A-N.1.1.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

The Number System

Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

Solve real-world and mathematical problems involving the four operations with rational numbers.

► Eligible Content

A-N.1.1.1 Apply properties of operations to add and subtract rational numbers, including real-world contexts.

► Instructional Supports

- Ask students to rewrite an expression using addition, subtraction, and grouping symbols in multiple ways. For example, give students the expression $6\frac{1}{5} - 3\frac{4}{5} - 1\frac{2}{5}$. Students could write the expressions $6\frac{1}{5} - (3\frac{4}{5} + 1\frac{2}{5})$, $(6\frac{1}{5} - 3\frac{4}{5}) + (-1\frac{2}{5})$, and $-3\frac{4}{5} - 1\frac{2}{5} + 6\frac{1}{5}$.
- Ask students to write and evaluate an expression that represents a given context involving addition or subtraction. For example, give students the context "Evan buys 45 pounds of dog food. He feeds his dog 2.5 pounds of dog food one week and 2.75 pounds the next week. How many pounds of dog food are left after the second week?" Students can write the expression $45 - 2.5 - 2.75$ and evaluate a result of 39.75.
- Ask students to use the strategy of creating pairs of additive inverses to help evaluate an expression. For example, ask students to evaluate the expression $6 + (-4)$. Students can represent the expression as $(4 + 2) - 4$ and then use the associative property and the commutative property to rewrite the expression as $2 + (4 + (-4))$, which is equivalent to $2 + 0$, or 2.

▶ Identifying and Addressing Common Misconceptions

- Students may not correctly distribute subtraction across parentheses. For example, ask students to rewrite the expression 7.4 – (3.8 + 3.7) without parentheses. Students who rewrite the expression as 7.4 – 3.8 + 3.7 may have this misconception. Ask students to evaluate the original expression and the rewritten expression and compare the results. An analogy of money may help students understand. Buying an item that costs $3.80 and another item that costs $3.70 means that both amounts are subtracted from the starting amount. The costs can either be totaled and subtracted all at once or be subtracted separately.
- Students may think that expressions that look different cannot be equivalent. For example, ask students to explain the relationship between the expressions 8.1 – (4.5 – 6.9) and 8.1 + 6.9 – 4.5 without performing any calculations. Students who do not identify the expressions as equivalent may have this misconception. Ask students to apply the order of operations to evaluate each expression.
- Students may think the commutative property applies to all operations. For example, ask students to rewrite the expression 8 – 5 using the commutative property. Students who write the expression as 5 – 8 may have this misconception. Give students the context "An object is 8 feet above water and falls 5 feet, and another object is 5 feet above the water and falls 8 feet." Ask students to determine whether each example would result in the object above or below the surface of the water. Then help them rewrite the expression as an addition expression before applying the commutative property.

▶ Enrichment

- Ask students to apply properties of operations to create equivalent expressions when given an expression involving an unknown number. For example, give students the expression $x + (-4.6)$. Students could write the expressions $x - 4.6$ or $-4.6 + x$.
- Ask students to explain or show why the commutative properties of addition and multiplication are still true when extended to rational numbers. Students could use a number line to show that when two numbers are added, the result is the same regardless of the order in which the numbers are added.

- Ask students to explain or show why the associative properties of addition and multiplication are still true when extended to rational numbers. Students could use a number line to show that the sum of three numbers is the same when either of the operations is performed first.

► Key Terms

absolute value, addition, additive inverse, negative, opposite, positive, properties of operations, rational, subtraction

A-N.1.1.3

► Reporting Category, Assessment Anchor, Anchor Descriptor

The Number System

Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

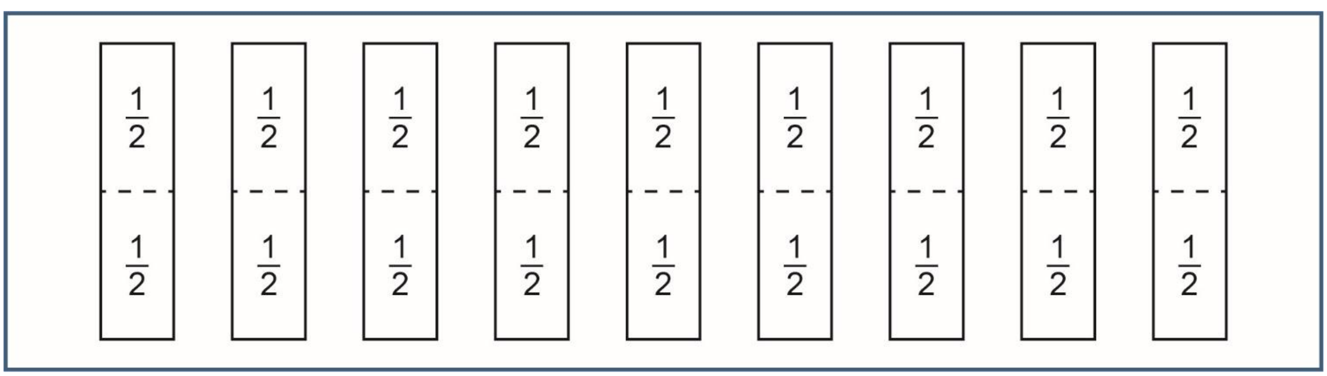
Solve real-world and mathematical problems involving the four operations with rational numbers.

► Eligible Content

A-N.1.1.3 Apply properties of operations to multiply and divide rational numbers, including real-world contexts; demonstrate that the decimal form of a rational number terminates or eventually repeats.

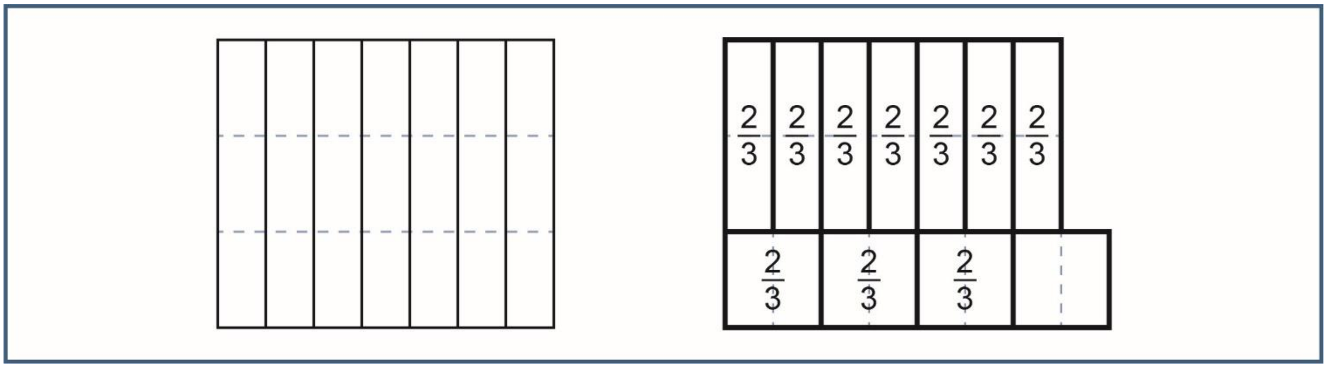
► Instructional Supports

- Ask students to demonstrate how to divide by a rational number using a visual model to show how to divide a group of whole objects into parts that are each smaller than a whole.
  - For example, have students create a visual model for $9 \div \frac{1}{2}$. Students could draw 9 bars and show each bar being cut into pieces that are each $\frac{1}{2}$ of a whole, as shown.



