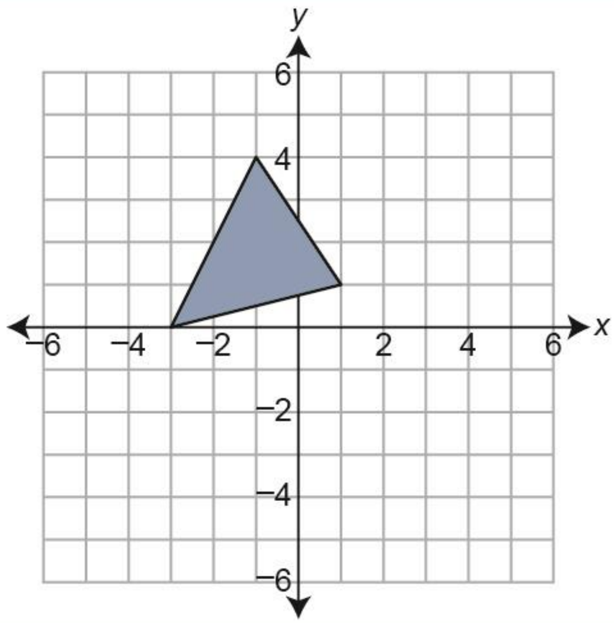
► Eligible Content

C-G.1.1.1 Identify and apply properties of rotations, reflections, and translations.

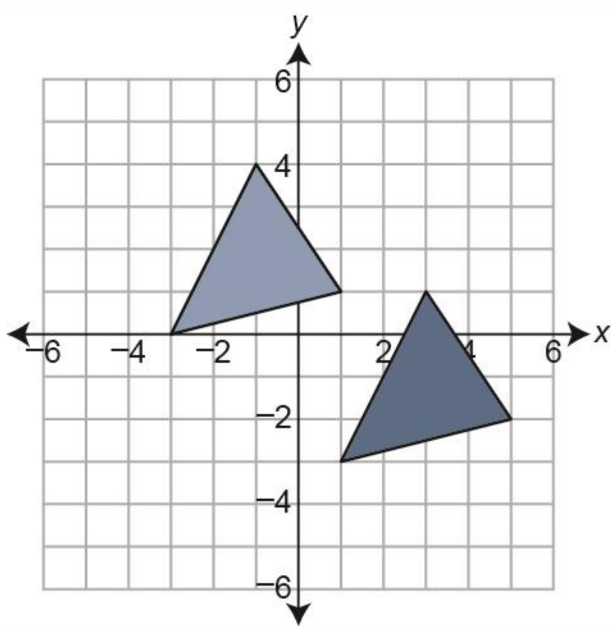
► Instructional Supports

- Ask students to demonstrate that reflecting a line segment does not change the length of the line segment. For example, ask students to draw a line segment and a dotted line on a sheet of paper. Have students use a marker that transfers when wet, or something similar, to draw the line segment. Instruct each student to fold the paper along the dotted line, apply pressure to the fold so that the marker transfers to the other side of the paper, and open the paper again to reveal the image of the original line segment. Then give students rulers to use to verify that the image and the pre-image of the line segment are the same length.
- Ask students to demonstrate that rotating a line results in another line. For example, ask students to draw a line on a coordinate grid. Give students tracing paper and ask students to place the piece of tracing paper on top of the coordinate grid, to trace the original line, and to make a small "+" at the origin to help keep track of how far the object has rotated. Using the origin as the center of rotation, students should place a pencil on the origin and then slowly rotate the tracing paper clockwise until the "+" realigns with itself (90 degrees). Have students repeat the same process for 180 degrees clockwise and 270 degrees clockwise. Students may also experiment with counterclockwise rotations. Observe that no matter how a line is rotated, the resulting figure is still a line.

- Ask students to demonstrate that translating a figure formed by line segments does not change the length of the segments or shape of the original figure. For example, give students the following figure.

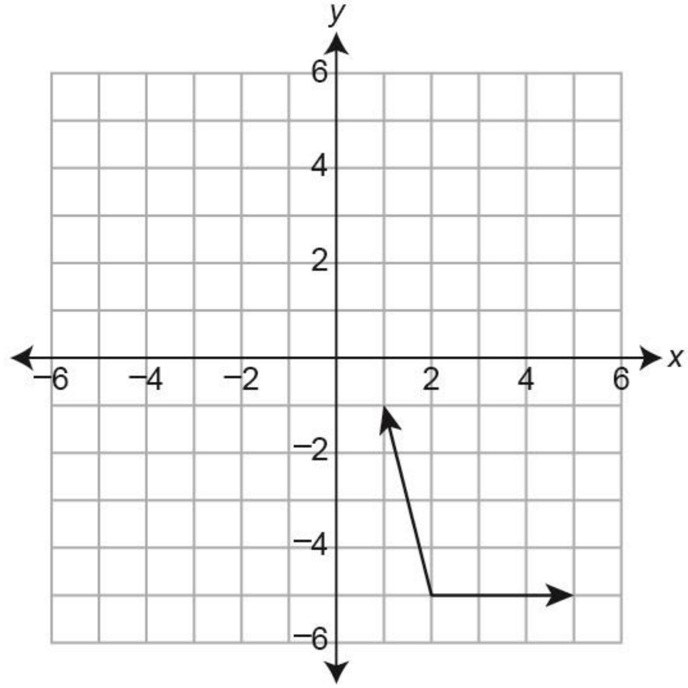


Have students plot each vertex of the pre-image 3 units down and 4 units right on the coordinate grid. Then connect the new points to create the image of the original figure.

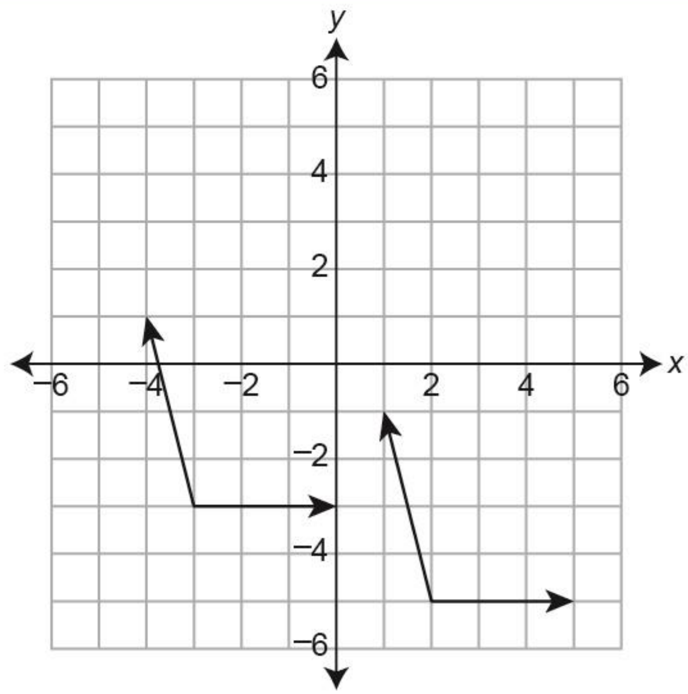


Use tracing paper to copy the image and observe that the side lengths in the pre-image are the same lengths as the side lengths in the image.

- Ask students to demonstrate that reflecting an angle or figure does not change the measure of the angle or any of the angle measures of the figure. For example, ask students to draw an angle in quadrant 1 on a coordinate grid. Have students use graph paper and a marker that will transfer when wet, or something similar. Instruct students to fold the paper along the *x*-axis, apply pressure to the fold so that the marker will transfer to the other side of the paper, and open the paper to reveal the image of the original angle. Use a protractor to verify that the angles of the image and the pre-image have the same measure.
- Ask students to demonstrate that rotating an angle or figure does not change the measure of the angle or any of the angle measures of the figure. For example, ask students to draw a figure on a coordinate grid. Give students tracing paper and ask them to place the tracing paper on top of the coordinate grid and trace the figure. Help students create a center of rotation by placing their pencil somewhere on the tracing paper, and then slowly rotate the tracing paper without rotating the original figure. Students should observe that no matter how the figure is rotated, the measure of each angle in the rotated figure is equal to the measure of its corresponding angle in the original figure.
- Ask students to demonstrate that translating an angle or figure does not change the measure of the angle or any of the angle measures of the figure. For example, give students the following graph.

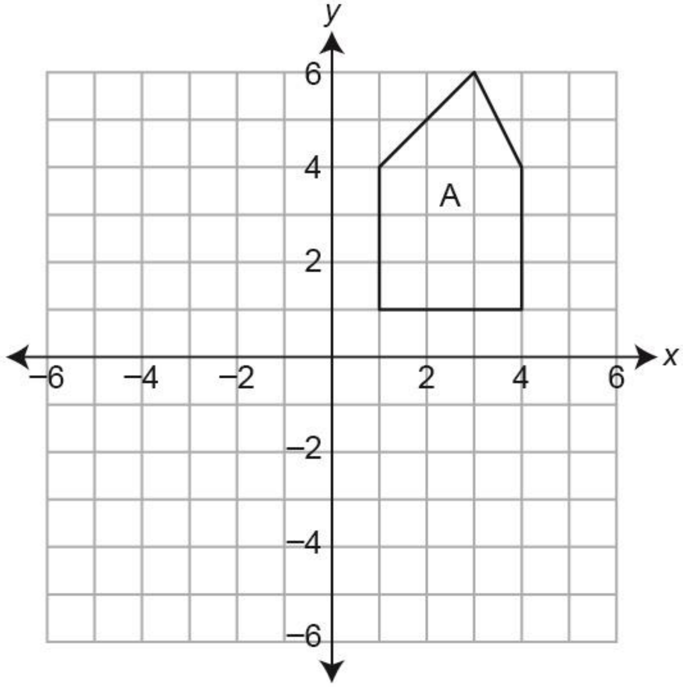


Have students then plot a point that is 2 units up and 5 units left of the vertex of the angle. Then, have them plot points 2 units up and 5 units left of each of the “endpoints” of the rays in the angle. Students should then connect the new points to create the image of the original angle.

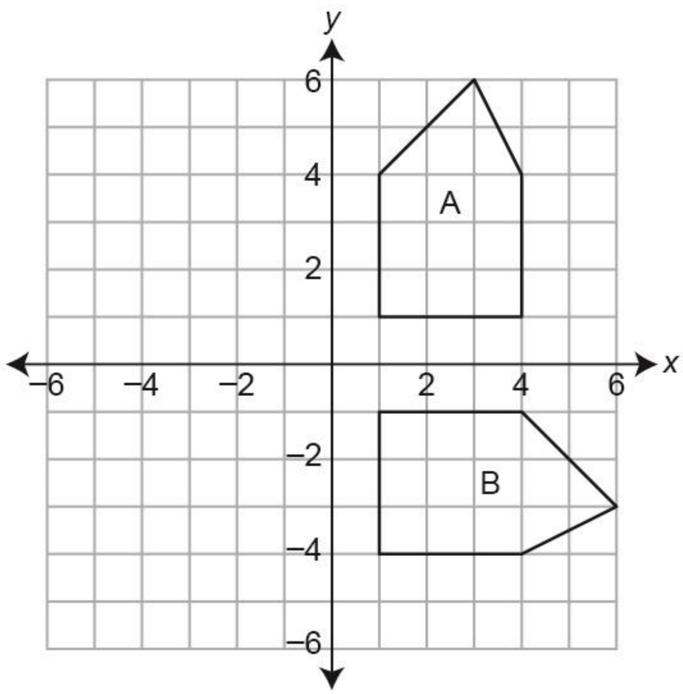


Ask students to use tracing paper to copy the image and show that the pre-image and the image have the same measure. Observe that no matter how the original angle is translated, its image has the same measure.

- Ask students to demonstrate that reflecting a pair of parallel lines, or a figure with a pair of parallel sides, will create lines or sides that are still parallel to one another. For example, ask students to draw a pair of parallel line segments by tracing along both long edges of a ruler. Have students use a marker that will transfer when wet, or something similar. Instruct each student to fold the paper in half, apply pressure to the fold so that the marker will transfer to the other side of the paper, and open the paper again to reveal the image of the original pair of parallel lines. Ask students to use the ruler to confirm that the new lines are parallel to each other. The ruler can be laid in between the new lines so that both lines are aligned with the edges of the ruler.
- Ask students to demonstrate that rotating a pair of parallel lines, or a figure with a pair of parallel sides, will create lines or sides that are also parallel to one another. For example, give students a coordinate grid and a figure with a pair of parallel sides on the grid. Have students use tracing paper to simulate a rotation by placing the piece of tracing paper on top of the coordinate grid and tracing the original figure. Have students make a small "+" at the origin to help keep track of how far the object on the paper has been rotated.

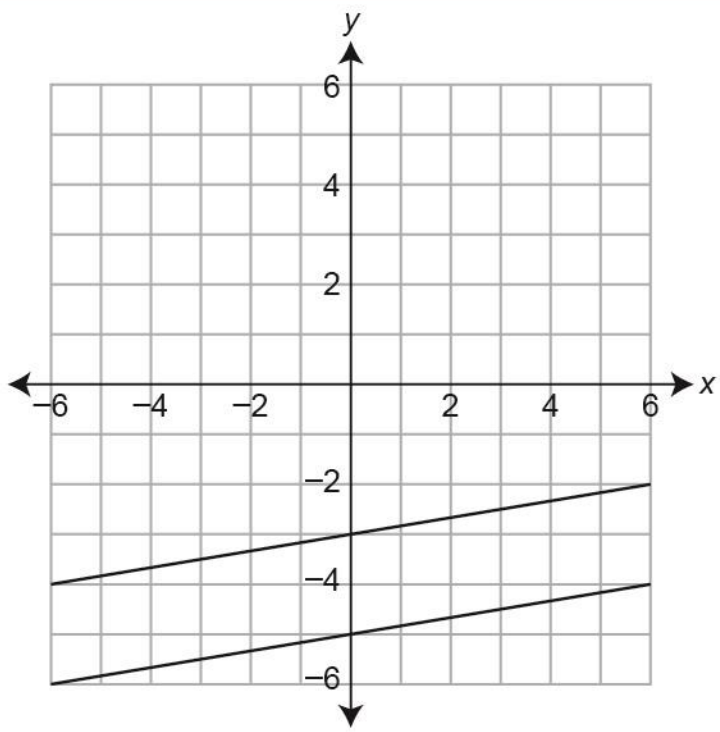


Using the origin as the center of rotation, students should place a pencil on the origin and then slowly rotate the tracing paper. Have students start by observing the lines when rotated 90 degrees clockwise.

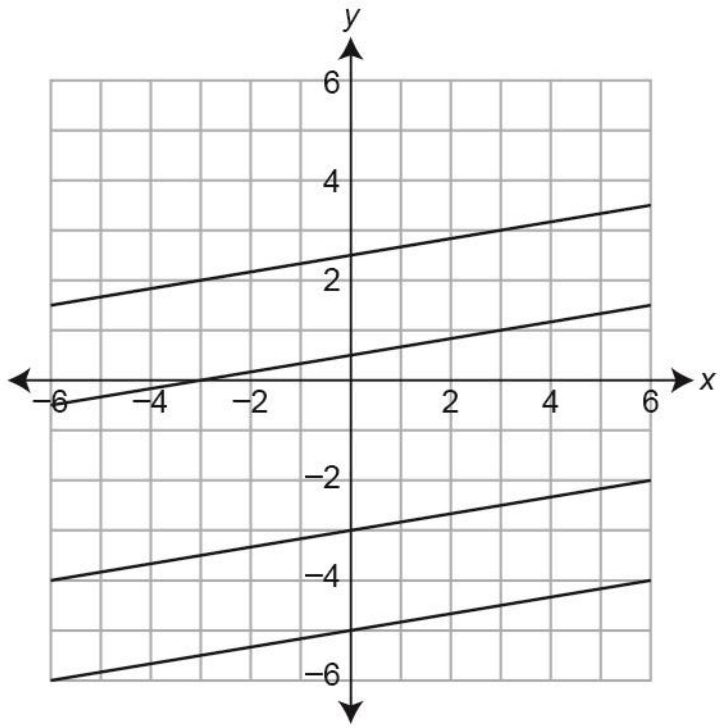


Then, have students repeat the same process to rotate the figure 180 degrees clockwise and 270 degrees clockwise. Students may also experiment with counterclockwise rotations. Students should observe that the distance between the parallel sides in each figure is exactly 3 units. Help students see that because the two parallel sides do not get any closer to each other, the sides do not intersect when extended and therefore are parallel.

- Ask students to demonstrate that translating a pair of parallel lines, or a figure with a pair of parallel sides, will create lines or sides that are also parallel to each other (as well as both original lines as long as the translation does not translate one of the original lines onto the other). For example, give students the following diagram.



Ask students to choose two points on each line (four points in total), to plot the image of all four points after a shift of 5 units up and 3 units left on the coordinate grid, and to connect the new points to create the image of each original line.

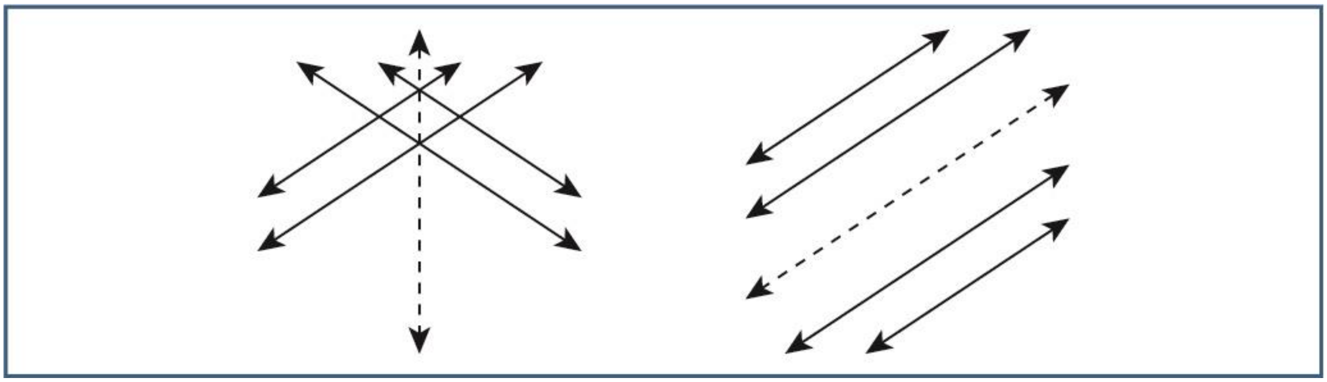


Help students confirm that both new lines are parallel by tracing the two original lines onto tracing paper and showing that the traced lines can be overlaid onto the two new lines.

► Identifying and Addressing Common Misconceptions

- Students may think that the length of a line segment changes after a rotation in the same way that "tilting" a rectangle into a parallelogram creates a new slant height that is different from the original height. Give students a line segment on a coordinate grid and ask students to observe how rotating the entire paper affects the length of the segment. Ask students to measure the length of the given line segment and complete each different type of transformation. Next, ask students to measure the length of each image to verify that the length did not change.
- Some students think that rotating an angle changes its measure. Ask students what the measure of a 60-degree angle is after it has been rotated 90 degrees. Some students may add the angles and say 150 degrees. Give students a protractor printed on a sheet of paper and ask them to draw a 60-degree angle. Ask them to rotate the paper and observe that the alignment of the angle on the protractor stays the same.

- Some students may think that a set of parallel lines reflected over another line is always parallel to the original set of lines (as well as each other). Ask students if they can find a way to reflect a line so that it is not parallel to its image. Some students may not be able to find such a reflection. Show the students that the only time the pre-image lines are parallel to the image lines is when the line of reflection is also parallel to the pre-image lines. Two possible illustrations are shown.

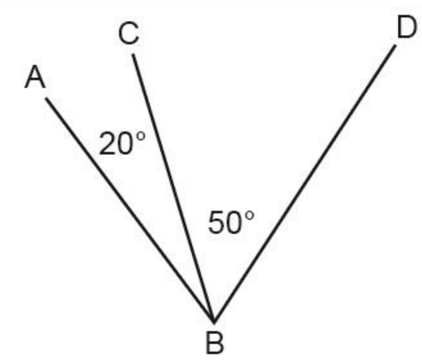


- Some students may think that a set of parallel lines rotated about a point is parallel to the original set of lines (as well as each other). Ask students whether a pair of parallel lines that has been rotated 90 degrees is parallel to the original pair of lines. Some students may think that the lines are parallel. Show students that the only time the pre-image lines are parallel to the image lines is when the rotation is a multiple of 180 degrees and the lines do not map onto each other or themselves.

► Enrichment

- Ask students to apply a combination of two or more reflections, rotations, or translations and show that the lengths of the segments in the image are the same lengths as the lengths of the segments in the pre-image.
- Give students the pre-image and image of a line segment and ask them to find 2 or more sequences of transformations that result in the given image.
- Ask students to describe transformations of lines that create images that are either parallel or perpendicular to the pre-image line.

- Ask students to explain why it is impossible to transform a square into a non-square rhombus using reflections, rotations, and translations. Students should find that because all three transformations preserve angle measures, any transformation will always have 90-degree angles.
- Ask students to apply additivity of angle measures to transformations. For example, give students the following figure.



Ask students to determine the measure of the image of angle ABD after a 90-degree clockwise rotation.

- Give students an angle and ask students to find a transformation where the pre-image and the image together form a triangle or a quadrilateral. Discuss what types of triangles and quadrilaterals can be formed in this way.
- Ask students to show that combinations of 2 or more different transformations applied to parallel lines also result in parallel lines.
- Give students a pair of parallel lines and ask them to find as many transformations as possible that will map one line onto the other. Ask them to explain why there is only one reflection that will map one line onto the other but infinitely many rotations and translations.
- Give students a pair of parallel lines and ask them to find a line of reflection such that the image and pre-image combined make a square.

► Key Terms

center of rotation, clockwise, counterclockwise, image, line, line of reflection, line segment, pre-image, reflection, rigid motion, rotation, transformation, translation, angle, parallel

C-G.1.1.2

▶ Reporting Category, Assessment Anchor, Anchor Descriptor

Geometry

Demonstrate an understanding of geometric transformations.

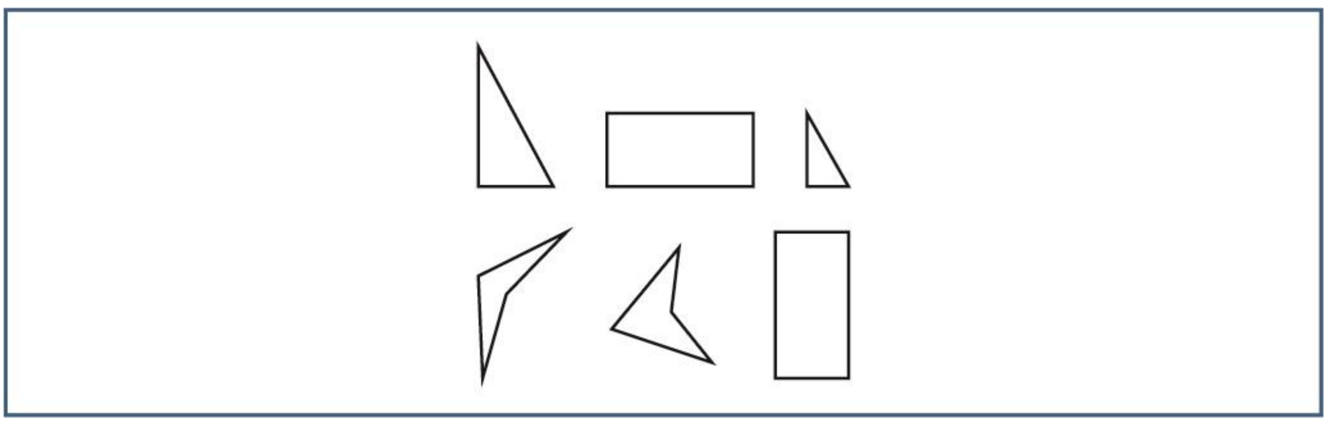
Apply properties of geometric transformations to verify congruence or similarity.

▶ Eligible Content

C-G.1.1.2 Given two congruent figures, describe a sequence of transformations that exhibits the congruence between them.

▶ Instructional Supports

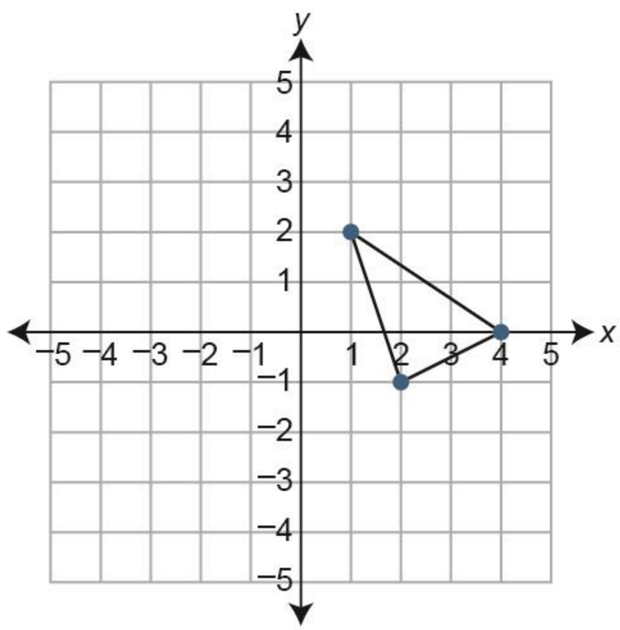
- Ask students to identify shapes in a set that are congruent to one another and to explain why. For example, give students the following set of shapes and ask them to identify the shapes that are congruent and explain how they know.



  Students should find the two rectangles to be congruent because they can be transformed into each other using rotations, reflections, and translations.

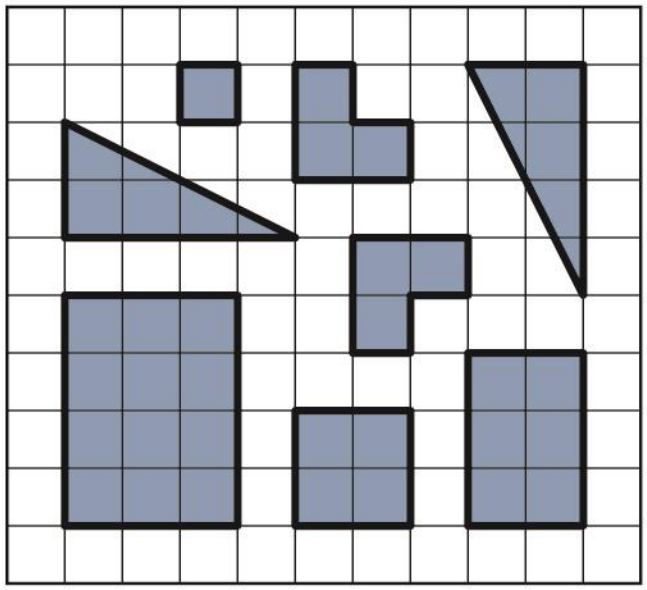
- Ask students to demonstrate that reflecting, rotating, or translating a figure creates a new figure whose sides and angles have the same measures as the sides and angles in the original figure. For example, give students a sheet of tracing paper and ask students to draw a line down the middle of the sheet and a shape on one half of the sheet. Instruct students to fold the paper along the line and trace their figure onto the other half of the tracing paper. Have students unfold the paper to reveal the image of the original figure. Ask students to confirm that the side lengths and angle measures are equal by measuring each side and angle of both the pre-image and the image.

- Ask students to show and explain why any combination of reflections, rotations, and translations will always create a figure with side lengths and angle measures that are the same as in the original figure. For example, give students the following figure and ask them to explain why rotating the figure 90 degrees clockwise around the origin, then translating the result 3 units to the left, and then reflecting the result over the *x*-axis creates an image that has the same side lengths and angle measures as the pre-image.



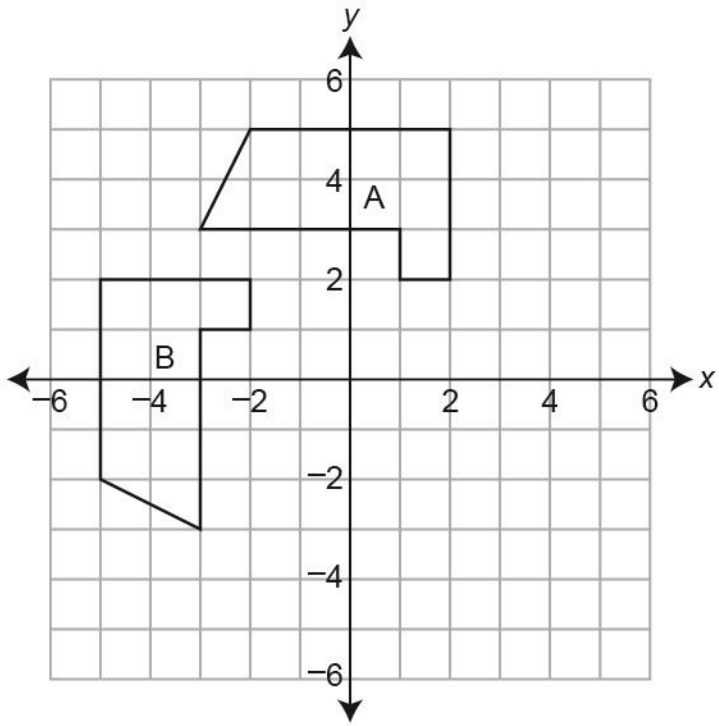
Students should explain that because reflections, rotations, and translations always result in line segments that are the same lengths and angles that have the same measures as the pre-image, each of these three transformations results in an image with the exact same side lengths and angle measures as the pre-image. Therefore, the result after each transformation has the same side and angle measures as the original. Students may complete each transformation and verify that the side lengths and angle measures are unchanged after each step.

- Give students a set of shapes and ask them to identify which shapes can be transformed into each other using reflections, rotations, and translations. For example, give students the following set of shapes.



Students should find that the two triangles are congruent and the two "L" shapes (each composed of 3 square units) are congruent. Therefore, both triangles can be transformed into each other and both "L" shapes can be transformed into each other using reflections, rotations, and translations. Note that the fact that the shapes are congruent indicates that such a sequence of transformations is possible, even if students cannot identify a specific sequence.

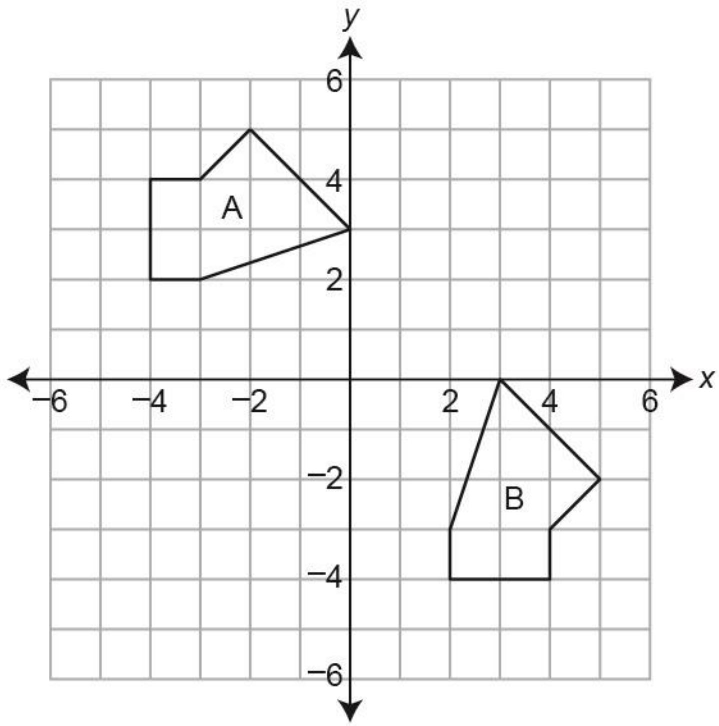
- Ask students to prove that two given figures on a coordinate grid are congruent by describing a transformation or sequence of transformations that maps one figure onto the other. For example, give students the following figures and ask them to determine a sequence of transformations that maps figure A onto figure B.



One possible sequence of transformations is a single 90-degree counterclockwise rotation about the origin. A second possible sequence is a 90-degree clockwise rotation about the origin followed by a reflection across both the *x*- and *y*-axes.

▶ Identifying and Addressing Common Misconceptions

- Some students may think that shapes are not congruent when they are oriented differently. Give students the following figure and ask whether the shapes in the figure are congruent and to explain their reasoning.



Some students may not recognize that the shapes are congruent because they are oriented differently. Help students identify corresponding sides and angles of each figure. Then ask them to measure the sides and angles to confirm that the figures are congruent. Figure B may not appear obviously congruent to figure A, but figure B can be created by rotating figure A 90 degrees clockwise about the origin and reflecting the result across the *x*-axis.

- Some students may think that two figures only need to be the same type of shape rather than the exact same size to be congruent. For example, give students two rectangles with different dimensions and ask students to determine whether they are congruent. Some students may think that they are congruent because they are both rectangles. Ask students to measure the side lengths and remind students that congruence requires the exact same side lengths and angle measures.

► Enrichment

- Ask students to create multiple images of a figure to form letters or pictures. For example, give students a narrow rectangle and ask them to create multiple transformations whose images combine to make the letter "E."
- Ask students to find two or more different transformations or sequences of transformations that map a figure onto another figure.
- Ask students to create congruent figures using a rotation about a point that is not the origin or a reflection across a line that is not an axis of the coordinate grid.

► Key Terms

congruent, image, mirror image, pre-image, reflection, rigid motion, rotation, sequence, transformation, translation

C-G.1.1.3

▶ Reporting Category, Assessment Anchor, Anchor Descriptor

Geometry

Demonstrate an understanding of geometric transformations.

Apply properties of geometric transformations to verify congruence or similarity.

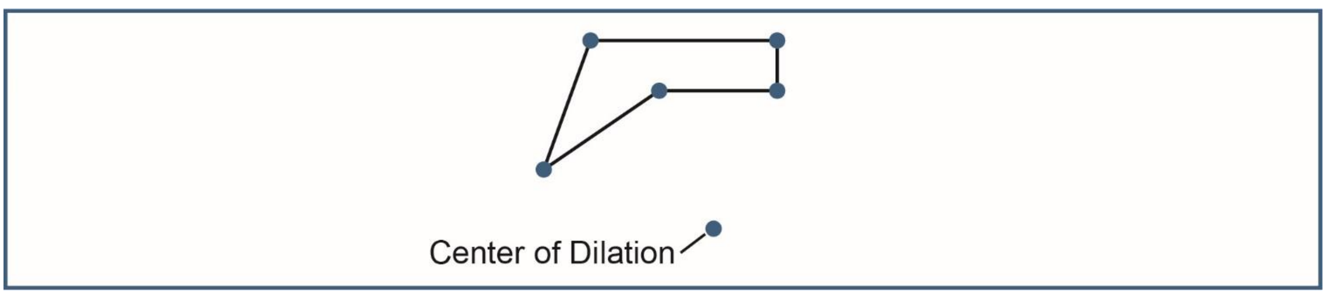
▶ Eligible Content

C-G.1.1.3 Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures, using coordinates.

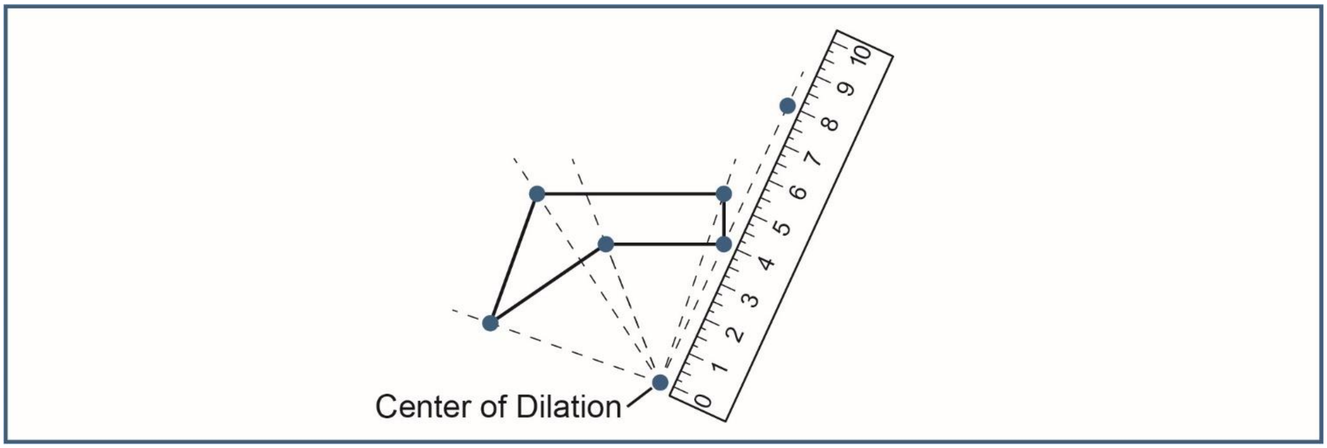
▶ Instructional Supports

Dilations

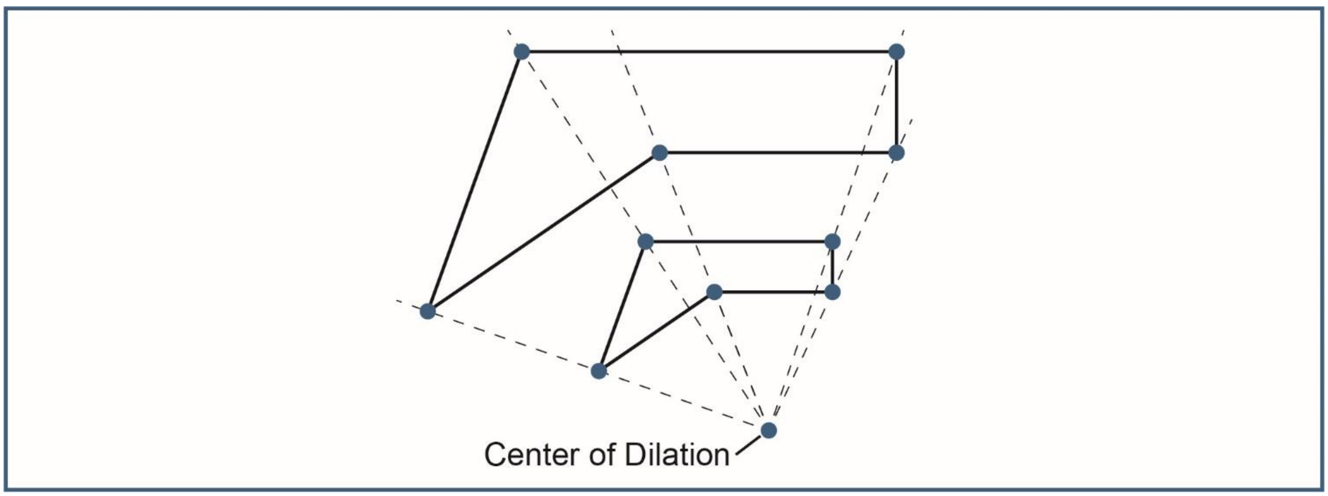
- Ask students to create a dilation given a figure and a scale factor. For example, give students the following figure and ask them to them to create a dilation from the given center with a scale factor of 2.



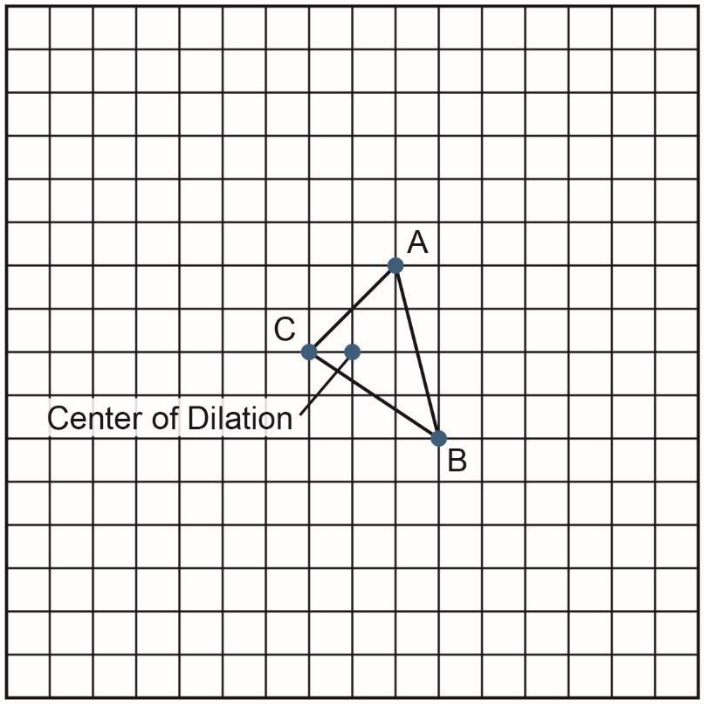
Ask students to place the end of a ruler at the center of dilation and to measure the distance from the center to a vertex of the figure.