Then students can count the number of pieces of size $\frac{1}{2}$ to get an answer of 18.

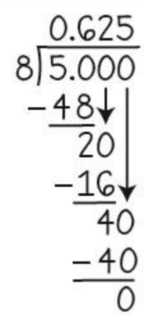
- Another example would be to have students create a visual model to show $7 \div \frac{2}{3}$. Students could create the visual model shown.



In the model, it takes 10 full sections that are each $\frac{2}{3}$ of the whole, with 1 remaining section that is $\frac{1}{2}$ of a $\frac{2}{3}$ section, for a total of $10\frac{1}{2}$ sections of size $\frac{2}{3}$. Therefore, the answer for $7 \div \frac{2}{3}$ is $10\frac{1}{2}$.

- Give students a multiplication or division equation and ask them to write the other three equations in the same fact family. For example, give students the equation $\frac{1}{4} \div -\frac{1}{5} = -\frac{5}{4}$. Students can create the equations $\frac{1}{4} \div -\frac{5}{4} = -\frac{1}{5}$, $-\frac{5}{4} \bullet -\frac{1}{5} = \frac{1}{4}$, and $-\frac{1}{5} \bullet -\frac{5}{4} = \frac{1}{4}$.
- Ask students to solve a division equation by rewriting it as a multiplication equation with an unknown factor. For example, the equation $-3 \div \frac{1}{6} = x$ can be rewritten as $x \bullet \frac{1}{6} = -3$. This can be thought of as answering the question "How many one-sixths are in –3?" Since $\frac{1}{6}$ taken 18 times is 3, then $\frac{1}{6}$ taken –18 times is –3. Therefore, $-3 \div \frac{1}{6} = -18$.
- Ask students to identify two strategies for evaluating a given expression involving multiplication of rational numbers. For example, give students the expression $12(\frac{1}{4} + -\frac{1}{2})$. Students can first calculate the sum within the parentheses to be $-\frac{1}{4}$ and then multiply that sum by 12 to find –3. Alternatively, students can distribute the 12 across the parentheses to get 3 + (–6), which also has a sum of –3.

- Ask students to solve a given division problem using two different methods. For example, give students the problem $\frac{4}{9} \div \frac{-2}{3}$. The problem can be solved by finding the quotient of the numerators and the quotient of the denominators: $\frac{4 \div -2}{9 \div 3} = \frac{-2}{3} = -\frac{2}{3}$. Alternatively, students can rewrite the division problem as multiplication by the reciprocal, or $\frac{4}{9} \bullet \frac{3}{-2}$, which yields the same result.
- Ask students to use a bar, called a vinculum, to express quotients of long division problems that are repeating decimals. For example, give students the repeating decimal quotients 0.292929 . . ., 1.337337337 . . ., and 0.888 . . . . Students can rewrite these as $0.\overline{29}$, $1.\overline{337}$, $0.\overline{8}$.
- Ask students to use long division to convert given rational numbers into their decimal equivalents.
    - Give students the rational number $\frac{5}{8}$. Students can convert this to a decimal form as shown.



Students should find that eventually there is a remainder of zero, so the decimal representation of $\frac{5}{8}$ terminates and is equivalent to 0.625.

- Next, give students the rational number $\frac{29}{45}$. Students can convert this to a decimal as shown.

$$
\begin{array}{r}
0.64 \\
45\overline{)\,29.00} \\
-\underline{270}\downarrow \\
200 \\
-\underline{180} \\
20
\end{array}
$$

Explain to students that the second occurrence of the remainder of 20 indicates that the decimal form will repeat rather than terminate, so the number has a decimal form of $0.6\overline{4}$.

- Ask students to convert an improper fraction into its decimal equivalent in multiple ways. For example, give students the rational number $\frac{34}{12}$.
    - Students can convert this to a decimal by setting up the division problem $34 \div 12$ to get the result $2.8\overline{3}$.
- Students may also convert $\frac{34}{12}$ into the mixed number $2\frac{5}{6}$ first and then determine the decimal equivalent of the fraction by finding the quotient of the division problem $5 \div 6$ to get the decimal $0.8\overline{3}$. Adding the whole number portion 2 to the decimal portion $0.8\overline{3}$ will give the result $2.8\overline{3}$.

► Identifying and Addressing Common Misconceptions

- Students may apply the distributive property across a set of parentheses containing multiplication or division. For example, ask students to evaluate the expression $\frac{3}{5}(3 \bullet 5)$. Students who calculate a product of $4\frac{4}{5}$ may have this misconception. Ask students to perform the operation within the parentheses as the first step in arriving at a solution and compare results.
- Students may invert the first rational number when using the relationship between division and multiplication by the reciprocal. For example, ask students to evaluate $\frac{7}{2} \div \frac{9}{4}$. Students who multiply $\frac{2}{7} \bullet \frac{9}{4}$ to get a result of $\frac{18}{28}$ may have this misconception. Ask students to use the same method to evaluate $\frac{10}{1} \div \frac{2}{1}$. Observe that this problem is equivalent to $10 \div 2$ and that inverting the first number does not give the correct answer of 5.