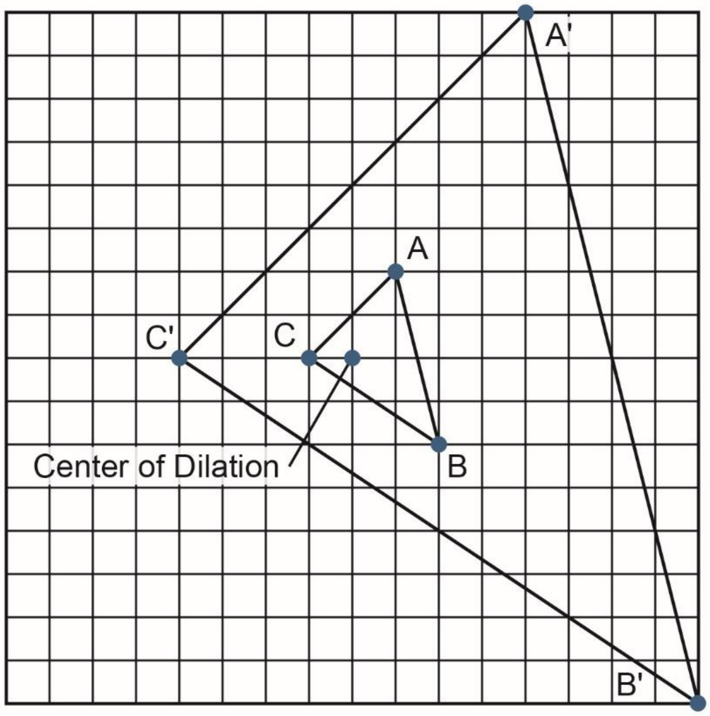
Then, instruct students to multiply the distance by 2 and create a new point at that distance next to the ruler. Have students repeat this process for each vertex of the figure. Ask students to connect the new vertices to create the image after the dilation.



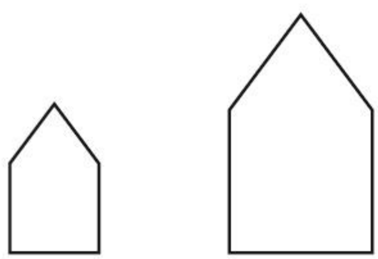
- Ask students to create a dilation of a given figure on a grid. For example, give students the following figure and ask them to create a dilation from the given center of dilation with a scale factor of 4.



Students should observe that point A is 1 space to the right and 2 spaces up from the center of dilation. Students should start at the center and move 1 space to the right and 2 spaces up 4 times to plot a new point. The new point should be 4 spaces to the right and 8 spaces up from the center of dilation. Ask students to repeat this process for the remaining two points and then connect the points to create the dilated triangle.

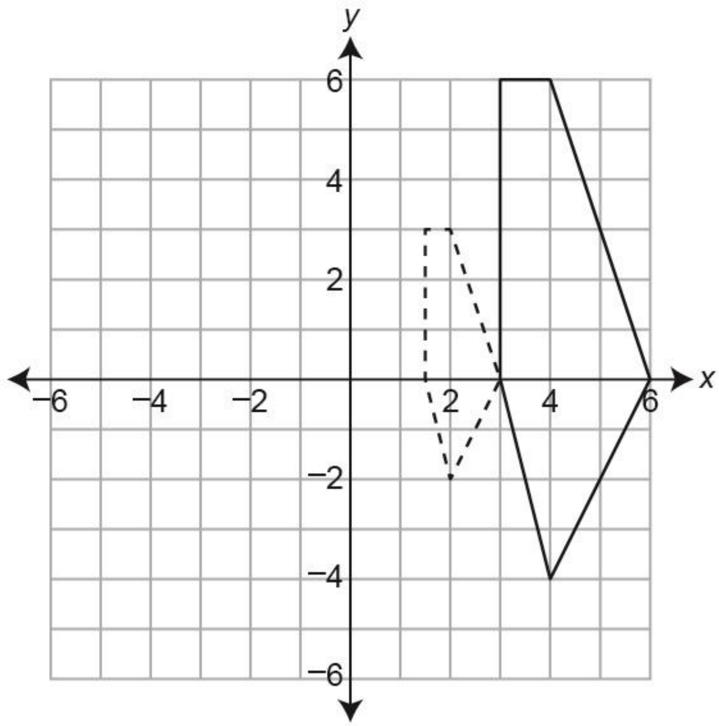


- Ask students to explore the effect of dilations on side lengths and angle measures. For example, give students the following figures.



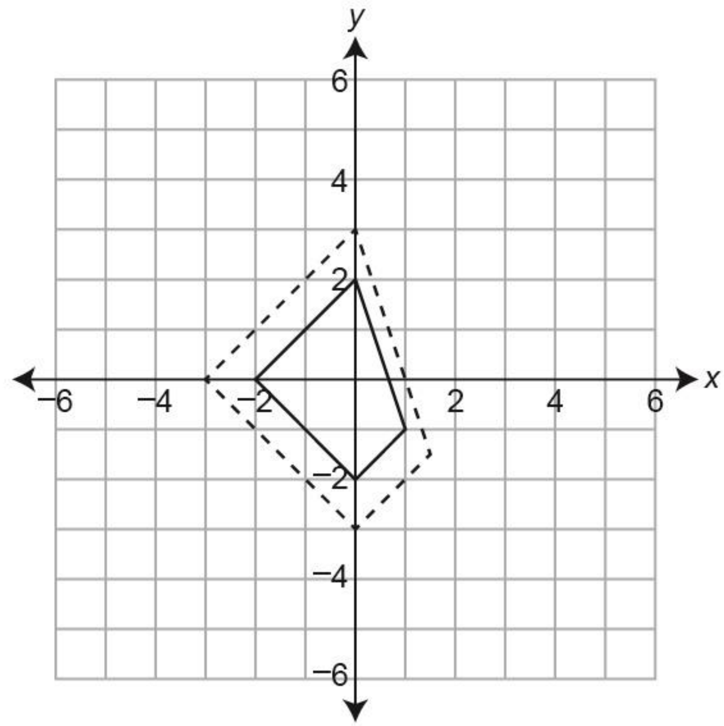
Tell students that the figure on the left was dilated by a scale factor of 1.6 to create the figure on the right. Instruct students to measure and record each of the sides and angles of both figures. Students should observe that all side lengths in the figure on the right are 1.6 times as long as the corresponding side lengths of the figure on the left and that this multiplier is the same as the dilation scale factor. Students should also find that the measures of the corresponding angles of both figures are equal.

- Ask students to describe the size of the image of a figure after a dilation by a scale factor *m*, where $0 < m < 1$. For example, give students the solid figure shown and ask them to dilate the figure by a scale factor of $\frac{1}{2}$ using the origin as the center of dilation.



Students should create the dotted figure shown. Have students repeat this process for any scale factor *m*, where $0 < m < 1$. Students should observe that no matter the value of *m*, the image is always smaller than the pre-image.

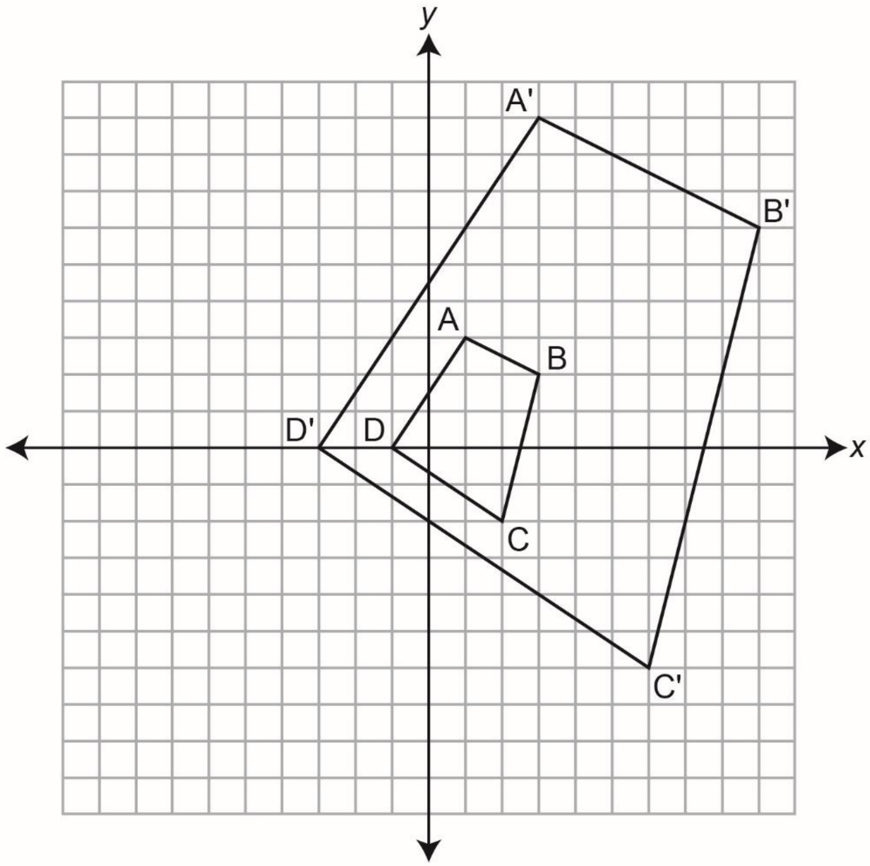
- Ask students to describe the size of the image of a figure after a dilation of a scale factor *m*, where $m > 1$. For example, give students the solid figure shown and ask them to dilate the figure by a scale factor of 1.5 using the origin as the center of dilation.



Students should create the dotted figure shown. Have students repeat this process for any scale factor m, where $m > 1$. Students should observe that whenever $m > 1$, the image is larger than the pre-image.

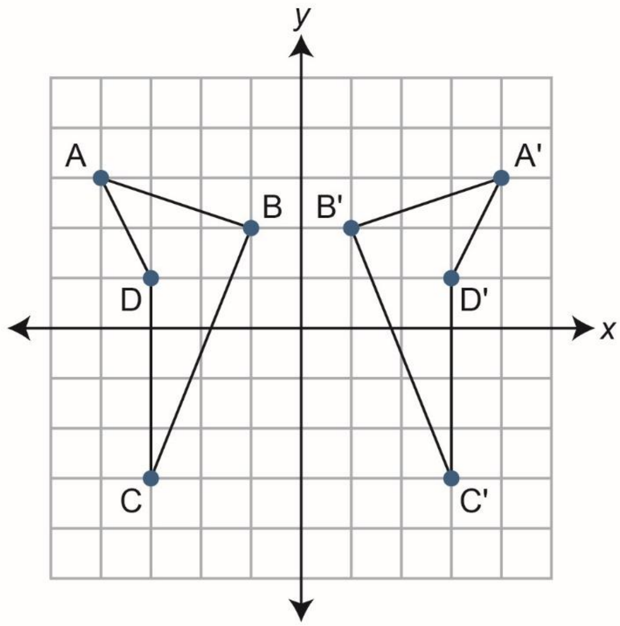
Effects of transformations on coordinates

- Ask students to explore the effects of dilations on coordinates. For example, give students a coordinate grid and ask them to draw a quadrilateral and record the coordinates of the vertices. Then, have students dilate their rectangle by a scale factor of 3 centered at the origin and record the coordinates of the vertices of the image. A possible quadrilateral with its dilation is shown.



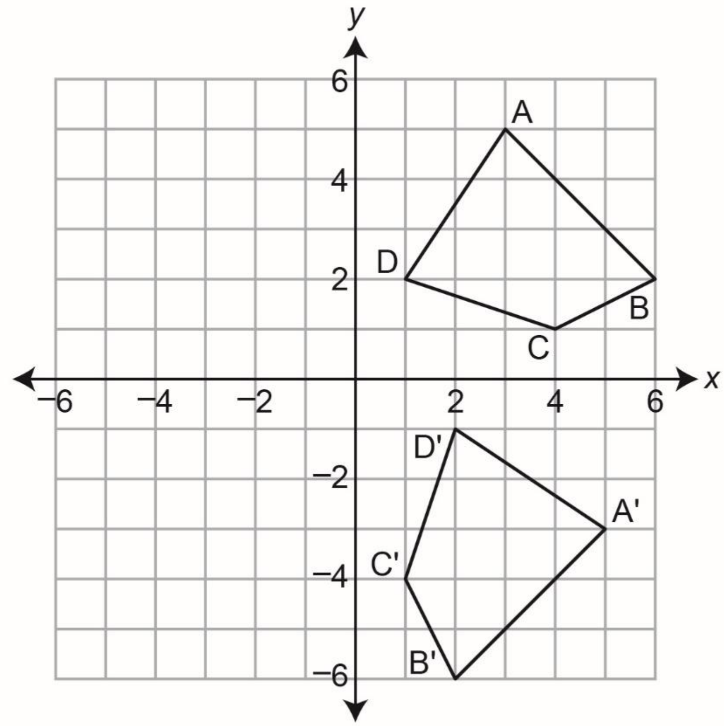
Ask students to compare the coordinates of the vertices of the figures. Students should notice that the coordinates of their image can all be attained by multiplying the coordinates of their pre-image by 3. Help students express this using the model $(x, y) \rightarrow (3x, 3y)$. In general, a figure can be dilated using the origin as the center of dilation by multiplying all the *x*- and *y*-coordinates of the pre-image by the dilation scale factor. If the dilation scale factor is *m*, the model $(x, y) \rightarrow (mx, my)$ can be used.

- Ask students to explore the effects of reflections on coordinates. For example, give students a coordinate grid and ask them to draw a quadrilateral and record the coordinates of the vertices. Then, have students reflect the quadrilateral across the *y*-axis and record the coordinates of all the vertices. A possible quadrilateral with its reflection is shown.



Ask students to compare the coordinates of the figures. Students should notice that the *y*-coordinate of each point is the same as the *y*-coordinate of its reflection and that the *x*-coordinate of each point is the opposite of the *x*-coordinate of its reflection. Help students express this using the model $(x, y) \rightarrow (-x, y)$. Repeat this process for reflections across the *x*-axis. When reflecting across the *x*-axis, the model $(x, y) \rightarrow (x, -y)$ can be used.

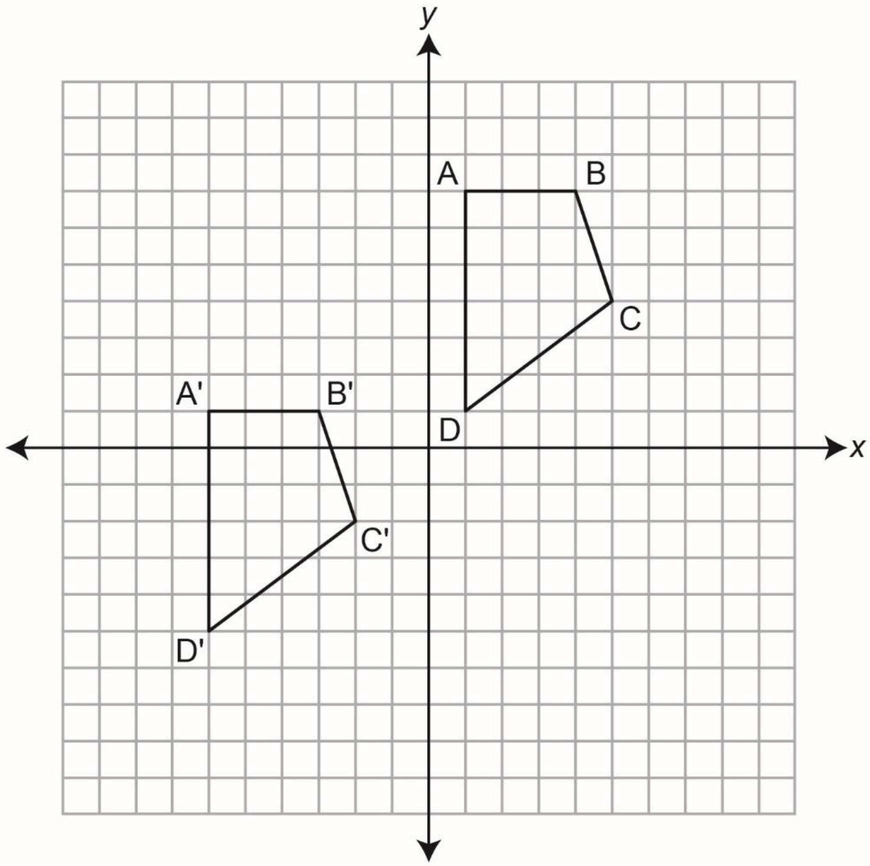
- Ask students to explore the effects of rotations on coordinates. For example, give students a coordinate grid and ask them to draw a quadrilateral and record the coordinates of the vertices. Then, have students rotate the quadrilateral 90 degrees clockwise around the origin and record the coordinates of all the vertices. A possible quadrilateral with its rotation is shown.



Ask students to compare the coordinates of the figures. Students should notice that the *x*-coordinate of each point in the image is equal to the *y*-coordinate of the corresponding point in the pre-image, and the *y*-coordinate of each point in the image is equal to the opposite of the *x*-coordinate of the corresponding point in the pre-image. Help students express this using the model $(x, y) \rightarrow (y, -x)$. Guide students to see that, in general, rotations around the origin can be expressed with the following models.

- 90 degrees clockwise or 270 degrees counterclockwise about the origin: $(x, y) \rightarrow (y, -x)$
- 180 degrees about the origin: $(x, y) \rightarrow (-x, -y)$
- 270 degrees clockwise or 90 degrees counterclockwise about the origin: $(x, y) \rightarrow (-y, x)$

- Ask students to explore the effects of translations on coordinates. For example, give students a coordinate grid and ask them to draw a quadrilateral and record the coordinates of the vertices. Then, have students translate the quadrilateral left 7 units and down 6 units and record the coordinates of all the vertices. A possible quadrilateral with its translation is shown.



Ask students to compare the coordinates of the figures. Students should notice that the x-coordinate of each point in the image is 7 less than the x-coordinate of the corresponding point in the pre-image and the y-coordinate of each point in the image is 6 less than the y-coordinate of the corresponding point in the pre-image. Help students express this using the model $(x, y) \rightarrow (x - 7, y - 6)$. Guide students to see that, in general, translations can be expressed with the following models.

- m units right: $(x, y) \rightarrow (x + m, y)$
- n units left: $(x, y) \rightarrow (x - n, y)$
- p units up: $(x, y) \rightarrow (x, y + p)$
- q units down: $(x, y) \rightarrow (x, y - q)$

▶ Identifying and Addressing Common Misconceptions

- Some students may think dilations change the angle measures of a figure as well as the side lengths. Ask students to determine the measure of a 60-degree angle after a dilation by a scale factor of 2. Some students may give an answer of 120 degrees. Ask students to describe what the dilation of a square by a factor of 2 would look like. Help students recognize that the image is still a square and therefore still has the same angle measures.
- Some students may think that reflections across an axis change the coordinate corresponding to that axis. That is, a reflection over the *x*-axis implies $(x, y) \rightarrow (-x, y)$ and a reflection over the *y*-axis implies $(x, y) \rightarrow (x, -y)$. Ask students to identify the coordinates of the point (2, 5) after a reflection across the *y*-axis. Some students may give the coordinates (2, –5). Ask students which direction the *y*-axis is aligned. Students should recognize that the *y*-axis is a vertical line. Ask students which direction the point (2, 5) moves after a reflection across the *y*-axis. Students should recognize that reflections across the *y*-axis moves a shape perpendicular to the *y*-axis in a left or right direction. Help students recognize that as you reflect across an axis, the point will move perpendicular to the direction of the axis. So reflecting across the *y*-axis, which is vertical, will move a point left or right, while reflecting across the *x*-axis, which is horizontal, will move a point up or down.