- Students may invert both rational numbers when using the relationship between division and multiplying by the reciprocal. For example, ask students to evaluate $\frac{1}{3} \div \frac{7}{9}$. Students who calculate an answer of $\frac{27}{7}$ may have this misconception. Ask students to try reversing the operation to generate a fact family.
- Students may think that a number with a repeating decimal is not a rational number. For example, ask students whether 0.25, $0.8\overline{3}$, and 0.4 are rational numbers. Students who do not include $0.8\overline{3}$ may have this misconception. Ask students to confirm that the fraction equivalents of the given numbers are $\frac{1}{4}$, $\frac{5}{6}$, and $\frac{2}{5}$. Observe that fractions are rational numbers.
- When converting a fraction to decimal form, students may terminate a repeating decimal instead of repeating the value or values in the decimal. For example, ask students to convert $\frac{16}{33}$ to a decimal. Students who find an answer of 0.48 may have this misconception. Ask students to write 0.48 as the fraction $\frac{48}{100}$ and compare this fraction with $\frac{16}{33}$ by finding a common denominator. Students should find that the two fractions are not equal.
- Students may not recognize that integers are included in the set of rational numbers. Ask students to identify the rational numbers in the set –70, $\frac{-3}{4}$, $\frac{49}{50}$, 10, and $-\frac{9}{30}$. Students who do not include –70 and 10 as rational numbers may have this misconception. Ask students to represent the integers as fractions in multiple ways, such as writing 10 as $\frac{10}{1}$ and $\frac{20}{2}$, and to compare each to the definition of a rational number.
- Some students may think all fractions will have decimal values that are less than 1. Ask students to find the decimal equivalent of $\frac{33}{12}$. Students who give an answer of 0.275 may have this misconception. Ask students if the original fraction is an improper fraction and have them convert it to a mixed number before performing the division of the fractional part.

► Enrichment

- Ask students to explore how a division expression can be rewritten with multiplication to apply the commutative and associative properties. For example, ask students how they could apply the commutative property and associative properties to the expression $(\frac{1}{5} \bullet \frac{5}{2}) \div \frac{1}{2}$. Students can observe that the division operation cannot be reordered because the value of the expression would change; however, when rewritten as an expression using multiplication, $(\frac{1}{5} \bullet \frac{5}{2}) \bullet \frac{2}{1}$, the numbers may be multiplied in any order.

- Ask students to justify steps used to evaluate an expression. For example, give students the following table and ask them to fill in the missing reasons.

	Expression	**Reason**
Step 1	$-\frac{1}{5} \bullet (\frac{1}{9} \div \frac{-1}{3}) \bullet 15$	Given
Step 2	$-\frac{1}{5} \bullet 15 \bullet (\frac{1}{9} \div \frac{-1}{3})$	?
Step 3	$-\frac{1}{5} \bullet 15 \bullet (\frac{1}{9} \bullet -3)$	Multiply by the reciprocal
Step 4	$-\frac{1}{5} \bullet 15 \bullet -\frac{1}{3}$	Multiply
Step 5	$-3 \bullet -\frac{1}{3}$	Multiply
Step 6	1	?

Students should justify step 2 with “Commutative property” and step 6 with “Multiplicative inverse” and explain their thinking. Ask students to create their own tables with missing reasons to trade with partners.

- Ask students to investigate the relationship between division expressions and fractions when variables are involved. For example, ask students to relate the expression $a \div (a \bullet b)$ to a fraction. Students can determine that the expression is equivalent to the fractions $\frac{a}{a \bullet b}$ or $\frac{1}{b}$.

- Ask students to construct a multiplication or division fact family and then alter the equations so that only multiplication or only division is used. For example, ask students to construct a fact family relating –3, –4, and 12. Students can construct the fact family $-3 \bullet -4 = 12$, $-4 \bullet -3 = 12$,

$12 \div -4 = -3$, and $12 \div -3 = -4$. Then ask students to rewrite equations from the fact family so that all equations are multiplication equations. Students can leave the first two equations as they are and rewrite the last two as $12 \cdot \frac{1}{-4} = -3$ and $12 \cdot \frac{1}{-3} = -4$.

- Ask students to investigate patterns when using long division to find repeating decimal equivalents to given fractions. For example, give students the fractions $\frac{1}{9}$, $\frac{2}{9}$, and $\frac{4}{9}$. Students could observe that the repeating decimal in each case is the numerator of the fraction.
- Ask students to explore patterns in the prime factorization of the denominators of fully simplified fractions and whether the decimal forms of the fractions terminate. For example, ask students to extend the following table with various fractions and to discuss any patterns they notice.

Fraction Written in Lowest Terms	Prime Factorization of Denominator	Decimal Form of Fraction	Terminates (Yes or No)
$\frac{3}{4}$	$2 \cdot 2$	0.75	Yes
$\frac{1}{12}$	$2 \cdot 2 \cdot 3$	0.083333 . . .	No
$\frac{7}{20}$	$2 \cdot 2 \cdot 5 \cdot 5$	0.35	Yes

Help students observe that when the denominator of a fully simplified fraction has only 2s and/or 5s in its prime factorization, the fraction will have a decimal form that terminates.

- Ask students to create and explain an example of a decimal that does not terminate and does not repeat. One possible example is 0.101001000100001000001 . . ., where the number of 0s between each pair of 1s increases by one each time.

► Key Terms

divide, fraction, multiply, properties of operations, rational

A-R.1.1.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

Ratios and Proportional Relationships

Demonstrate an understanding of proportional relationships.

Analyze, recognize, and represent proportional relationships and use them to solve real-world and mathematical problems.

► Eligible Content

A-R.1.1.1 Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units.

► Instructional Supports

- Ask students to compute a unit rate involving two fractions. For example, ask students to find the typing speed, in pages per minute, of a person who types $\frac{2}{3}$ page in $\frac{3}{5}$ minute. This rate can be represented with the complex fraction $\frac{\frac{2}{3}\text{ page}}{\frac{3}{5}\text{ minute}}$. Three approaches for determining the unit rate of pages per minute, which means the number of pages typed in **one** minute, are shown.
    - Students create an equivalent fraction with a denominator of 1. This can be accomplished by multiplying the numerator and denominator by $\frac{5}{3}$ because $\frac{3}{5} \bullet \frac{5}{3} = 1$. The equivalent fraction with a denominator of 1 can then be expressed as $\frac{\frac{2}{3} \bullet \frac{5}{3}\text{ pages}}{\frac{3}{5} \bullet \frac{5}{3}\text{ minutes}} = \frac{\frac{10}{9}\text{ pages}}{1\text{ minute}}$. Therefore, the unit rate is $\frac{10}{9}$ pages typed in **one** minute, or $\frac{10}{9}$ pages per minute.
    - Another approach that results in a denominator of 1 is to divide the numerator and denominator by the denominator itself, in this case $\frac{3}{5}$, because $\frac{3}{5} \div \frac{3}{5} = 1$. The resulting

complex fraction of $\frac{\frac{2}{3}\text{ pages}}{1\text{ minute}}$ highlights the unit of 1 in the denominator. It also indicates that dividing the numerator by the denominator of the original rate results in a unit rate.

- Students can also simplify the complex fraction by multiplying it by $\frac{15}{15}$, with 15 being the least common multiple of the denominators.

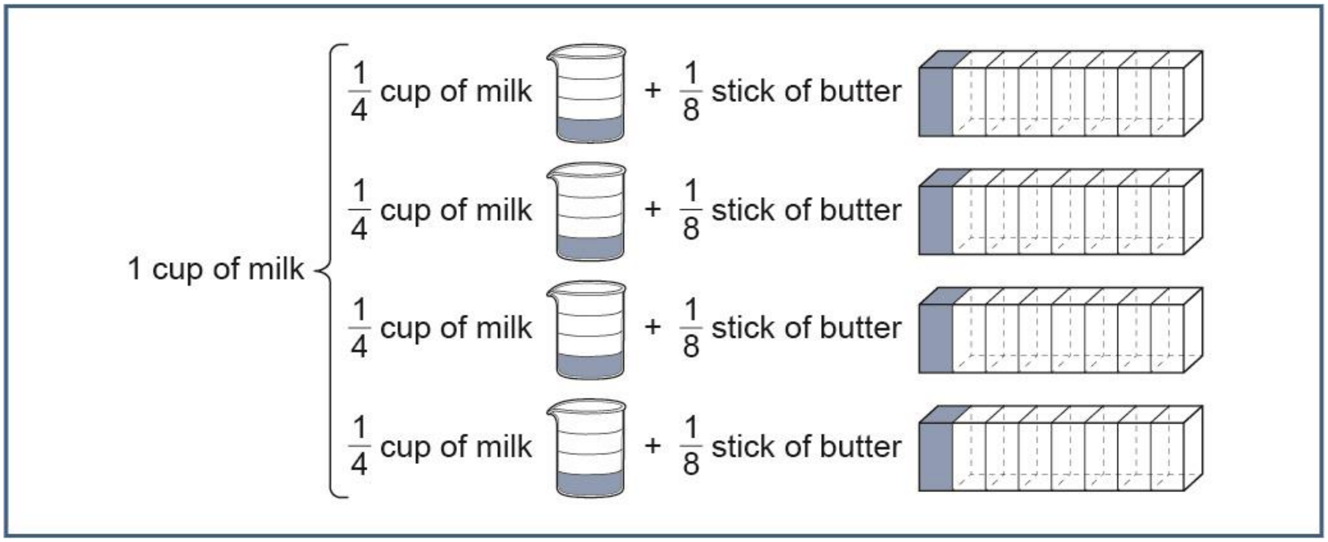
- Ask students to solve unit rate problems. For example, give students the situation “A landscaping company can mow $\frac{2}{5}$ acre in $\frac{1}{3}$ hour. How many acres can the landscaping company mow in 2 hours?” Two possible solution strategies are shown.
  - Students can set up a proportion with the same type of units placed across from one another horizontally.

$$\frac{\frac{2}{5}\text{ acre}}{\frac{1}{3}\text{ hour}} = \frac{x\text{ acres}}{2\text{ hours}}$$

    Students can first simplify the complex fraction by dividing the numerator and denominator to get $\frac{6}{5}$. The proportion can then be written as $\frac{6}{5} = \frac{x}{2}$. Students then multiply both sides of the equation by 2 to get the solution of $x = \frac{12}{5}$, which is equivalent to $2\frac{2}{5}$. Therefore, the landscaping company can mow $2\frac{2}{5}$ acres in 2 hours.

  - Students can solve by calculating the unit rate. Students calculate the unit rate by dividing $\frac{2}{5}$ acre by $\frac{1}{3}$ hour, resulting in the unit rate of $\frac{6}{5}$ acres per hour. To find the amount the company can mow in 2 hours, the unit rate can be multiplied by 2 to get $\frac{6}{5} \bullet 2 = \frac{12}{5} = 2\frac{2}{5}$ acres.

- Ask students to solve a unit rate problem using a visual model. For example, give students the situation “A recipe uses $\frac{1}{4}$ cup of milk and $\frac{1}{8}$ stick of butter. How many sticks of butter are needed when 3 cups of milk are used for the recipe?” Students can set up the following model to find the unit rate.



There are 4 groups of $\frac{1}{4}$ cup of milk in 1 cup of milk, so there are 4 groups of $\frac{1}{8}$ stick of butter for each 1 cup of milk. This is equivalent to $\frac{4}{8}$ or $\frac{1}{2}$ stick of butter for each 1 cup of milk. Therefore, the unit rate is $\frac{1}{2}$ stick of butter per 1 cup of milk. Students multiply the unit rate by the amount of milk to find the amount of butter. Doing so results in $\frac{1}{2} \bullet 3 = \frac{3}{2}$. When 3 cups of milk are used for the recipe, $1\frac{1}{2}$ sticks of butter are needed.

► Identifying and Addressing Common Misconceptions

- Some students may incorrectly set up proportions. Ask students to set up a proportion to solve the problem "Dennis drives $\frac{3}{4}$ mile in $\frac{5}{6}$ minute. At this rate, how many miles can he drive in 10 minutes?" Some students may create the proportion $\frac{\frac{3}{4}}{\frac{5}{6}} = \frac{10}{x}$. Ask students to include the units in their proportion. Help them recognize that the quantity on the left side of their equation has units of "miles per minute," while the quantity on the right side has units of "minutes per mile." On the other hand, using $\frac{x}{10}$ on the right side of the equation results in the same units on both sides of the equation.

- Some students may incorrectly simplify complex fractions. Ask students to simplify $\frac{\frac{2}{3}}{\frac{2}{5}}$. Some students may try to "cancel" the 2s and give an answer of $\frac{3}{5}$. Ask students to calculate $\frac{2}{3} \div \frac{2}{5}$. Then help students recognize that canceling the 2s does not leave $\frac{3}{5}$ but rather $\frac{\frac{1}{3}}{\frac{1}{5}}$; the 3 and the 5 remain in the denominators of their respective fractions.

► Enrichment

- Ask students to simplify a complex fraction, first by dividing fractions and then by converting the numerator and denominator into decimals and dividing the decimals. Then ask them to confirm that both methods give the same result.
- Ask students to continue to explore both possible unit rates in a given context. For example, give students the situation "A runner can run $\frac{2}{5}$ lap on a track in $\frac{1}{8}$ minute." Students can calculate the unit rate in laps per minute $\left(\frac{16}{5}\right)$ or in minutes per lap $\left(\frac{5}{16}\right)$. Students should recognize that the rates are reciprocals of each other.