▶ Enrichment

- Ask students to describe the effect of a sequence of two or more different dilations using the same center. In particular, help students recognize that the sequence of dilations is equivalent to a dilation whose scale factor is the product of the scale factors in the sequence.
- Ask students to create algebraic models for sequences of transformations.
- Ask students to explore the effect of a dilation using a scale factor that is negative.

▶ Key Terms

center of dilation, congruent, coordinate, dilation, image, pre-image, preserve, reflection, rigid motion, rotation, similar, transformation, translation, vertex

C-G.1.1.4

► Reporting Category, Assessment Anchor, Anchor Descriptor

Geometry

Demonstrate an understanding of geometric transformations.

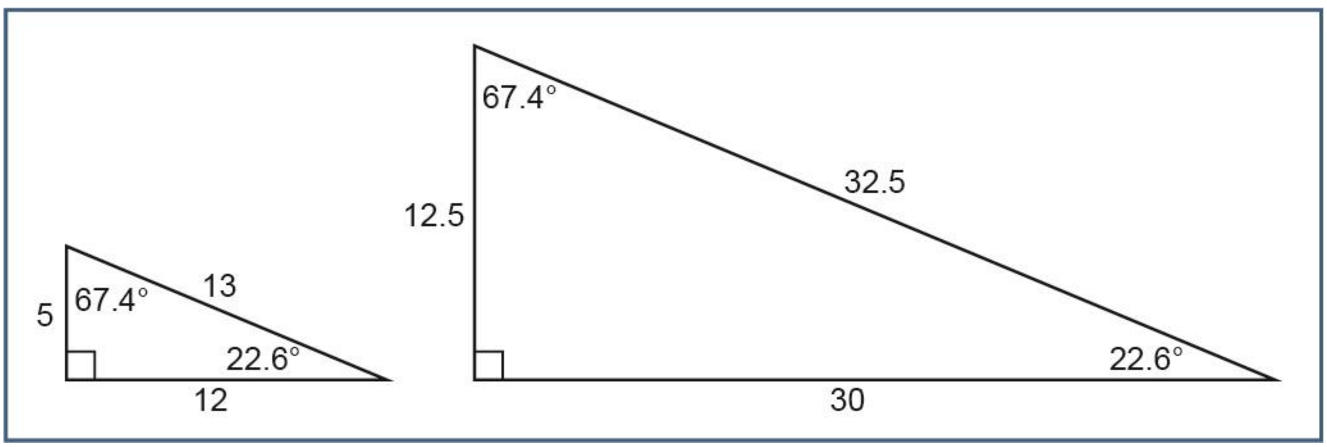
Apply properties of geometric transformations to verify congruence or similarity.

► Eligible Content

C-G.1.1.4 Given two similar two-dimensional figures, describe a sequence of transformations that exhibits the similarity between them.

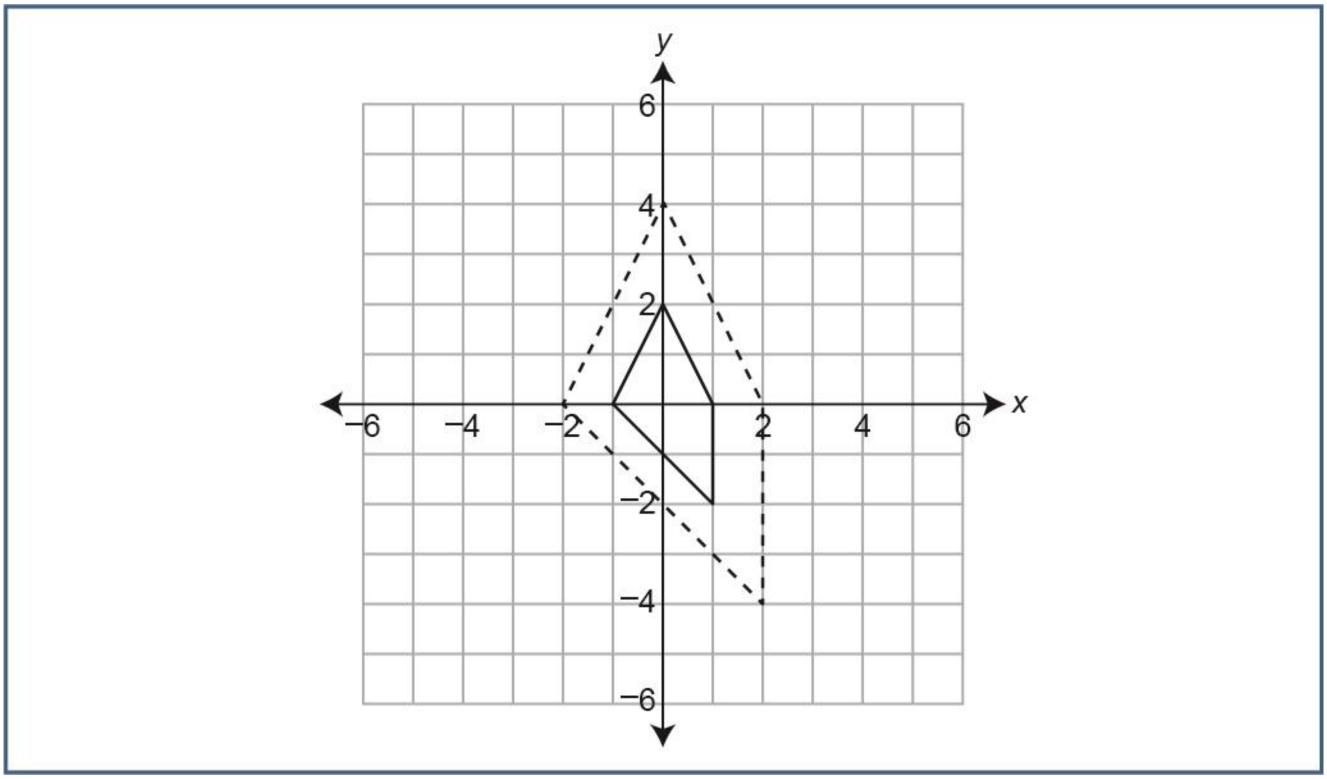
► Instructional Supports

- Ask students to determine whether two given shapes are similar. For example, give students the following triangles and ask them to determine whether the triangles are similar and to explain how they found their answer.



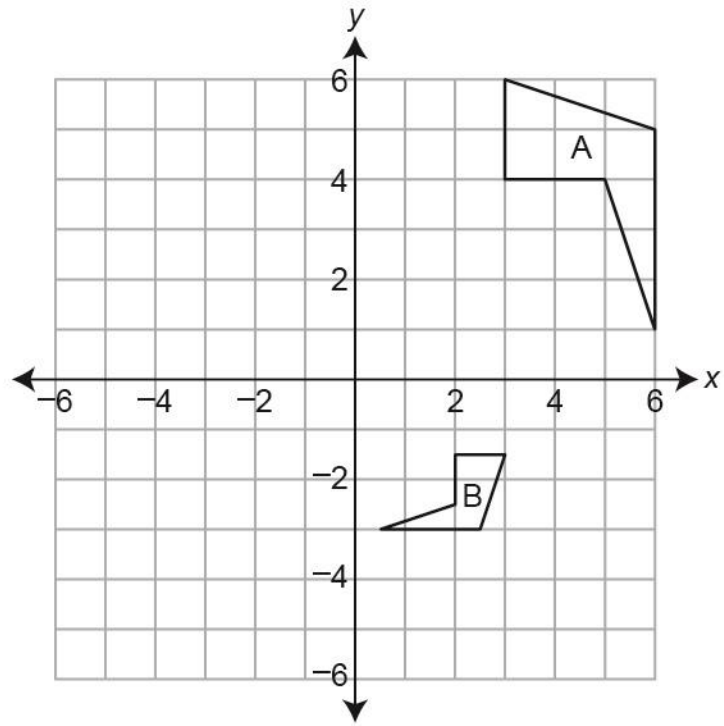
  Students should calculate that each of the side lengths of figure B is 2.5 times as large as the corresponding side length of figure A. Also, students should note that the corresponding angle measures of each figure are congruent. Therefore, students should determine that the triangles are similar.

- Give students a figure on a coordinate grid and ask them to perform a transformation or sequence of transformations on it and then determine whether the image and pre-image are similar. For example, give students the solid figure shown and ask students to dilate the figure with a scale factor of 2 about the origin.



Students should create the dashed figure. Ask students to use a ruler and protractor to measure each side length and each angle measure of both figures. Students should find that each side length of the image is 2 times as large as the corresponding side length of the pre-image and each pair of corresponding angle measures are equal. Therefore, they should conclude that the two figures are similar. Help students see that, in general, any combination of reflections, rotations, translations, and dilations results in an image that is similar to its pre-image.

- Give students two similar figures on a coordinate grid and ask them to determine a transformation or sequence of transformations that maps one figure onto the other. For example, give students the following graph.



Have students determine a transformation or set of transformations that could be used to map one figure onto the other. One possible sequence is that figure A can be mapped onto figure B by rotating figure A 90 degrees clockwise about the origin and then dilating figure A with a scale factor of $\frac{1}{2}$ centered at the origin.

► Identifying and Addressing Common Misconceptions

- Some students may think that two figures are similar if they share similar attributes. Give students two rectangles that are not similar and ask students to explain whether they are similar. Some students may note that the rectangles share some similar characteristics (e.g., right angles, shape type) and therefore are similar figures. Help students understand that the term "similar" is used differently in math from how it is used in everyday language. In math, the term "similar" does not just mean having attributes in common; there must be a sequence of transformations between the figures.
- Some students may think that similar figures must be different sizes. Show students two congruent figures and ask them if the figures are similar. Some students may say that they are

not. Ask students to measure the side lengths and find a common scale factor between corresponding sides. Help them recognize that the scale factor is 1.

► Enrichment

- Ask students to create similar figures using dilations not centered at the origin.
- Ask students to describe projections and shadows in terms of dilations.

► Key Terms

congruent, corresponding angles, corresponding sides, dilation, exhibit, image, pre-image, preserve, reflection, rigid, rotation, similar, transformation, translation

C-G.2.1.1 & C-G.2.1.2

▶ Reporting Category, Assessment Anchor, Anchor Descriptor

Geometry

Understand and apply the Pythagorean Theorem.

Solve problems involving right triangles by applying the Pythagorean theorem.

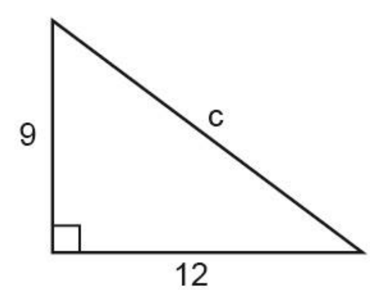
▶ Eligible Content

C-G.2.1.1 Apply the converse of the Pythagorean theorem to show a triangle is a right triangle.

C-G.2.1.2 Apply the Pythagorean theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.

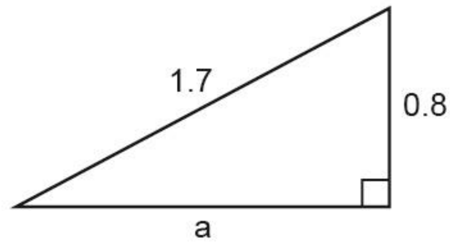
▶ Instructional Supports

- Ask students to find the hypotenuse of a right triangle when the lengths of both legs of the right triangle are known. For example, give students the right triangle shown.



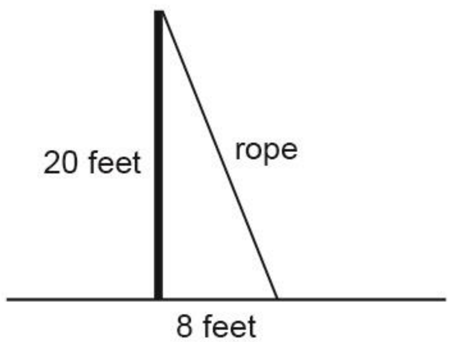
Entering the values for the side lengths of the triangle into the Pythagorean theorem $a^2 + b^2 = c^2$ creates the equation $9^2 + 12^2 = c^2$. Therefore, $c^2 = 225$ and $c = 15$.

- Ask students to find an unknown leg of a right triangle when the length of the other leg and the hypotenuse of the right triangle are known. For example, give students the right triangle shown.



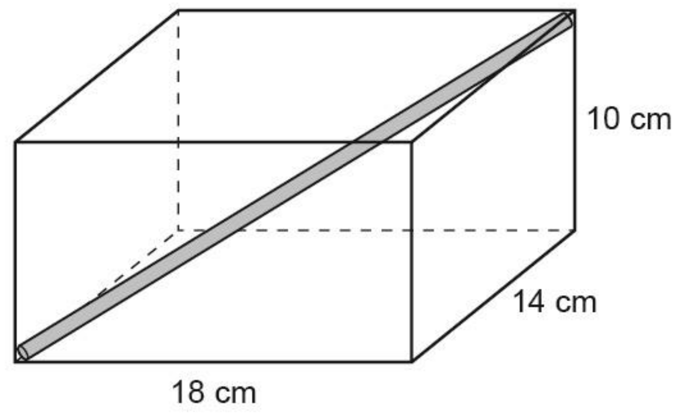
The hypotenuse has a length of 1.7 and replaces *c* in the Pythagorean theorem. The leg length of 0.8 can replace either *a* or *b*. Using these values creates the equation $a^2 + 0.8^2 = 1.7^2$. Solving for a^2 results in $a^2 = 1.7^2 - 0.8^2 = 2.25$. Therefore, $a = 1.5$.

- Ask students to use the Pythagorean theorem to find an unknown length in a real-world situation. For example, ask students to find the length of a rope that is tied to the top of a 20-foot pole and staked to the ground 8 feet from the base of the pole.

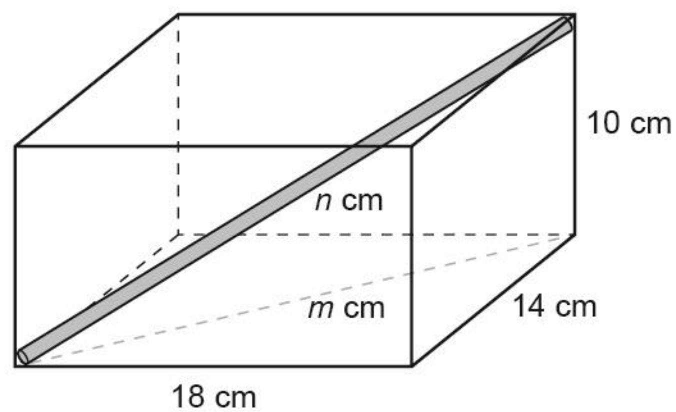


The rope and pole form a right triangle with the ground. Therefore, the Pythagorean theorem can be used to find the length of the rope. Students create the equation $20^2 + 8^2 = r^2$ and solve for r^2 to get $r^2 = 464$. Therefore, the length of the rope is approximately 21.54 feet (21 feet, 6.5 inches).

- Ask students to use the Pythagorean theorem to find lengths in three-dimensional shapes. For example, ask students to find the length of the longest possible metal pole that can fit into a box that is 18 centimeters long, 14 centimeters wide, and 10 centimeters tall.



Students use the Pythagorean theorem twice in this problem to find the length of the pole. The hypotenuse of the first triangle is a leg of the second triangle.



The length of the pole, n, can be found using the Pythagorean theorem and the equation $m^2 + 10^2 = n^2$. The value of m^2 can be found using the Pythagorean theorem and the equation $14^2 + 18^2 = m^2$. Students solve for m^2 to get $m^2 = 520$, which can be used in the first equation to create the equation $520 + 10^2 = n^2$. Therefore, $n^2 = 620$. The longest metal pole that can fit into the box is $n = \sqrt{620} \approx 24.9$ centimeters long.

► Identifying and Addressing Common Misconceptions

- Some students may always substitute the given lengths in for a and b regardless of whether the given lengths are legs. Ask students to find the length of a leg of a right triangle given that the hypotenuse and the other leg have lengths of 8 and 5, respectively. Some students may set up the equation $5^2 + 8^2 = c^2$ or $8^2 + 5^2 = c^2$. Ask students to explain what each of the variables in the Pythagorean theorem represents and to identify the hypotenuse of the right triangle. Help students understand that c represents the length of the hypotenuse and that the length of the hypotenuse cannot be substituted anywhere else in the equation.

- Some students may add or subtract the bases before squaring the values or cancel out the squares when taking the square root of both sides of their equation. Ask students to find the value of a in the equation $a^2 + 5^2 = 13^2$. Students who give an answer of 8 may have made one of these errors. Have students rewrite their equation by squaring all the numbers before solving for a.

► Enrichment

- Ask students to use the Pythagorean theorem in combination with other formulas to solve problems. For example, give students the volume and the radius of a cone and ask them to find the slant height of the cone.

► Key Terms

dimension, hypotenuse, leg, Pythagorean theorem, right triangle, unit

C-G.2.1.1 & C-G.2.1.2

► Reporting Category, Assessment Anchor, Anchor Descriptor

Geometry

Understand and apply the Pythagorean Theorem.

Solve problems involving right triangles by applying the Pythagorean theorem.

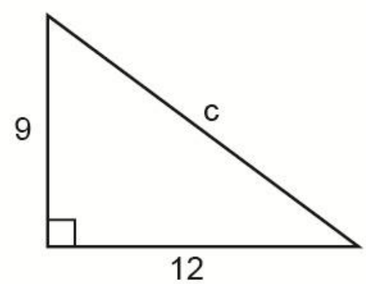
► Eligible Content

C-G.2.1.1 Apply the converse of the Pythagorean theorem to show a triangle is a right triangle.

C-G.2.1.2 Apply the Pythagorean theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.

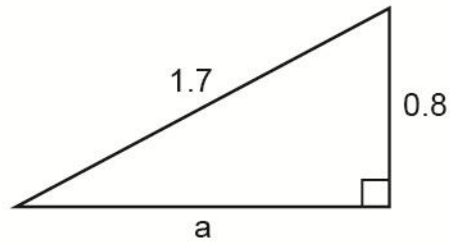
► Instructional Supports

- Ask students to find the hypotenuse of a right triangle when the lengths of both legs of the right triangle are known. For example, give students the right triangle shown.



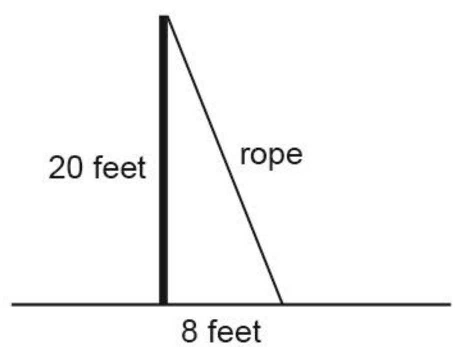
Entering the values for the side lengths of the triangle into the Pythagorean theorem $a^2 + b^2 = c^2$ creates the equation $9^2 + 12^2 = c^2$. Therefore, $c^2 = 225$ and $c = 15$.

- Ask students to find an unknown leg of a right triangle when the length of the other leg and the hypotenuse of the right triangle are known. For example, give students the right triangle shown.