► Key Terms

ratio, unit, unit rate

A-R.1.1.2

► Reporting Category, Assessment Anchor, Anchor Descriptor

Ratios and Proportional Relationships

Demonstrate an understanding of proportional relationships.

Analyze, recognize, and represent proportional relationships and use them to solve real-world and mathematical problems.

► Eligible Content

A-R.1.1.2 Determine whether two quantities are proportionally related (e.g., by testing for equivalent ratios in a table, or graphing on a coordinate plane and observing whether the graph is a straight line through the origin).

► Instructional Supports

- Ask students to determine whether two quantities have a proportional relationship. For example, ask students to determine whether a 5-pound bag of potatoes that costs $3 and a 10-pound bag of potatoes that costs $6 represents a proportional relationship and to explain why. A possible student response is shown.

  To know whether the weight and cost of the potatoes represents a proportional relationship, I calculated the weight-to-cost ratio for both quantities. The ratios for each quantity are 5:3 and 10:6. The ratios are equivalent because the ratio of 10:6 can be simplified to 5:3. So the weight and cost of the potatoes represents a proportional relationship.

  Another example could be to ask students to determine whether a 2-pound bag of sugar that costs $3.75 and a 5-pound bag of sugar that costs $6.25 represents a proportional relationship. A possible student response is shown.

To determine whether the weight of the sugar and the cost of the sugar represent a proportional relationship, I found the ratio of cost per pound for each bag. The ratio for the 2-pound bag of sugar is 1.875:1, and the ratio for the 5-pound bag of sugar is 1.25:1. The ratios are not equivalent, so the relationship is not proportional.

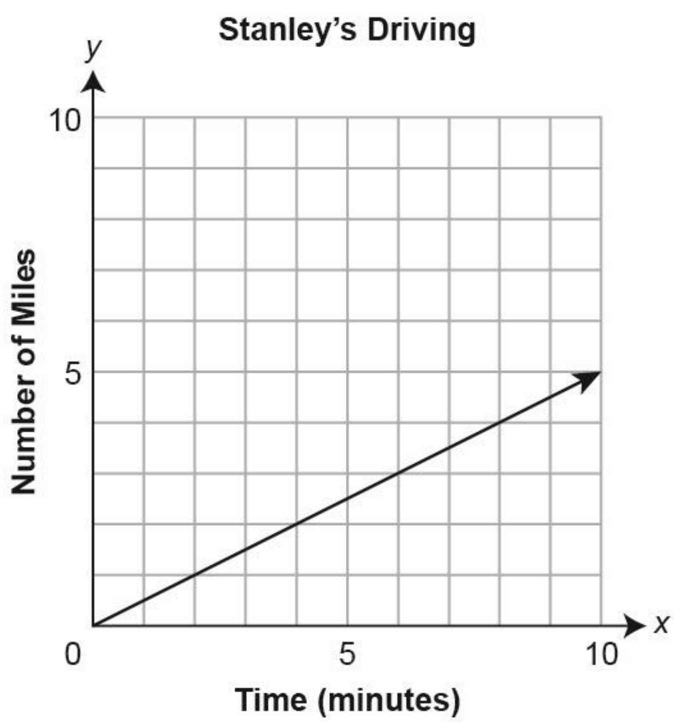
- Give students a proportional relationship and ask them to write additional ratios that fit the relationship. For example, give students the situation "Renting a bicycle for 1 hour costs $13, and renting a bicycle for 3 hours costs $39." Students can determine that these quantities represent a proportional relationship by comparing the ratios of time to cost. The ratios are 1:13 and 3:39, which are equivalent. Because the ratio 5:65 is also equivalent, paying $65 to rent a bicycle for 5 hours is another ratio in the proportional relationship.
- Ask students to determine whether the relationship shown in a given table is proportional. For example, give students the following table showing the amount of money LaMarcus earned from work and ask them to determine whether the relationship between the number of hours worked and the amount of money earned is proportional.

LaMarcus's Earnings from Work

Time Worked (hours)	2	3	5	8
Money Earned (dollars)	24	36	60	96

Students observe the ratios in the table are 2:24, 3:36, 5:60, and 8:96. Students then determine that all the ratios are equivalent to 1:12. Therefore, the relationship in the table does represent a proportional relationship in which LaMarcus earned $12 for every hour worked.

- Ask students to determine whether a given graph represents a proportional relationship. For example, give students the following graph.



The graph shows a straight line passing through the origin. Therefore, the graph indicates that the relationship between number of minutes and number of miles driven is proportional. Students can describe the relationship as either "Stanley travels 1 mile for every 2 minutes of driving" or "for every 2 minutes Stanley has driven, he has traveled 1 mile."

► Identifying and Addressing Common Misconceptions

- Some students may not check all the ratios in a table when determining whether a relationship is proportional. Ask students to determine whether the following table shows a proportional relationship.

x	y
2	12
4	24
6	30
8	40

Some students may notice that the first two ratios are equal and conclude that the relationship is proportional. Ask students to calculate the ratio between the y-value and the x-value for the other two points as well. Remind students that in a proportional relationship, **all** ordered pairs in the table need to have the same ratio.

- Some students may not include the requirement of passing through (0, 0) when identifying proportional relationships. Ask students to identify the proportional relationships when presented with a graph showing
  - a relationship that is proportional,
  - a relationship that is linear but not proportional, and
  - a relationship that is nonlinear.

  Some students may identify all linear relationships as proportional. Ask students to calculate and compare the ratio of the y-value to the x-value at several points along the graph of each relationship.

- Some students may look for an additive relationship in a table when determining whether a relationship is proportional. Ask students to determine whether the following table shows a proportional relationship.

x	y
1	3
4	6
7	9
10	12

  Some students may notice that the difference between the y-coordinate and x-coordinate of each pair is 2 and conclude that the table shows a proportional relationship. Ask students to graph the values and help them recognize that while the points form a straight line, that line does not pass through the origin. In fact, any table with an additive relationship between the x-coordinates and the y-coordinates cannot be a proportional relationship, because the coordinates of the origin, (0, 0), do not have an additive relationship.

► Enrichment

- Ask students to use proportional relationships to solve problems. For example, give students the following tables and ask them to determine how much more it would cost to buy 21 orange trees from Nursery B than from Nursery A.