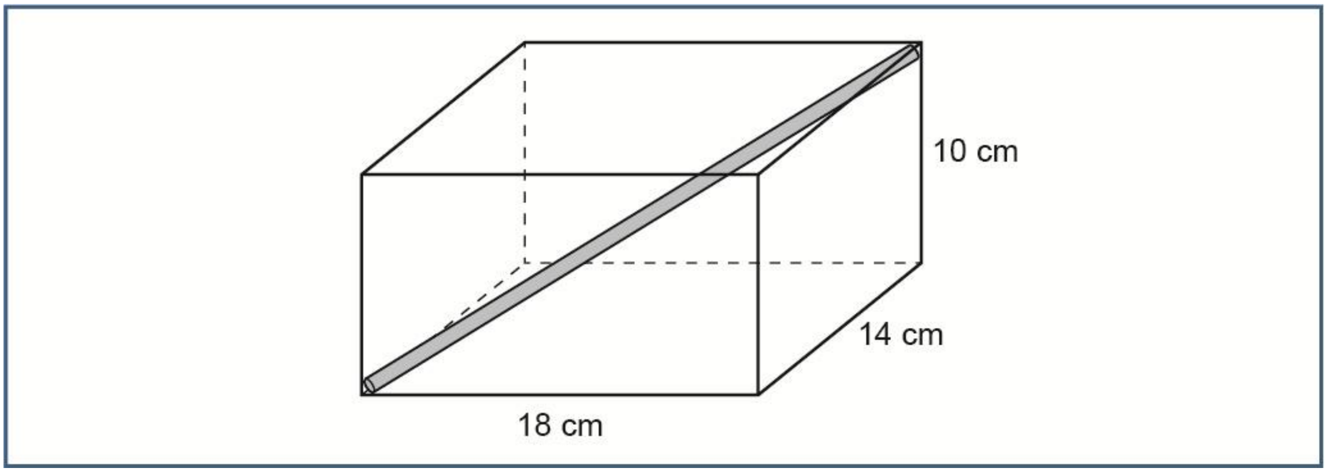
The hypotenuse has a length of 1.7 and replaces c in the Pythagorean theorem. The leg length of 0.8 can replace either a or b. Using these values creates the equation $a^2 + 0.8^2 = 1.7^2$. Solving for a^2 results in $a^2 = 1.7^2 - 0.8^2 = 2.25$. Therefore, $a = 1.5$.

- Ask students to use the Pythagorean theorem to find an unknown length in a real-world situation. For example, ask students to find the length of a rope that is tied to the top of a 20-foot pole and staked to the ground 8 feet from the base of the pole.

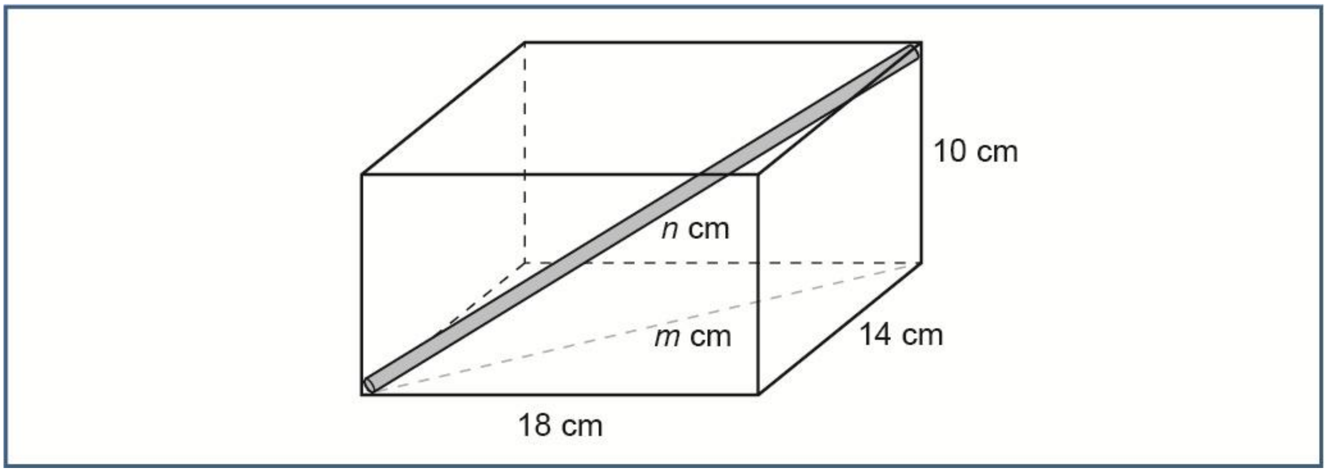


The rope and pole form a right triangle with the ground. Therefore, the Pythagorean theorem can be used to find the length of the rope. Students create the equation $20^2 + 8^2 = r^2$ and solve for r^2 to get $r^2 = 464$. Therefore, the length of the rope is approximately 21.54 feet (21 feet, 6.5 inches).

- Ask students to use the Pythagorean theorem to find lengths in three-dimensional shapes. For example, ask students to find the length of the longest possible metal pole that can fit into a box that is 18 centimeters long, 14 centimeters wide, and 10 centimeters tall.



Students use the Pythagorean theorem twice in this problem to find the length of the pole. The hypotenuse of the first triangle is a leg of the second triangle.



The length of the pole, n, can be found using the Pythagorean theorem and the equation $m^2 + 10^2 = n^2$. The value of m^2 can be found using the Pythagorean theorem and the equation $14^2 + 18^2 = m^2$. Students solve for m^2 to get $m^2 = 520$, which can be used in the first equation to create the equation $520 + 10^2 = n^2$. Therefore, $n^2 = 620$. The longest metal pole that can fit into the box is $n = \sqrt{620} \approx 24.9$ centimeters long.

► Identifying and Addressing Common Misconceptions

- Some students may always substitute the given lengths in for a and b regardless of whether the given lengths are legs. Ask students to find the length of a leg of a right triangle given that the hypotenuse and the other leg have lengths of 8 and 5, respectively. Some students may set up the equation $5^2 + 8^2 = c^2$ or $8^2 + 5^2 = c^2$. Ask students to explain what each of the variables in the Pythagorean theorem represents and to identify the hypotenuse of the right triangle. Help students understand that c represents the length of the hypotenuse and that the length of the hypotenuse cannot be substituted anywhere else in the equation.

- Some students may add or subtract the bases before squaring the values or cancel out the squares when taking the square root of both sides of their equation. Ask students to find the value of a in the equation $a^2 + 5^2 = 13^2$. Students who give an answer of 8 may have made one of these errors. Have students rewrite their equation by squaring all the numbers before solving for a.

▶ Enrichment

- Ask students to use the Pythagorean theorem in combination with other formulas to solve problems. For example, give students the volume and the radius of a cone and ask them to find the slant height of the cone.

▶ Key Terms

dimension, hypotenuse, leg, Pythagorean theorem, right triangle, unit

C-G.2.1.3

► Reporting Category, Assessment Anchor, Anchor Descriptor

Geometry

Understand and apply the Pythagorean Theorem.

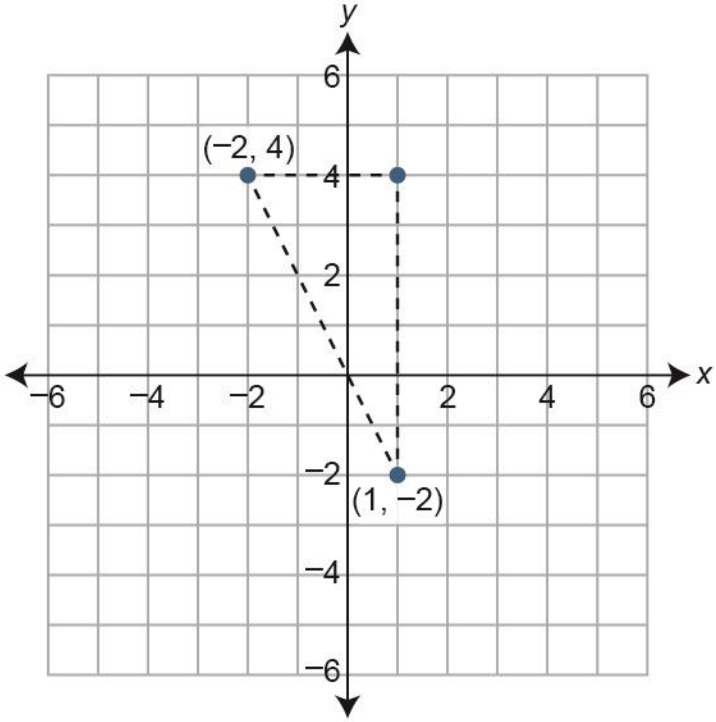
Solve problems involving right triangles by applying the Pythagorean theorem.

► Eligible Content

C-G.2.1.3 Apply the Pythagorean theorem to find the distance between two points in a coordinate system.

► Instructional Supports

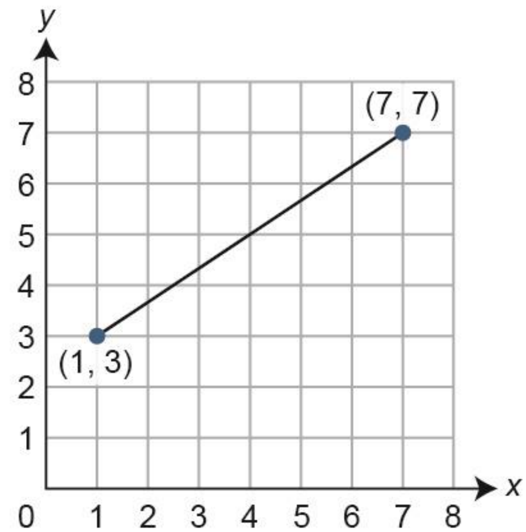
- Ask students to find the distance between two points on a coordinate grid. For example, ask students to find the distance between the points (1, –2) and (–2, 4). Have students plot both points. Then ask them to draw a vertical line through one point and a horizontal line through the other point. The intersection of these two lines creates a third point. The three points form a right triangle.



Students can use the grid lines to count the lengths of the legs of the right triangle to be 3 and 6. Then, students can use the Pythagorean theorem to create the equation $6^2 + 3^2 = c^2$. Therefore, the distance between the original two points is approximately 6.7 units.

► Identifying and Addressing Common Misconceptions

- Some students may think the distance between two points on a coordinate grid is the sum of the vertical and horizontal differences between the coordinates of the points. Ask students to find the distance between the points (1, 5) and (3, 8). Some students may give an answer of 5 (2 units horizontally and 3 units vertically). Ask students to plot the points and draw the shortest path between the points. Ask students to compare the length of the segment connecting the points to the length of a path going 2 units horizontally and 3 units vertically between the points. Help students understand that the direct path is shorter. If necessary, ask students to determine the shortest distance between two points in the classroom and illustrate why the direct path is shorter by counting paces.
- Some students may count the distance between points by counting the number of grid lines that a line crosses. Give students the following graph and ask students the distance between the points.



Some students may give an answer of 8 because the segment shown passes through 8 grid lines or corners. Pick a portion of the segment between grid lines and ask students whether the piece of the segment is the same length as a unit on the grid. Help them recognize that grid lines indicate unit distances only when moving directly along the grid and not diagonally.

► Enrichment

- Ask students to find the location on a coordinate grid of all the points a specific distance from a given point and determine the shape created.
- Ask students to find the distance between two points on a three-dimensional coordinate grid.
- Ask students to create the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, which can be used to find the distance between any two points on a coordinate grid.

► Key Terms

coordinate system, distance, hypotenuse, leg, line segment, Pythagorean theorem, right triangle, vertex

C-G.3.1.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

Geometry

Solve real-world and mathematical problems involving volume.

Apply volume formulas of cones, cylinders, and spheres.

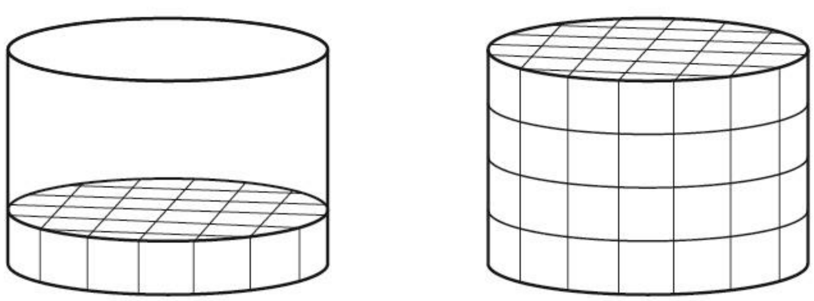
► Eligible Content

C-G.3.1.1 Apply formulas for the volumes of cones, cylinders, and spheres to solve real-world and mathematical problems. Formulas will be provided.

► Instructional Supports

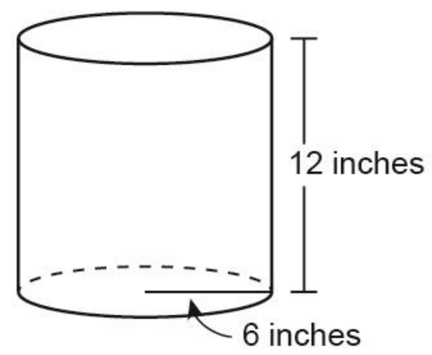
Cylinders

- Ask students to extend their knowledge of volumes of prisms to find the volumes of cylinders. For example, ask students to find the volume of a cylinder with a height of 4 inches and a radius of 3 inches. Remind students that the volume of a prism can be found by multiplying the area of its base by its height. Students identify the base of the cylinder as a circle with radius 3 and find the area to be 9π square inches. Help students understand that this means that the base of the cylinder can be covered with a layer of 1-inch cubes (or parts of cubes) having a volume of 9π cubic inches.



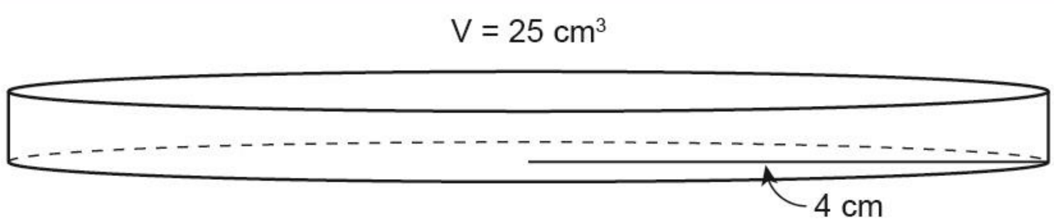
Because the cylinder has a height of 4 inches, the cylinder can be filled with 4 such layers, resulting in a total volume of 36π cubic inches.

- Ask students to use the formula $V = \pi r^2 h$ to find the volume of a cylinder. For example, ask students to find the volume of a cylinder with a height of 12 inches and a radius of 6 inches.



The formula can be used to find the volume, V, in cubic units, of a cylinder with a radius of r units and a height of h units. Ask students to substitute the given values into the formula, resulting in the equation $V = \pi \cdot 6^2 \cdot 12$. Students should calculate that the cylinder has a volume of 432π cubic inches.

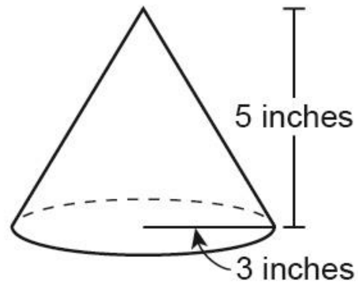
- Ask students to find a missing dimension of a cylinder given the volume and the other dimensions of the cylinder. For example, ask students to find the height of a cylinder with a volume of 25 cubic centimeters and a radius of 4 centimeters.



Ask students to insert the known values into the volume equation to find the unknown height. Students write the equation $25 = \pi \cdot 4^2 \cdot h$ and solve for h to determine that the cylinder has a height of $\frac{25}{16\pi} \approx 0.497$ centimeters.

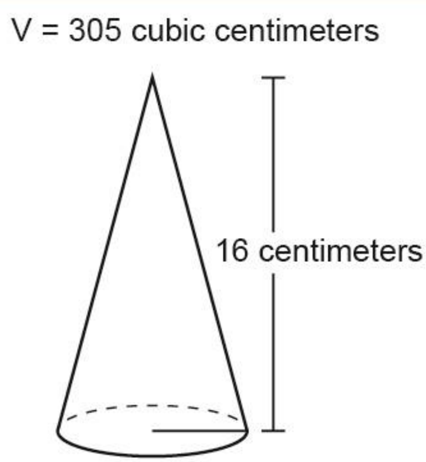
Cones

- Ask students to find the volume of a cone. For example, ask students to find the volume of a cone with a height of 5 inches and a radius of 3 inches.



The formula $V = \frac{1}{3}(\pi r^2 h)$ can be used to find the volume, V, in cubic units, of a cone with a radius of r units and a height of h units. Therefore, $V = \frac{1}{3} \cdot \pi \cdot 3^2 \cdot 5$ for the given cone. Students should calculate that the cone has a volume of 15π cubic inches.

- Ask students to find a missing dimension of a cone given the volume and the other dimensions of the cone. For example, ask students to find the diameter of a cone with a volume of 305 cubic centimeters and a height of 16 centimeters.

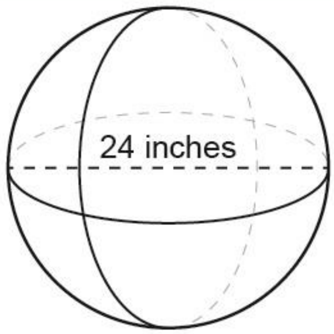


Students insert the known values into the volume equation to find the radius, resulting in the equation $305 = \frac{1}{3} \cdot \pi \cdot r^2 \cdot 16$. The equation can be solved for the radius by first simplifying to

$\frac{915}{16\pi} = r^2$. Students should calculate that the cone has a radius of $\frac{\sqrt{915}}{4\sqrt{\pi}} \approx 4.27$ centimeters. Therefore, the cone has a diameter of $\frac{\sqrt{915}}{2\sqrt{\pi}} \approx 8.53$ centimeters.

Spheres

- Ask students to find the volume of a sphere given the dimensions of the sphere. For example, ask students find the volume of a sphere with a diameter of 24 inches.

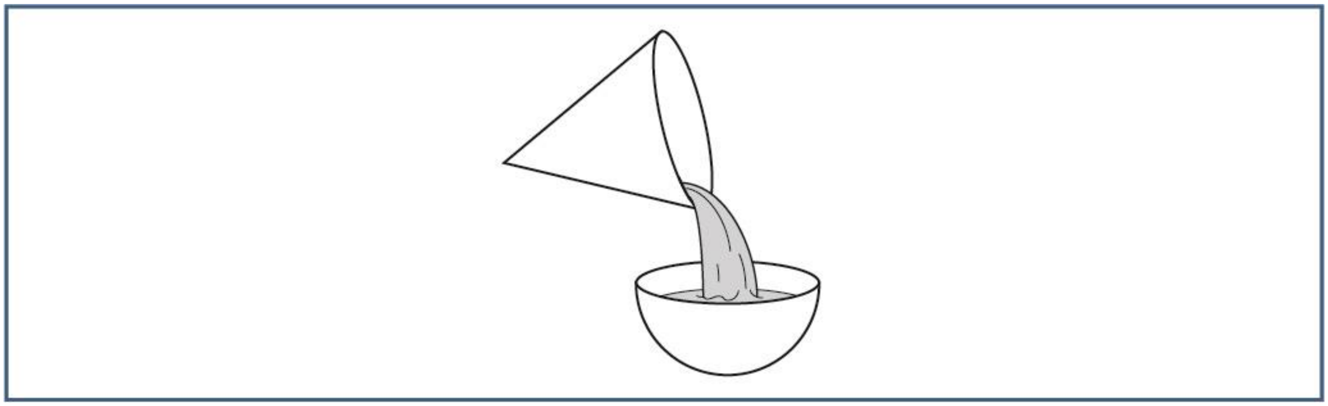


Students use the formula $V = \frac{4}{3}\pi r^3$ to find the volume, V, in cubic units, of a sphere with a radius of r units. Students determine that the radius of the sphere is 12 inches, which is used to create the equation $V = \frac{4}{3} \cdot \pi \cdot 12^3$. The cone has a volume of $2{,}304\pi$ cubic inches.

- Ask students to find the radius of a sphere given the volume of the sphere. For example, ask students to find the radius of a sphere with a volume of 147.456 cubic centimeters. Ask students to insert the given volume into the volume equation to find the radius. The resulting equation is $147.456 = \frac{4}{3} \cdot \pi \cdot r^3$. Students multiply both sides of the equation by $\frac{3}{4}$ and divide both sides of the equation by π, creating the equation $\frac{110.592}{\pi} = r^3$. Students calculate that the sphere has a radius of $\frac{4.8}{\sqrt[3]{\pi}} \approx 3.277$ centimeters.

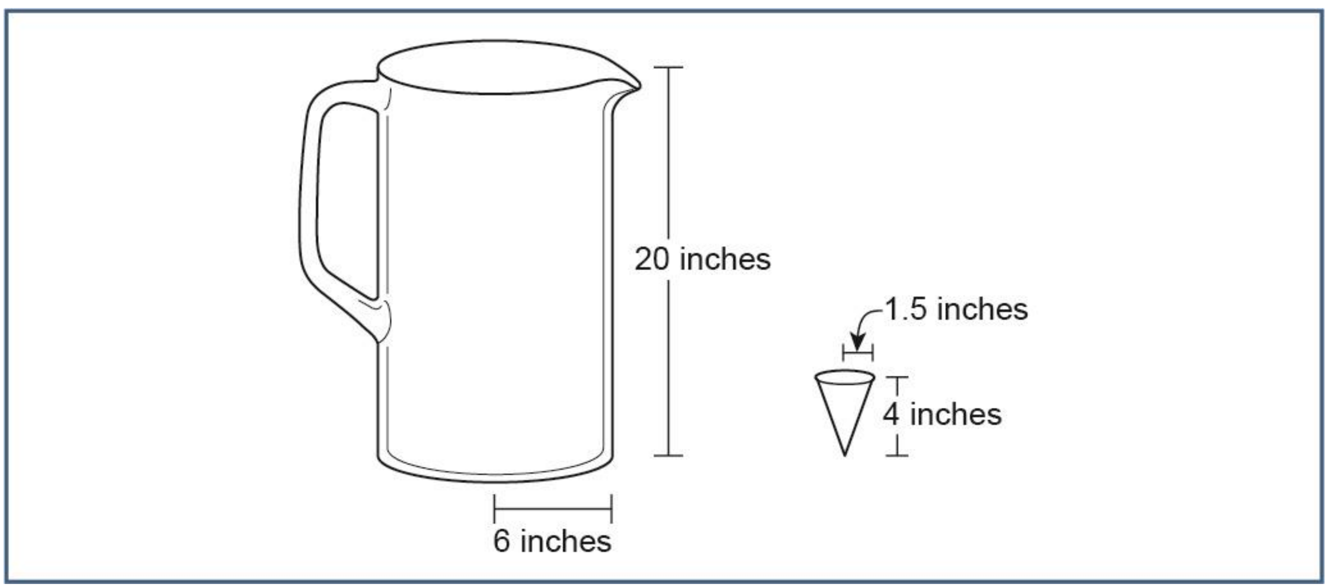
Cylinders, cones, and spheres

- Ask students to investigate or verify connections between the different volume formulas. For example, give students conical and hemispherical containers that have the same radius, with the height of the cone being twice the radius. Ask students to compare the volumes of the containers by filling them with water, sand, or rice. Students can observe that both containers have the same volume because the cone can be filled and have its contents poured into the hemisphere, filling it perfectly.



The volume of the hemisphere is half that of a sphere, $\frac{2}{3}\pi r^3$. The volume of the cone is $\frac{1}{3}\pi r^2(2r) = \frac{2}{3}\pi r^3$. The matching formulas confirm what is seen in the experiment.

- Ask students to use the formulas for the volumes of cones, cylinders, and spheres to solve real-world problems. For example, ask students to find the number of cone-shaped paper cups with a radius of 1.5 inches and a height of 4 inches that could be filled from a cylindrical water jug with a radius of 6 inches and a height of 20 inches that is 70% full of water.



Students find the amount of water in the jug by substituting the appropriate numbers into the volume formula for a cylinder and multiplying the total volume by 0.7. There are

$\pi \cdot 6^2 \cdot 20 \cdot 0.7 = 504\pi$ cubic inches of water in the jug. Next, students use the volume formula for a cone to find that each cup can hold $\frac{1}{3} \cdot \pi \cdot 1.5^2 \cdot 4 = 3\pi$ cubic inches of water. Finally, students divide the amount of water in the jug by the amount of water each cup can hold. There is enough water in the cylindrical jug to fill $504\pi \div 3\pi = 168$ conical cups of water.

► Identifying and Addressing Common Misconceptions

- Some students may mistake which volume formula goes with which figure. Ask students to explain how to find the volume of each figure and what measurements are required for each figure. Ask students to explain why the formula for the volume of a prism works and help them recognize how the formula for the volume of a cylinder is related to the formula for the volume of a prism. For the other two shapes, ask students to identify the measurements needed to calculate their volumes. Students should recognize that a sphere has only a radius and no height, while a cone has both. Help them use this fact to match the correct shape with the correct formula.
- Some students may use the diameter instead of the radius. Ask students to find the volume of a cylinder with a diameter of 12 units and a height of 10 units. Some students may give a volume of $1{,}440\pi$ cubic units. Ask students to explain the meaning of each variable in the volume formula. Then ask students to identify the meaning of the 12 and the 10. Help students be aware of the mismatch between the information required and the information given.

► Enrichment

- Ask students to develop a formula for the surface area of a cylinder.
- Ask students to explore the relationship between triangles and cones. Triangles take the bottom edge of a rectangle and taper it to a single point, while cones take the bottom face of a cylinder and taper it to a single point. Triangles are two-dimensional figures and have one-half the area of the corresponding rectangle. Cones are three-dimensional figures and have one-third the volume of the corresponding cylinder.
- Ask students to find volumes of composite figures involving cones, cylinders, and spheres.

► Key Terms

cone, constant, cylinder, diameter, pi, radius, sphere, volume

D-S.1.1.1

▶ Reporting Category, Assessment Anchor, Anchor Descriptor

Statistics and Probability

Investigate patterns of association in bivariate data.

Analyze and interpret bivariate data displayed in multiple representations.

▶ Eligible Content

D-S.1.1.1 Construct and interpret scatter plots for bivariate measurement data to investigate patterns of association between two quantities. Describe patterns such as clustering, outliers, positive or negative correlation, linear association, and nonlinear association.

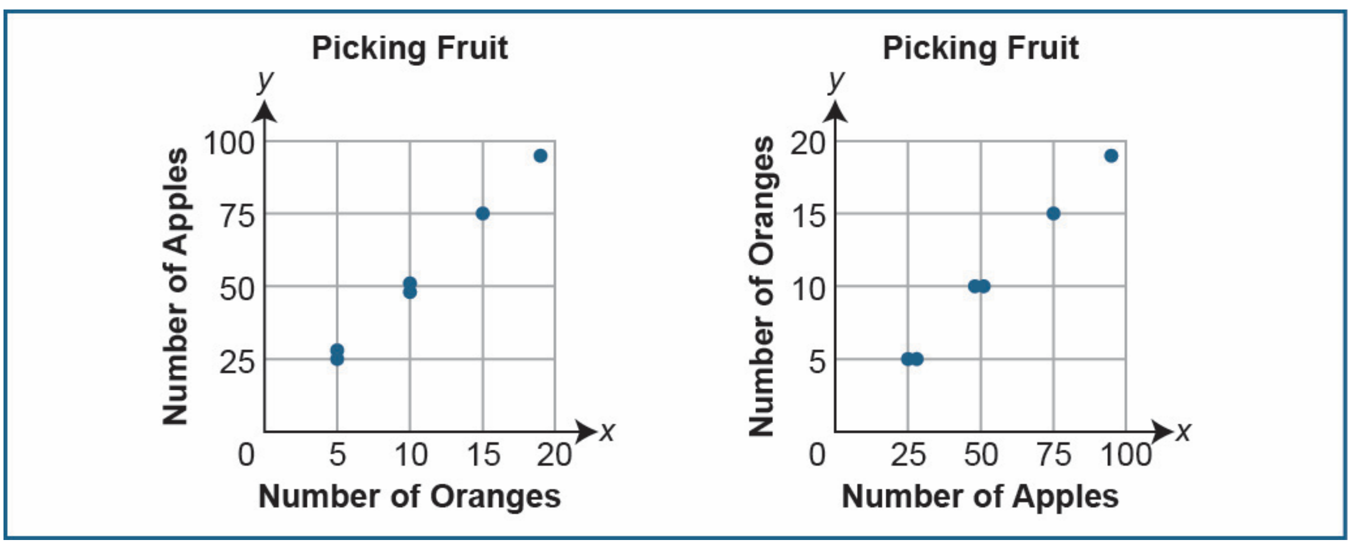
▶ Instructional Supports

Constructing and interpreting scatter plots

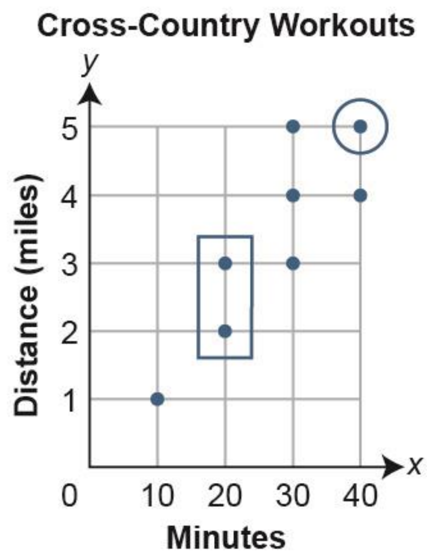
- Ask students to create a scatter plot given a set of bivariate data. For example, the table shown indicates the numbers of minutes Colin spent picking oranges and apples on different occasions.

Number of Oranges	5	10	19	5	10	15
Number of Apples	25	51	95	28	48	75

Two scatter plots are possible depending on which variable is used as the *x*-variable.



- Ask students to interpret the meaning of particular data points in a scatter plot. For example, the scatter plot shown represents 8 workouts completed by a cross-country runner, with three points indicated.



A possible student explanation could be that the circled point indicates the athlete ran 5 miles in 40 minutes. The two points inside the rectangle indicate the athlete ran 2 miles in 20 minutes during one workout and 3 miles in 20 minutes during another workout.

Associations between variables

- Ask students to distinguish between the dependent variable (y) and the independent variable (x) when presented with two related variables. For example, the owner of a boat rental shop compares the outside temperature and the number of boats rented. In this case, the temperature represents the independent variable. The temperature may affect the number of boats that are rented, but the number of boats that are rented does not have any effect on the temperature. Stated differently, the temperature is independent of the number of boats that are rented, but the number of boats that are rented may depend on the temperature.

- Ask students to describe an association between two related variables. For example, ask students to describe the association between the number of minutes students spent preparing for a science exam and the scores on the exam. Students describe the variables as having a positive association. The exam scores (y-variable) increase as the number of minutes spent preparing for the exam (x-variable) increases.

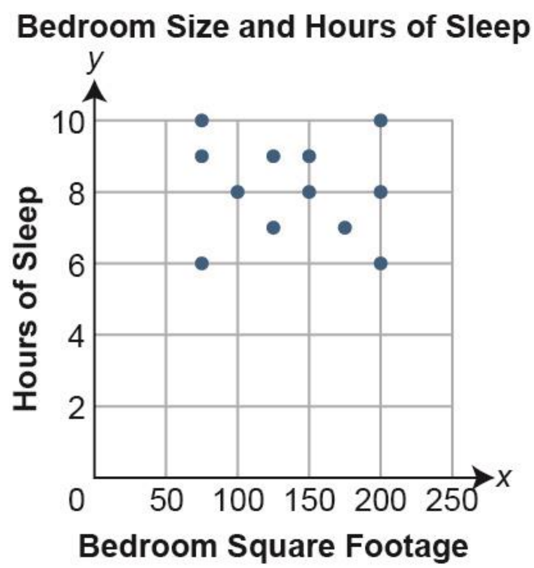
- Ask students to describe an association between two variables and to create a corresponding context. For example, ask students to find an example of two variables that have a negative linear association. Two possible responses are shown.
    - *the number of math problems remaining on an assignment vs. the number of minutes spent working on the assignment*
    - *the number of raspberries remaining on a raspberry bush vs. the number of minutes since a farmer started picking raspberries from the raspberry bush*

How scatter plots show associations between variables

- Ask students to explain how changes to the x-variable (independent variable) affect the y-variable (dependent variable) in scatter plots. For example, the following scatter plot shows the sale prices and square footages for 10 homes.

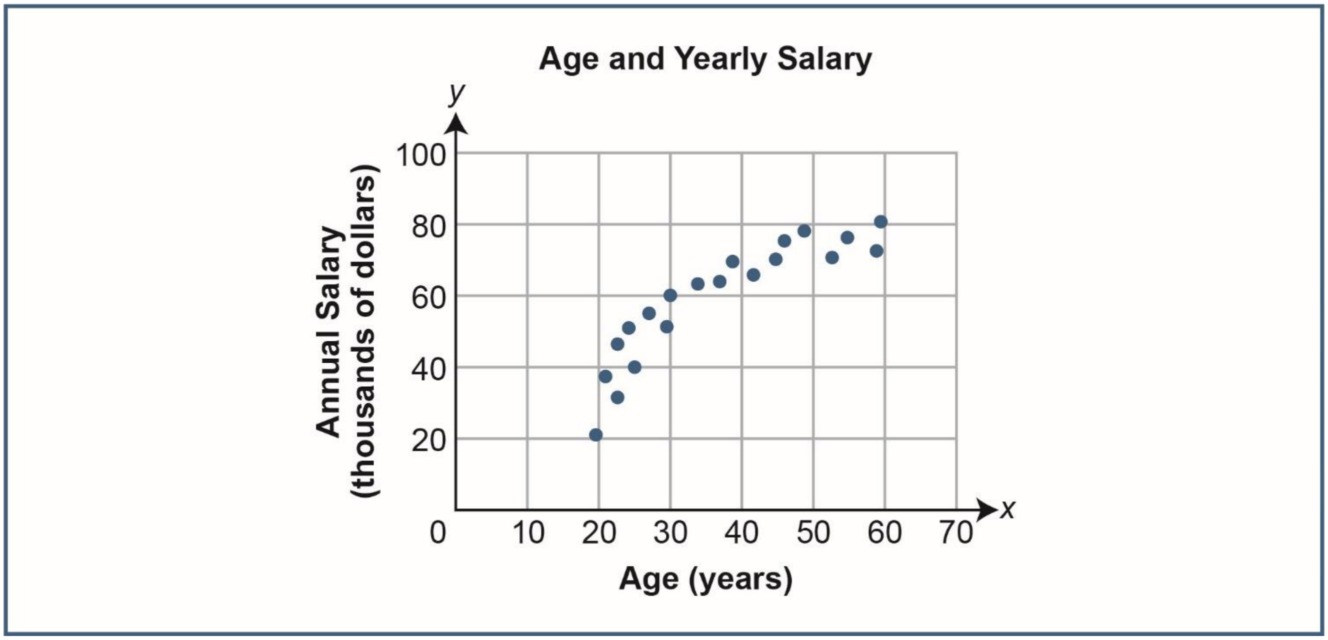
Students should note that the scatter plot shows a positive association between the sale price of a home and the number of square feet in the home. Based on the data shown in the scatter plot, the sale price of a home generally increases as the number of square feet in the home increases. This association could also be classified as a linear association, as the data points appear to cluster closely along a line that could be drawn through the data.

- Ask students to explain features of a scatter plot that show that there is no correlation between the variables. For example, give students the following scatter plot and ask them to articulate why there is no correlation between bedroom square footage and number of hours slept.



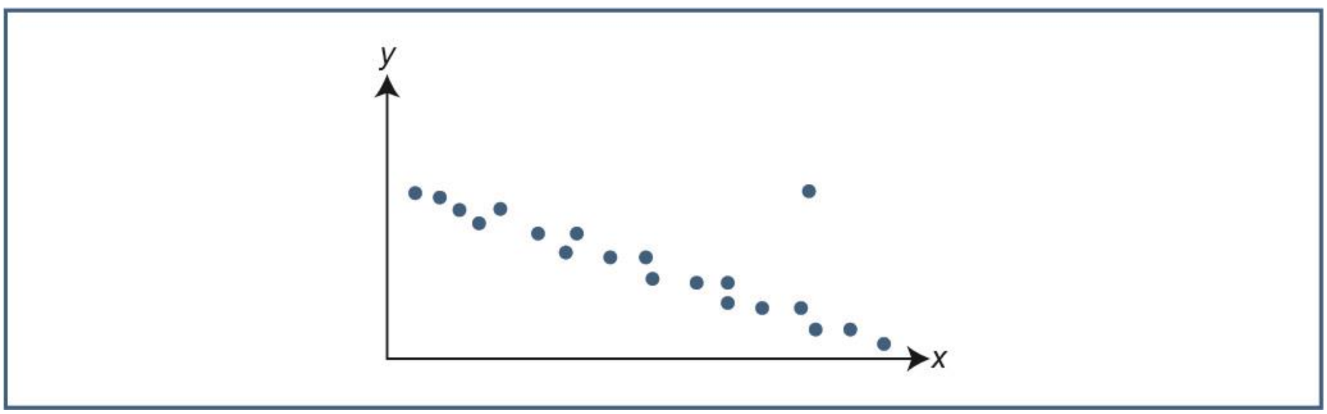
Students explain that there is no correlation between the two variables because there is no clear upward or downward trend in the plot. The average number of hours of sleep for each bedroom does not change by much as the square footage changes.

- Ask students to identify associations (i.e., linear or nonlinear and positive or negative) as well as outliers when provided with a scatter plot. For example, give students the scatter plot shown.



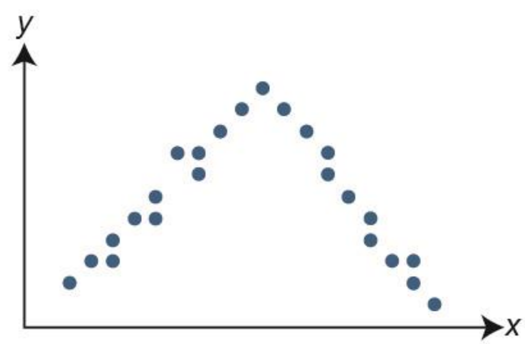
There is a positive association because, in general, the annual salary increases as the age of workers increases. The association is nonlinear because the graph shows a discernable curve. The scatter plot does not include any outliers.

- Ask students to create a scatter plot given a description of the association between two variables. For example, ask students to draw a scatter plot that indicates a negative linear association and has a single outlier. A possible student response is shown.



▶ Identifying and Addressing Common Misconceptions

- Some students may think that a scatter plot shows a linear association only if all the points lie exactly on a single line. Give students a scatter plot that shows a linear trend but is not perfectly linear and ask them if the data show a linear association between the variables. Some students may respond that it does not. Help students understand that variables that have a linear association need only to show a linear trend and do not need to be perfectly linear.
- Some students may think that a scatter plot can be interpreted as having both a positive and negative association. Give students the following scatter plot and ask them if the scatter plot shows a positive association or a negative association.



Some students may think that it shows both associations. Help students understand that although portions of the scatter plot appear to have a positive association or a negative association, the data in their entirety have neither a positive association nor a negative association.

- Some students may think that a scatter plot made up of linear parts is linear. Ask students whether the following plot shows a linear association.

Some students may respond that it does show a linear association. Students should understand that although some points in the data set create a linear pattern, the entirety of the data does not follow a single straight line.

► Enrichment

- Ask students to use a graphing calculator or spreadsheet to create a scatter plot.
- Ask students to explore how switching the *x*- and *y*-variables affects the associations shown in a scatter plot. Students should conclude that swapping the variables does not affect whether the association shown is linear, nonlinear, positive, or negative.
- Ask students to explore how transforming the *x*- and *y*-values affects the scatter plot. Students should notice that adding and subtracting the same number from all *x*- or *y*-values only shifts the locations of the points, but the shape of the plot stays the same. Students should also notice that although multiplying and dividing all the *x*- or *y*-values stretches or compresses the plot, the plot can still be made to look the same as the original by adjusting the scale on the axes.

► Key Terms

bivariate, clustering, data, linear association, negative association, nonlinear association, outlier, positive association, scatter plot, univariate

D-S.1.1.2

▶ Reporting Category, Assessment Anchor, Anchor Descriptor

Statistics and Probability

Investigate patterns of association in bivariate data.

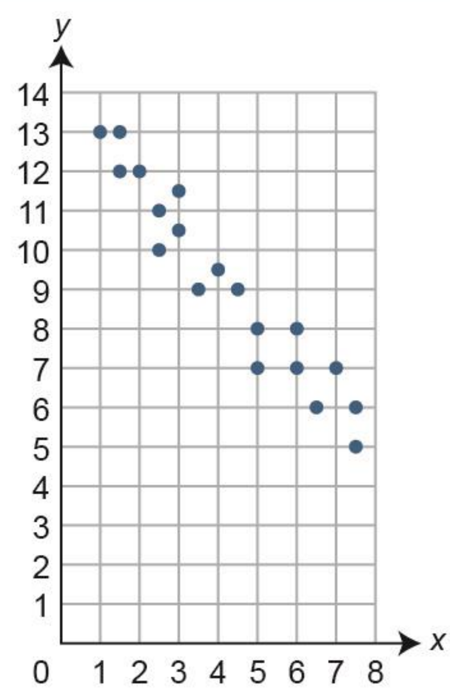
Analyze and interpret bivariate data displayed in multiple representations.