Nursery A

Number of Orange Trees	Total Cost
4	$150
6	$225
8	$300

Nursery B

Number of Orange Trees	Total Cost
3	$129
9	$387
15	$645

- Give students a table with missing values and ask them to determine what values would create a proportional relationship.
- Ask students to explore the transitivity of proportional relationships. For example, there is a proportional relationship between the time that Justin spends typing and the number of words that he types, and there is also a proportional relationship between the time that Justin spends typing and the number of pages that he types. Is it necessarily true that the relationship between the number of words Justin types and the number of pages that he types is a proportional relationship?

► Key Terms

coordinate plane, equivalent, graph, origin, proportional, quantity, ratio, table

A-R.1.1.3

▶ Reporting Category, Assessment Anchor, Anchor Descriptor

Ratios and Proportional Relationships

Demonstrate an understanding of proportional relationships.

Analyze, recognize, and represent proportional relationships and use them to solve real-world and mathematical problems.

▶ Eligible Content

A-R.1.1.3 Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.

▶ Instructional Supports

- Ask students to explain why the number paired with 1 in a proportional relationship identifies the constant of proportionality. For example, give students the following table showing the time and distance that Dale rides his bike.

Time (hours)	Distance (miles)
0	0
1	13
2	26
3	39

  Students determine that the ratios $\frac{13}{1}$, $\frac{26}{2}$, and $\frac{39}{3}$ represent the constant of proportionality. From this, students recognize that each ratio is equivalent to $\frac{13}{1}$, which is equal to 13. The denominator of 1 confirms that it is a unit rate. So the constant of proportionality indicates that Dale rides his bike 13 miles per hour.

- Ask students to find the constant of proportionality of a proportional relationship represented as an equation, graph, or table by finding the value that is paired with 1. Some examples are shown.

- Give students the equation $y = 12x$, which represents the money (y) that Sarai earns from working x hours at her job. To find the value paired with 1, students substitute 1 into the equation for x to get $y = (12)(1) = 12$. Therefore, the constant of proportionality is 12. Sarai earns $12 per hour.

- Give students the following table.

Number of Boxes	Number of Crayons
0	0
1	18
2	36
3	54

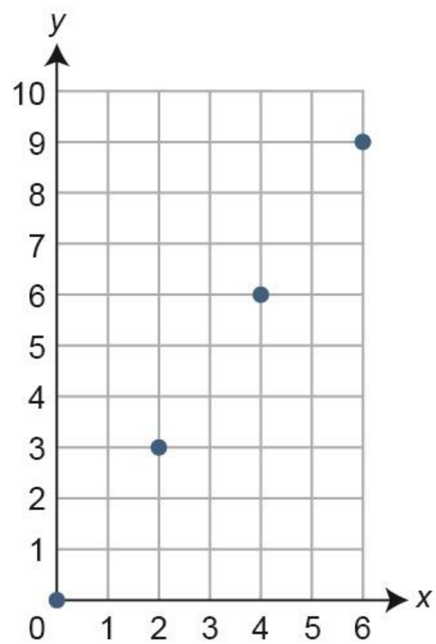
Students use the value paired with 1 to determine that the constant of proportionality is 18. There are 18 crayons per box.

- Ask students to find the constant of proportionality using ratios when given a proportional relationship represented as a graph, equation, table, or description. Some examples are shown.
  - Give students the table shown.

x	y
0	0
1	5
2	10
3	15

The constant of proportionality shown in the table is 5, as the ratio between every y-value and its corresponding x-value is equivalent to 5.

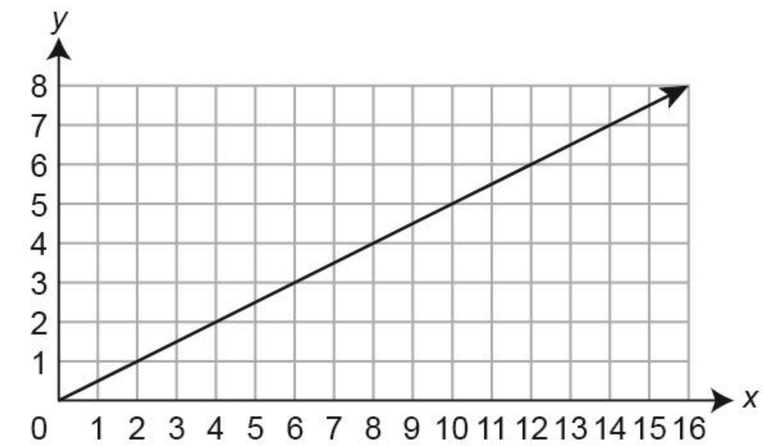
- Give students the graph shown.



The constant of proportionality may be found by finding the ratio between the y-coordinate of any point and its x-coordinate. Students can use the last point shown on the graph, (6, 9), to calculate the constant of proportionality as $\frac{9}{6}$ or $\frac{3}{2}$.

- Give students the equation $y = 10x$. Students may select some values of x to evaluate in the equation. For example, using $x = 1$ gives $y = 10(1) = 10$, and using $x = 2$ gives $y = 10(2) = 20$. Students can then use the ratio of each y-value to its corresponding x-value to find the constant of proportionality as $\frac{10}{1} = \frac{20}{2} = 10$.

• Ask students to create a table, graph, or equation that represents a given constant of proportionality. For example, give students the constant of proportionality $\frac{1}{2}$. Students create the graph shown.



A proportional relationship must contain the point (0, 0), and because the constant of proportionality is $\frac{1}{2}$, each y-value is one-half its corresponding x-value.

► Identifying and Addressing Common Misconceptions

- Some students may reverse the ratio when interpreting the constant of proportionality. Ask students to create a table for the relationship "Markers cost $3.00 per box." Some students may create the following table.

Number of Boxes	Cost of Boxes
3	1
6	2
9	3
12	4

  Ask students to explain the meaning of "per box." Help students relate "per box" to the idea of "for 1 box." The cost is 3 dollars per 1 box, not 3 boxes per 1 dollar.

- Some students may try to calculate the constant of proportionality as an additive relationship rather than as a multiplicative relationship. Ask students to find the constant of proportionality for a proportional relationship containing the point (1, 4). Some students may find $4 - 1 = 3$ and think that the constant of proportionality is 3. Remind students that proportional relationships are defined to be multiplicative relationships.

► Enrichment

- Ask students to write word problems that involve finding unit rates.
- Ask students to create slope triangles to find the constant of proportionality from a graph.

► Key Terms

constant of proportionality, coordinate plane, multiplicative comparison, slope triangle, unit rate

A-R.1.1.4

► Reporting Category, Assessment Anchor, Anchor Descriptor

Ratios and Proportional Relationships

Demonstrate an understanding of proportional relationships.

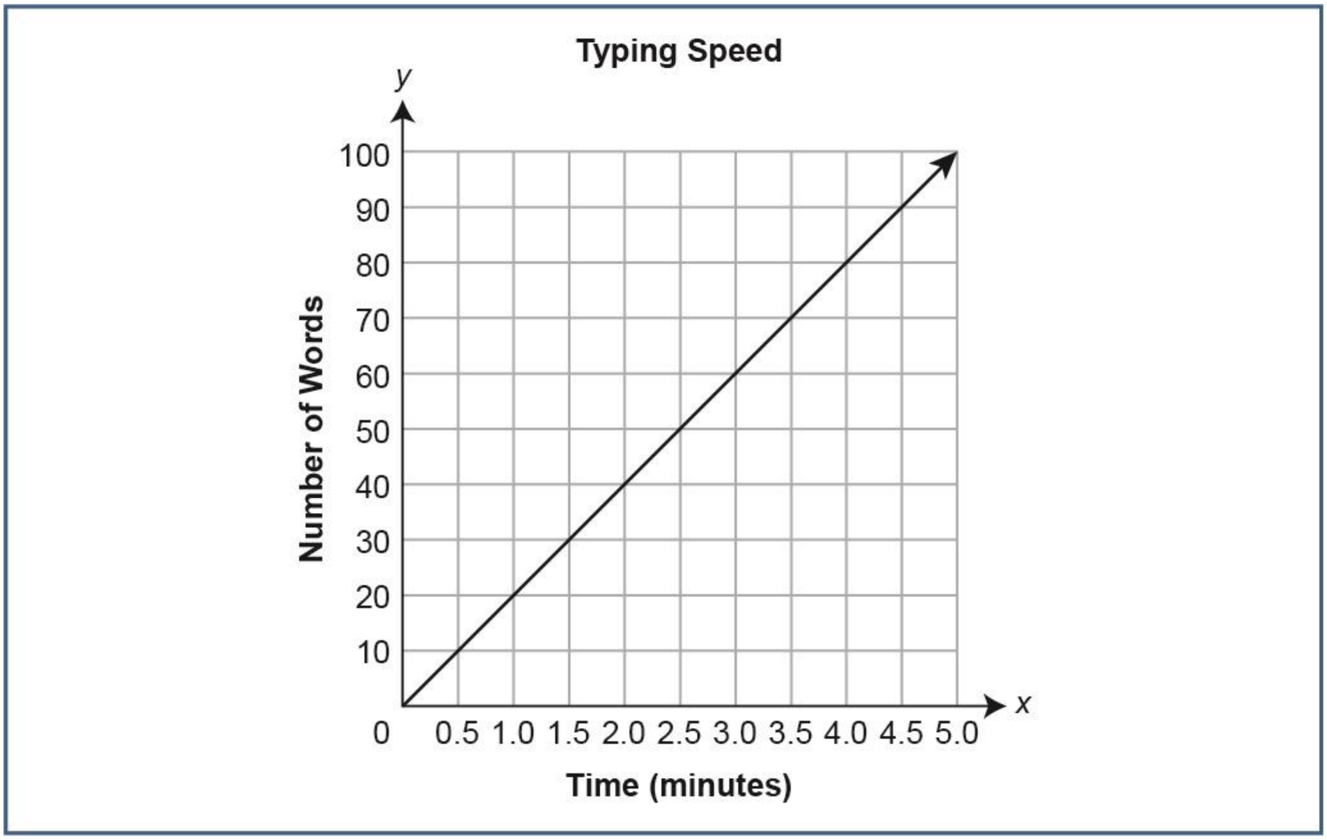
Analyze, recognize, and represent proportional relationships and use them to solve real-world and mathematical problems.

► Eligible Content

A-R.1.1.4 Represent proportional relationships by equations.

► Instructional Supports

- Ask students to write an equation with defined variables to represent a proportional relationship that is given as a verbal description. For example, give students the context "Rain is falling at a rate of 0.04 inches per hour." Students can represent this situation using the equation $r = 0.04h$, where r is the amount of rain, in inches, that has fallen and h is the number of hours the rain has been falling.
- Ask students to write an equation with defined variables to represent a proportional relationship that is given as a graph. For example, give students the following graph showing the number of words a student has typed over time.



The constant of proportionality can be identified on a graph where the x-value is 1, which, in this case, is the point (1, 20). The relationship on the graph can be represented by the equation $y = 20x$, where y is the number of words typed and m is the number of minutes spent typing.

- Ask students to write an equation with defined variables to represent a proportional relationship that is given as a table. For example, give students the table shown representing the height (h), in inches, of a plant based on the number of days (d) since it sprouted.

d	***h***
2	3
4	6
6	9

The constant of proportionality can be determined by the common ratio $\frac{h}{d}$ for all value pairs in the table, which is 1.5. Therefore, the equation $h = 1.5d$ can be used to represent this situation.

► Identifying and Addressing Common Misconceptions

- Some students may define the variables incorrectly. Give students the context "The distance, in miles, a bus travels can be represented by the equation $y = 59x$," and ask them to explain the meaning of each variable in context. Students who identify y as the time spent traveling and x as the distance traveled may have this misconception. Ask students to substitute a specific value for the time spent traveling, such as 1 hour, solve for the distance, and decide if the distance found makes sense.
- Some students may multiply the rate by the dependent variable. Ask students to write an equation representing the situation "A dog eats 2 cups of food each day." Students who write the equation $2f = d$, where f is the amount of food and d is the number of days, may have this misconception. Ask students to use the context to first determine a pair of values for d and f without writing an equation, such as 7 days and 14 cups of food. Then ask students to write an equation and to test their values in their equation.
- Students may not associate equations of the form $y = mx$ with proportional relationships. Ask students to write an equation for the situation "A store sells 4 cans of soup for 5 dollars." Students who write an equation such as $y = 4x + 5$ or $y = 5x + 4$ may have this misconception. Ask students to plot several points from their equation on a coordinate grid, draw a line through the points, and compare the line to the requirements for a proportional relationship.

► Enrichment

- Ask students to explore the effects of changing the constant of proportionality on the graph of a proportional relationship. For example, give students a graph with the line representing $y = 4x$. Ask students to graph the line representing a proportional relationship with a constant of proportionality that is half that of the given line and to graph the line representing a proportional relationship with a constant of proportionality that is twice that of the given line. Ask students to make observations about how the points on each line are related. Students can observe that the y-value of each point on the line representing a constant of proportionality that is half that of the given line is half the value of the corresponding point on the original line. Similarly, students can observe that the y-value of each point on the line representing a constant of proportionality that is twice that of the given line is twice the value of the corresponding point on the original line.