▶ Eligible Content

D-S.1.1.2 For scatter plots that suggest a linear association, identify a line of best fit by judging the closeness of the data points to the line.

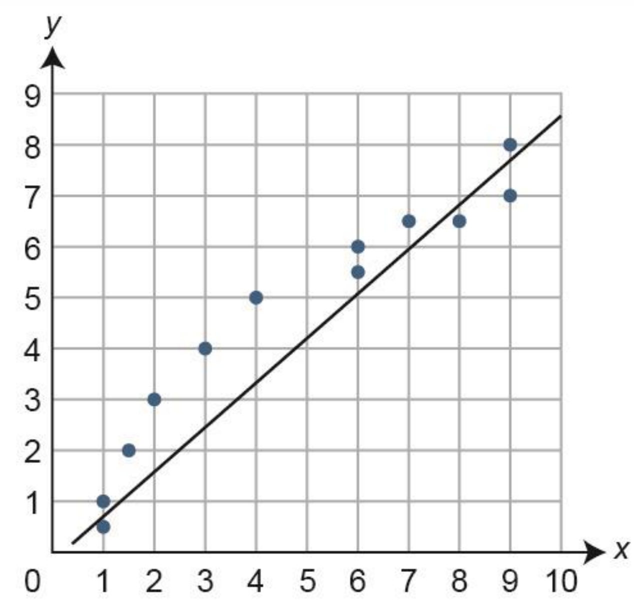
▶ Instructional Supports

- Ask students to explain whether there is a linear association between two variables on a scatter plot. For example, give students the scatter plot shown.



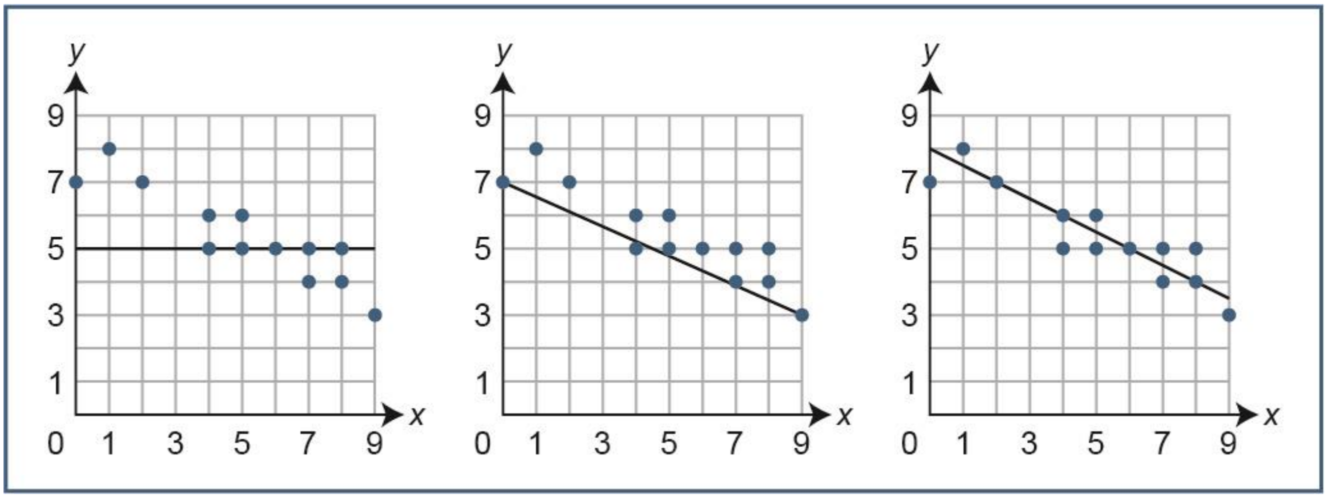
A straight line can be drawn that passes through or close to most of the data points. Therefore, there does appear to be a linear association. More specifically, because the plot shows a distinct downward trend, the association appears to be a negative linear association.

- Ask students to assess how well a given line fits data shown in a scatter plot. For example, give students the following scatter plot and line.



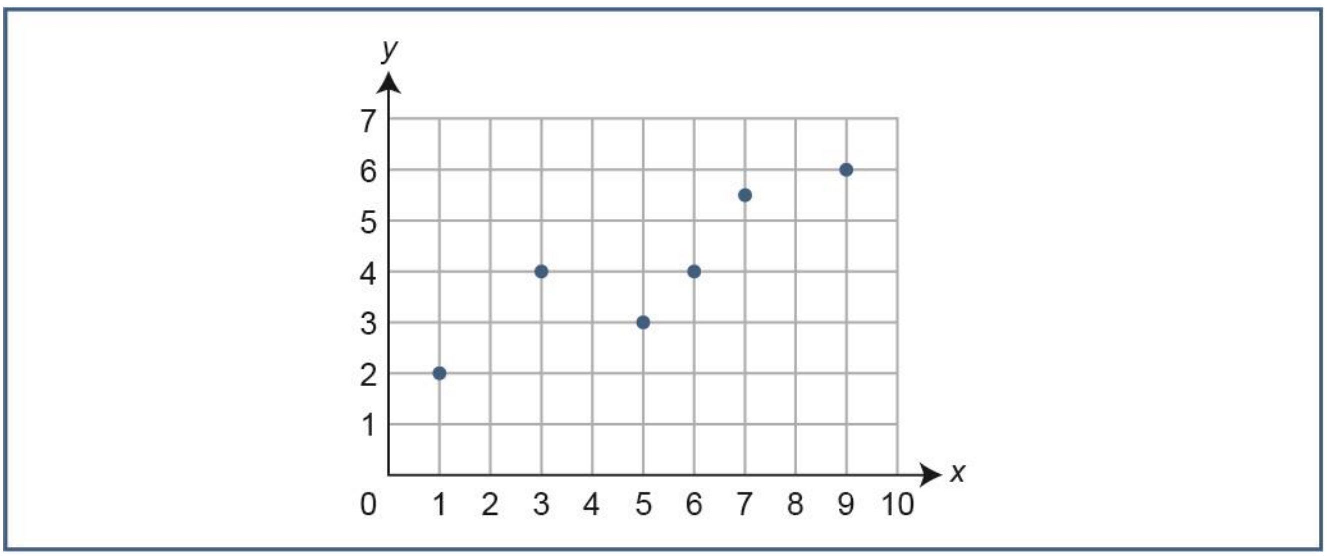
Although the straight line follows the general upward trend of the points, most of the points are above the line. The points that are below the line are located at either end of the line. Therefore, a curved graph might be more appropriate than a straight line.

- Ask students to examine a scatter plot shown with several different lines and evaluate and explain which line provides the best fit. For example, give students the following scatter plot shown with three different lines.

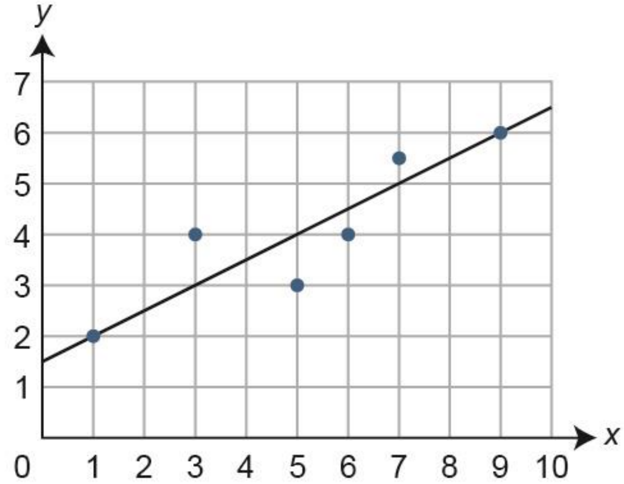


Although the line in the first graph passes exactly through 5 of the points, it does not follow the general downward trend of the points. The line in the second graph does follow the general downward trend of the points, but the majority of the points are above the line. The third line is the best of the three because it captures the general downward trend of the points and has about the same number of points above and below the line.

- Ask students to draw a line of best fit on a scatter plot and then use coordinates to estimate the slope and y-intercept of the line. For example, give students the following scatter plot.

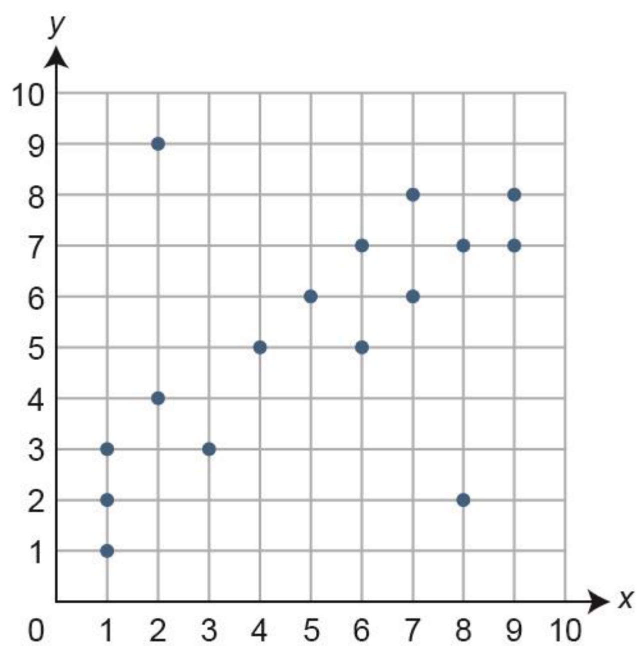


A possible line of best fit is shown.



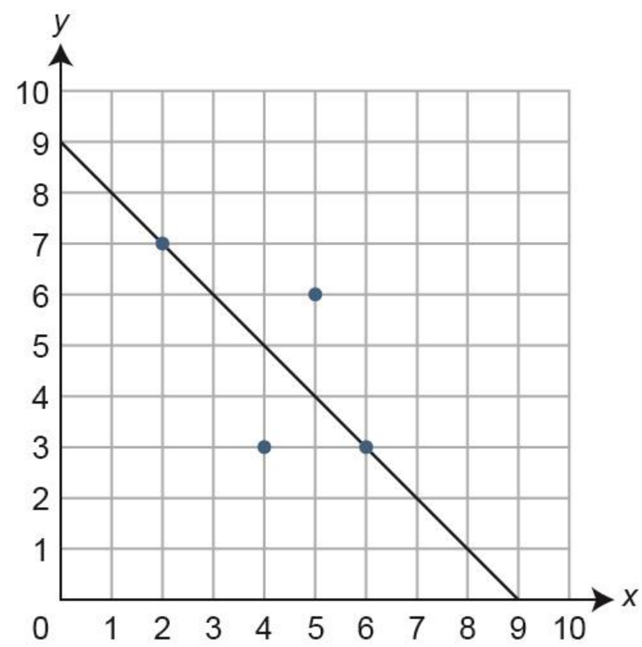
The y-intercept of the line appears to be about halfway between 1 and 2. Students may use the fact that the line appears to pass through (1, 2) and (9, 6) to estimate the slope of the line to be about $\frac{1}{2}$. Therefore, the line of best fit has a y-intercept of $1\frac{1}{2}$ and a slope of $\frac{1}{2}$.

- Ask students to identify outliers on a scatter plot. For example, give students the following scatter plot.



Students identify the points (2, 9) and (8, 2) as outliers.

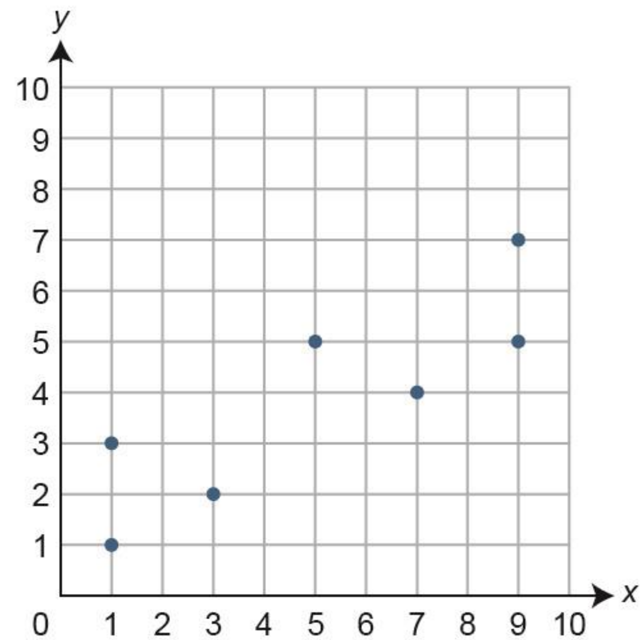
- Ask students to investigate the effect of outliers on the line of best fit. For example, give students the following scatter plot and line of best fit.



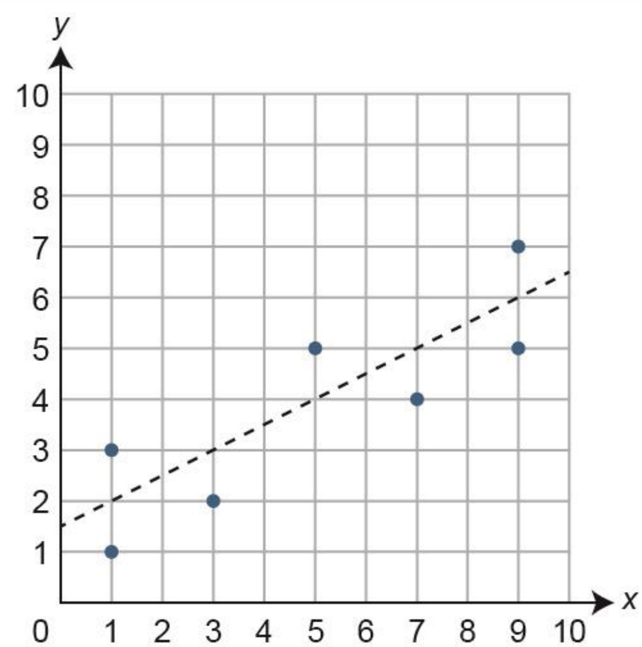
Ask half the students to plot the additional point (3, 7) and the other half to plot the point (9, 7). Then instruct all students to adjust the line of best fit accordingly. Students should recognize that adding the point (3, 7) changes the line of best fit much less then adding the point (9, 7).

► Identifying and Addressing Common Misconceptions

- Some students may think that a line of best fit must pass through one or more data points. Give students the following scatter plot and ask them to draw a line of best fit.

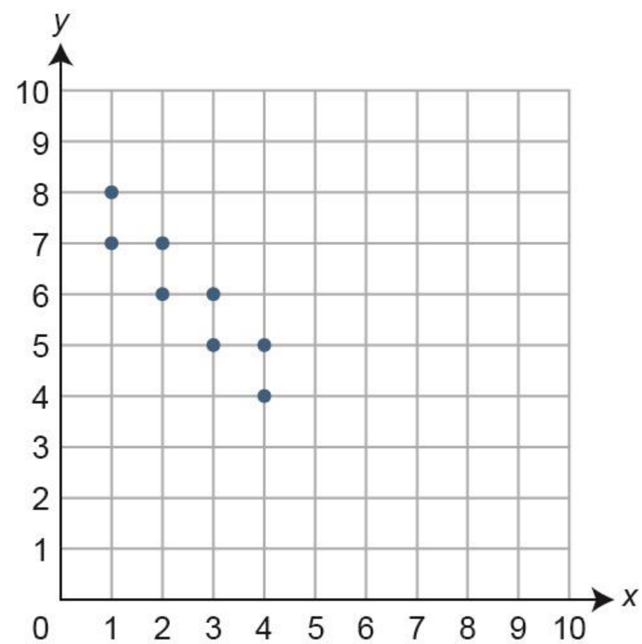


Some students may draw a line that passes through some of the points. Show students the following line and ask them to compare the line with their line.



Help students compare the overall distance of each line with the entire set of points and explain that a line that is fairly close to all points is generally considered to be better than a line that is very close to some points and very far from some points.

- Some students may assume that a line of best fit must pass through grid line intersections. Give students the following scatter plot and ask them to draw a line of best fit.

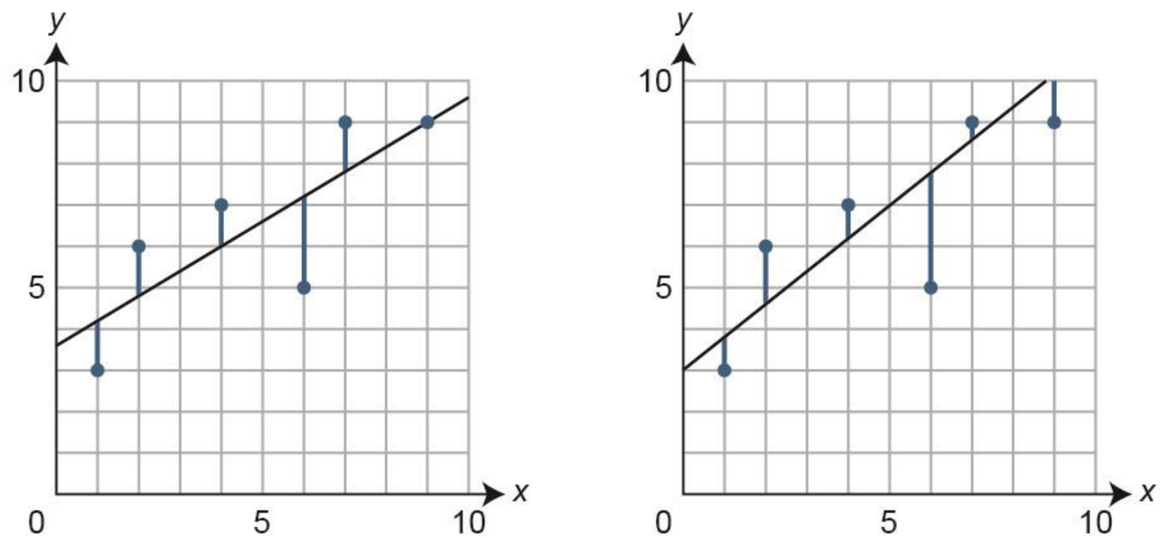


Some students may draw a line that passes through corners on the grid. Give students the same plot but without grid lines and ask them to draw a line of best fit. Ask them to compare the plots to see if the line has changed. If possible, show the difference between the plots by using tracing paper to overlay. Emphasize that because adding or removing grid lines does not change the points in the plot, the grid lines should also have no effect on which line best fits the data.

► Enrichment

- Provide students with a straight line on a grid and ask them to create multiple different sets of unique data points that could have the line as a line of best fit.

- Give students a scatter plot and two lines and ask them to compare the fit of the lines by measuring the vertical distance between each data point and the line. For both lines, ask students to add up these vertical distances.



1.2 + 1.2 + 1 + 2.2 + 1.2 + 0 = 6.8   0.8 + 1.4 + 0.8 + 2.8 + 0.4 + 1.2 = 7.4

Compare the two sums and discuss that the first line with a sum of 6.8 is a better fit than the second line with a sum of 7.4. This is very similar to the actual process of determining the line of best fit. In the actual process, the vertical distances are squared before being added.

- Give students a scatter plot that does not show a linear association and ask them to estimate what curve best fits the points. In addition, illustrate the concept of overfitting a scatter plot by using a graphing utility to show a curve that passes through every single point in the scatter plot.

▶ Key Terms

association, bivariate, clustering, data, line of best fit, outlier, quantitative, scatter plot

D-S.1.1.3

► Reporting Category, Assessment Anchor, Anchor Descriptor

Statistics and Probability

Investigate patterns of association in bivariate data.

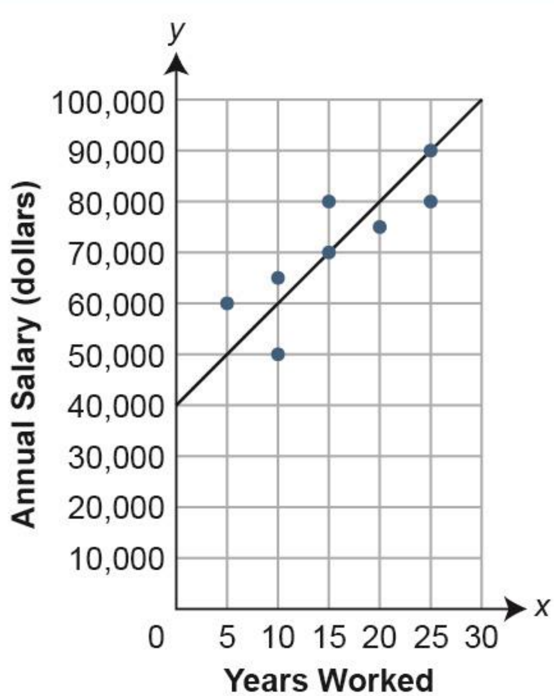
Analyze and interpret bivariate data displayed in multiple representations.

► Eligible Content

D-S.1.1.3 Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept.

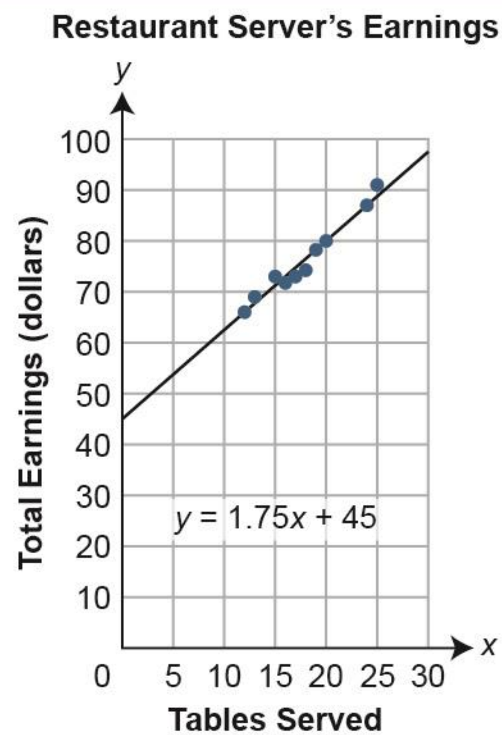
► Instructional Supports

- Give students a linear model and ask them to identify and interpret the slope and *y*-intercept. For example, a manager creates a scatter plot and draws an estimated line of best fit to represent the numbers of years worked by employees along with the corresponding annual salaries.



In this case, the line of best fit crosses the y-axis at 40,000. The slope can be estimated using the points (15, 70,000) and (25, 90,000) to be about 2,000. The equation of the line of best fit is therefore $y = 2{,}000x + 40{,}000$. The y-intercept of 40,000 indicates that a new employee is likely to earn about $40,000 per year. The slope of 2,000 suggests that an employee is likely to earn about 2,000 additional dollars for each year worked.

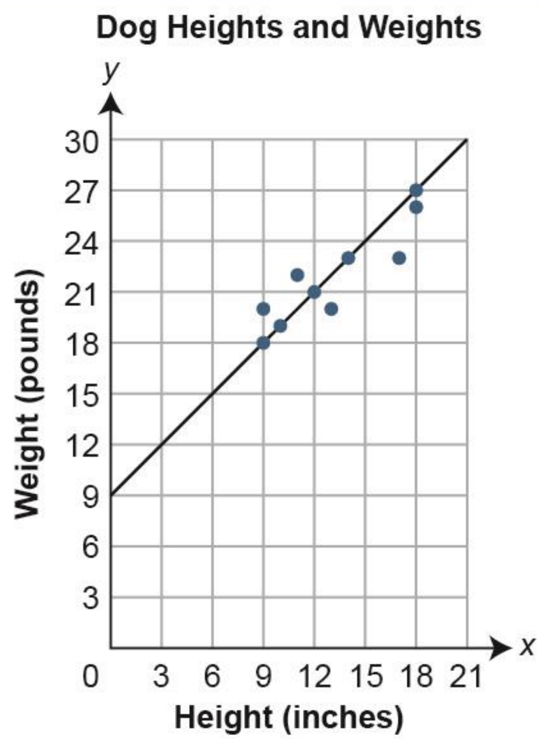
- Provide students with a linear model and ask them to use it to make predictions. For example, give students the following linear model showing a restaurant server's total earnings during each of the last ten shifts based on the number of tables served. Ask them to predict the amount of money the server would earn from serving 30 tables.



Students predict the amount of money earned from serving 30 tables by substituting $x = 30$ (since the number of tables is on the x-axis) into the equation of the line. Students then calculate $y = 1.75(30) + 45 = 52.5 + 45 = 97.5$. Therefore, students predict that the server would earn about $97.50 when serving 30 tables during a shift.

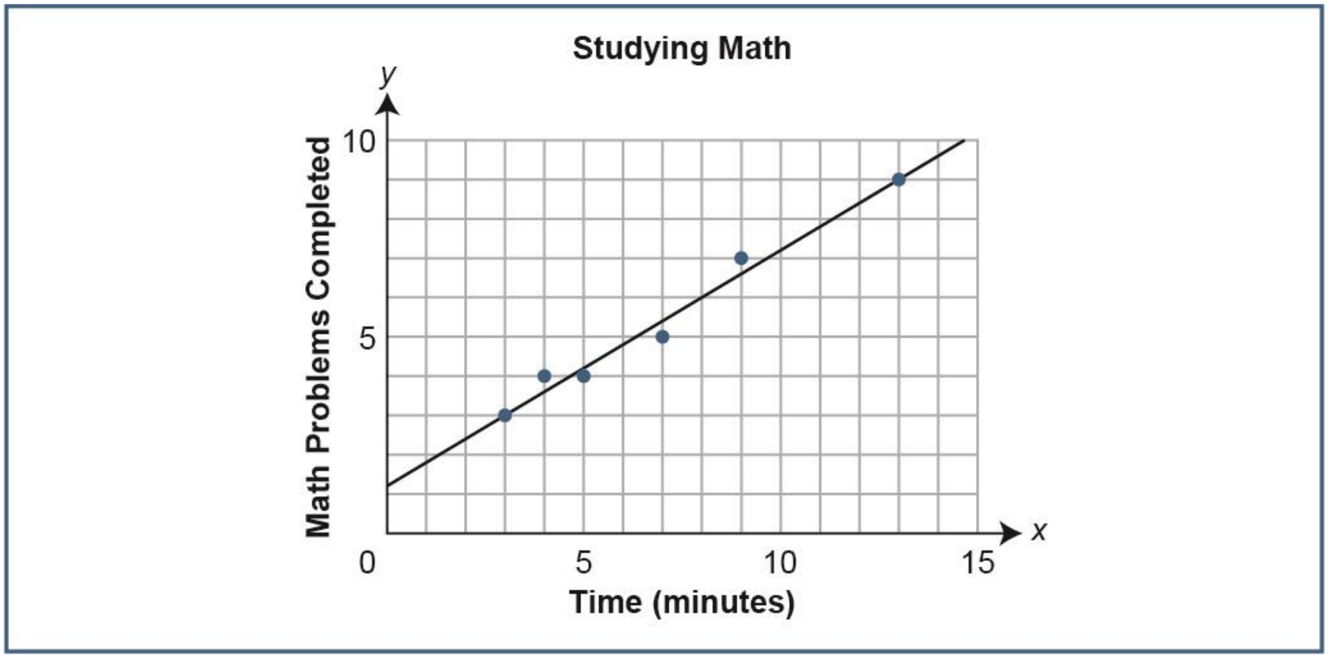
► Identifying and Addressing Common Misconceptions

- Some students may think that a line of best fit for bivariate data always results in predictions that are exact. Provide students with the following linear model and ask them to estimate how likely it is for a 15-inch-tall dog to weigh 25 pounds.



Some students may think this is not possible because a 15-inch-tall dog should weigh 24 pounds based on the line. Ask students to explain the meaning of one of the dots that is not on the line. Remind students that each dot represents the height and weight of a dog, several of which do not match the equation perfectly.

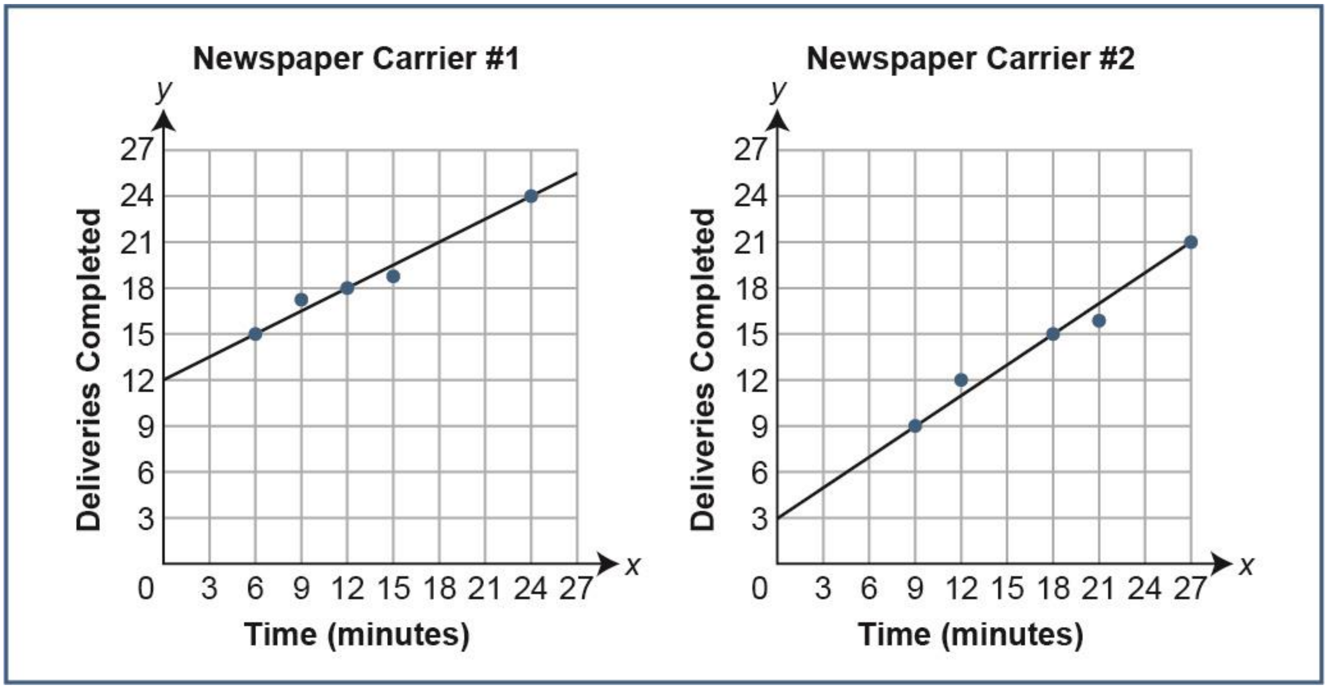
- Some students may think that predictions from lines of best fit can be made only for values that are located within the bounds of the observed data. Give students the following linear model and ask them whether the model can be used to predict the number of math problems completed in an 11-minute study session and in a 16-minute study session.



Some students may think that the model can be used to make predictions about the 11-minute study session but not the 16-minute study session. Remind students that the linear model can be used to make predictions within the observed data and, within reasonable boundaries, outside of the observed data.

► Enrichment

- Provide students with two different linear models and ask them to make comparative predictions. For example, the following linear models show how long it takes two different newspaper carriers to make their deliveries.



Ask students to use the two lines of best fit to predict the time span over which the two carriers deliver the same number of newspapers.

- Provide students with a scatter plot and ask them to sketch a line of best fit and write an equation for the line.
- Provide students with a scatter plot and a line of best fit and ask them to interpret, for each point in the scatter plot, whether the line of best fit overestimates or underestimates the *y*-value.

► Key Terms

bivariate, clustering, data, decreasing association, increasing association, scatter plot

D-S.1.2.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

Statistics and Probability

Investigate patterns of association in bivariate data.

Understand that patterns of association can be seen in bivariate categorical data by displaying frequencies and relative frequencies in a two-way table.

► Eligible Content

D-S.1.2.1 Construct and interpret a two-way table summarizing data on two categorical variables collected from the same subjects. Use relative frequencies calculated for rows or columns to describe possible associations between the two variables.

► Instructional Supports

- Ask students to create a survey with questions that have categorical responses, give the survey to the class, and then create a two-way table that shows the results. For example, the results of a survey of a class are shown.
    - In the class, 9 students have short hair and have a video game console.
    - In the class, 4 students have short hair and do not have a video game console.
    - In the class, 5 students have long hair and have a video game console.
    - In the class, 7 students have long hair and do not have a video game console.

  A corresponding two-way table is shown.

	Short Hair	Long Hair	Total
Console	9	5	14
No Console	4	7	11
Total	13	12	25

- Ask students to interpret a two-way table and answer questions about it. For example, the two-way table shows music group participation for all 200 eighth-grade students at a school.

	Wears glasses	Does not wear glasses	Total
Band	20%	20%	40%
Orchestra	15%	17.5%	32.5%
Choir	12.5%	15%	27.5%
Total	47.5%	52.5%	100%

Some possible questions to ask students, along with possible responses, are shown.

- What percentage of the eighth-grade students who were surveyed are in band?

  Students determine from the table that 40% of the eighth-grade students are in band.

- How many eighth-grade students who wear glasses are in the choir?

  Students determine there are 25 eighth-grade students who wear glasses in the choir because $0.125 \cdot 200 = 25$.

- What percentage of the eighth-grade students who were surveyed are students who do not wear glasses and who are **not** in the orchestra?

  Students determine that the students who don't wear glasses and are not in the orchestra are the students who do not wear glasses and are in either band or choir. Therefore, 35% of the eighth-grade students who do not wear glasses and who are not in the orchestra because $20 + 15 = 35$.

- Give students a partially completed two-way table and ask them to complete it. For example, the two-way table below shows information about whether customers at a coffee shop ordered cream or sugar in their coffee.

	Sugar	**No Sugar**	**Total**
Cream	24		
No Cream	27		50
Total		48	

One possible way to begin completing the table is to recognize that the total number of customers who ordered sugar is 51 because the sum of customers who ordered coffee with sugar and cream and customers who ordered coffee with sugar and without cream is equal to 24 + 27. As a result, the total number of customers is 99 because there are 51 customers who ordered coffee with sugar and 48 customers who ordered coffee without sugar. The table now appears as shown.

	Sugar	**No Sugar**	**Total**
Cream	24		
No Cream	27		50
Total	**51**	48	**99**

There are 49 customers who ordered coffee with cream because the total number of customers is 99, and 50 of those customers ordered coffee without cream. The difference of 99 and 50 is 49. The table now appears as shown.

	Sugar	**No Sugar**	**Total**
Cream	24		**49**
No Cream	27		50
Total	51	48	99

There are 25 customers who ordered coffee with cream but no sugar because 49 – 24 = 25. Similarly, there are 23 customers who ordered coffee without cream or sugar because 50 – 27 = 23. The completed two-way table is shown.

	Sugar	**No Sugar**	**Total**
Cream	24	**25**	49
No Cream	27	**23**	50
Total	51	48	99

- Ask students to explain the difference between a two-way table that uses frequencies and a two-way table that uses proportions. For example, provide students with the partially completed two-way table shown below and ask them to explain what the number 33 indicates depending on whether the table shows frequencies or proportions.

	Cat	**Dog**	**Total**
Single Color	33		
Multiple Colors			
Total			

Students explain that if the table shows frequencies, then 33 indicates there are 33 cats (i.e., data points) that are a single color. If the table shows proportions, then 33 indicates that 33% of the animals are cats of a single color.

- Give students a two-way table that uses frequencies and ask them to convert it to a table that uses proportions or vice versa. For example, the following table shows information about participants in a community musical.

	Adult	Child	Total
New	6	12	18
Experienced	6	16	22
Total	12	28	40

The table indicates that there were 40 total participants. Therefore, the values in the table can be converted to proportions by dividing each value by 40. The converted table appears as shown.

	Adult	Child	Total
New	15%	30%	45%
Experienced	15%	40%	55%
Total	30%	70%	100%

- Give students variables and ask them to explain why there is or is not an association between them. For example, give students the variables of dog color, dog weight, and dog height. It is unlikely that the color of a dog is associated with either the weight or height of the dog. However, there is likely to be an association between dog height and dog weight. In a two-way table, such an association could be shown by a larger number of dogs that fall into a tall height range with a high weight range as well as a larger number of dogs that fall into a short height range with a low weight range.
- Give students a two-way table and ask them to determine whether there is an association between the variables shown. For example, give students the following two-way table containing information about playing and watching basketball.

	Plays Basketball	Does Not Play Basketball	Total
Watches Basketball	36	15	51
Does Not Watch Basketball	4	45	49
Total	40	60	100

When looking at those who play basketball, there are many more individuals who watch basketball. When looking at those who do not play basketball, there are many more individuals who do not watch basketball. Similarly, when looking at those who watch basketball, there are many more individuals who play basketball. And when looking at those who do not watch basketball, there are many more who do not play basketball. Therefore, the table does seem to suggest an association between playing basketball and watching basketball.

► Identifying and Addressing Common Misconceptions

- Some students may think that specific variables must appear in either the rows or the columns of a two-way table. Give students the following two-way table about the birds that a hiker observes and ask students to rearrange the table so that the color variables appear in the rows and the flying/on the ground variables appear in the columns.

	Not brown	Brown	Total
Flying	21	9	30
On the ground	14	6	20
Total	35	15	50

Some students may not think that such an arrangement is possible. Point to each number and ask them to explain the meaning of each number in the table. Then help students create the following table.

	Flying	On the ground	Total
Not brown	21	14	35
Brown	9	6	15
Total	30	20	50

Point to each number in the new table and ask students to explain the meaning of each number. Help students recognize that both tables contain the exact same information.

- Some students may think that filling in a partially completed two-way table must be done in a particular order. Give students the following partially completed two-way table and ask them to identify the possible first cells that can be determined.

	Seventh Grade	Eighth Grade	Total
Hot Lunch	31	19	
Cold Lunch			
Total	59		111

Some students may identify only one possible first step. Help students complete the table in two different ways by filling in the cells in different orders. Students should recognize that the completed two-way table is the same regardless of the order in which the cells are filled in. The completed table is shown.

	Seventh Grade	Eighth Grade	Total
Hot Lunch	31	19	50
Cold Lunch	28	33	61
Total	59	52	111

- Some students may think that every two-way table must reveal some type of association. Give students the following two-way table containing information about grade level and sports preference and ask them to identify whether the table shows any association between the variables.

	Soccer	**Baseball**	**Total**
Seventh Grade	80	60	140
Eighth Grade	40	30	70
Total	120	90	210

Some students may think that the table shows an association between the variables. Cover up everything in the table except the seventh-grade row and ask them to determine the percentage of seventh-grade students who prefer soccer. Then cover everything except the eighth-grade row and ask them to determine the percentage of eighth-grade students who prefer soccer. Lastly, cover everything except the bottom row and ask them to determine the percentage of all students in the survey who prefer soccer. Help students recognize that knowing whether a student is in seventh grade or in eighth grade does not make the student any more or less likely to prefer soccer.

► Enrichment

- Ask students to create a two-way table in which the numerical entries would be identical whether the table showed either frequencies or proportions.
- Give students a scatter plot and ask them to create a two-way table by dividing the variables into categories. For example, give students a scatter plot showing the heights and rebounds per game of basketball players. Ask students to create height categories of "under 6 feet," "6 feet to 6 feet, 6 inches," "between 6 feet, 6 inches and 7 feet," and "7 feet or over." Then ask students to create rebound categories of "less than 5 per game," "5 to 10 per game," and "more than 10 per game." As an additional challenge, give students some scatter plots that show different types of associations and ask students to predict how those associations will show in the two-way table.

- Provide students with a partially completed two-way table in which all the known frequencies are expressed with variables and then ask them to determine the unknown frequencies in terms of the known frequencies. For example, provide students with the following two-way table.

	Short Tail	Long Tail	Total
Dog	a		d
Cat	b		
Total		c	

Students use logic, addition, and subtraction to complete the table as shown.

	Short Tail	Long Tail	Total
Dog	a	$d - a$	d
Cat	b	$c - (d - a)$	$[(a + b) + c] - d$
Total	$a + b$	c	$(a + b) + c$

► Key Terms

bivariate, categorical, cell, column, data, frequency, relative frequency, row, two-way table