- Ask students to determine how a linear equation that does not represent a proportional relationship could be changed to represent a proportional relationship. For example, give students the equation $y = 0.875x + 1$. Students can determine that subtracting 1 from the right side of the equation would result in the proportional relationship $y = 0.875x$. Students can also explore further by graphing both equations and noting that the graphs form parallel lines, but one graph contains the origin and the other does not.
- Ask students to isolate different values in the equation of a proportional relationship. For example, give students the equation $y = –0.8x$ and ask students to isolate the constant of proportionality. Students can write the equivalent equation $\frac{y}{x} = –0.8$. Students can read the equation aloud as "the ratio of the y-values to the x-values is always a constant value of –0.8" and observe that this equation matches the definition of a proportional relationship.

► Key Terms

equation, multiplicative comparison, proportional relationship, ratio, unit rate

A-R.1.1.5

► Reporting Category, Assessment Anchor, Anchor Descriptor

Ratios and Proportional Relationships

Demonstrate an understanding of proportional relationships.

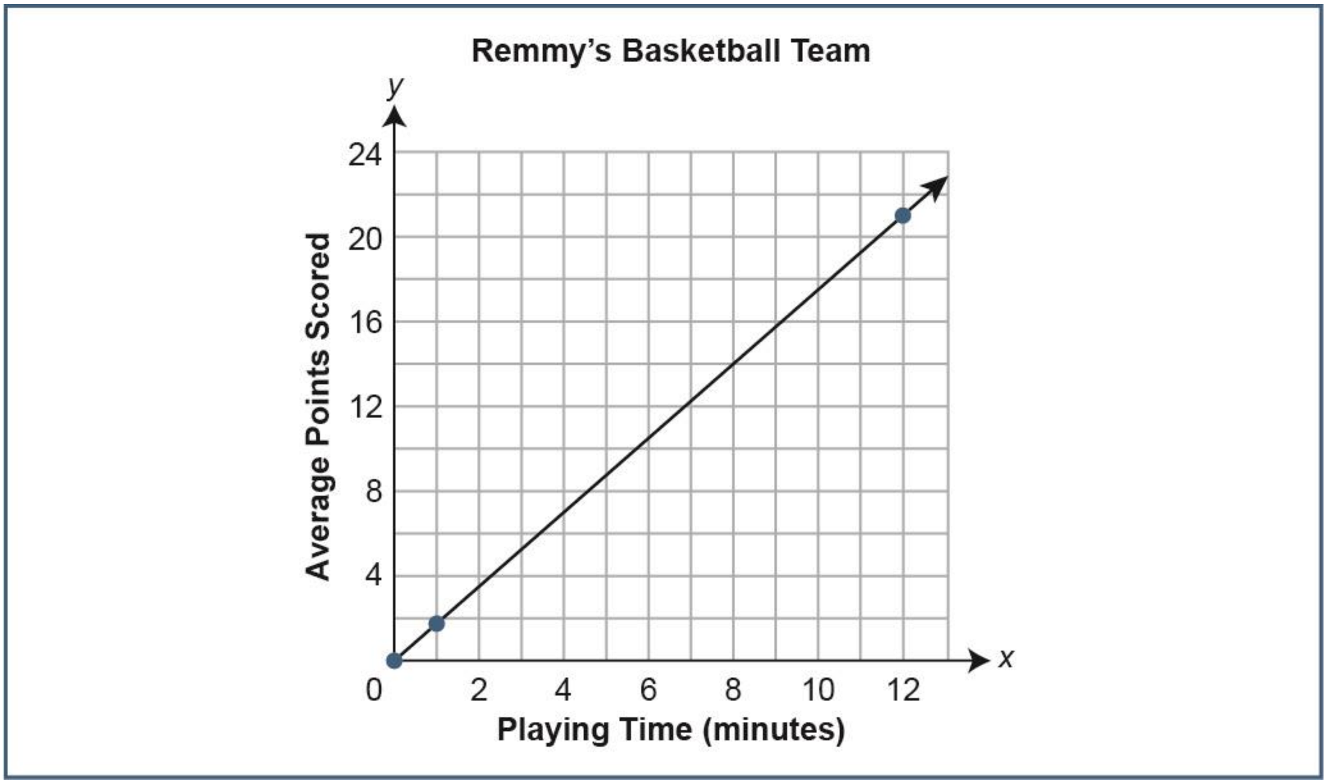
Analyze, recognize, and represent proportional relationships and use them to solve real-world and mathematical problems.

► Eligible Content

A-R.1.1.5 Explain what a point (x, y) on the graph of a proportional relationship means in terms of the situation, with special attention to the points (0, 0) and (1, r) where r is the unit rate.

► Instructional Supports

- Ask students to interpret points on the graph of a proportional relationship. For example, give students the following graph and ask them to explain the meanings of the points (0, 0), $(1, \frac{7}{4})$, and (12, 21).

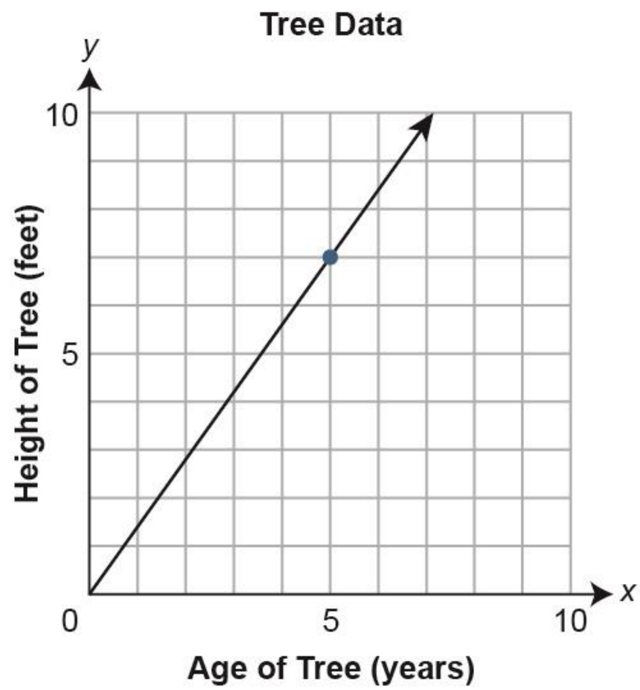


On this graph, the title is "Remmy's Basketball Team," the *x*-axis is labeled "Playing Time (minutes)," and the *y*-axis is labeled "Average Points Scored." Therefore, the first coordinate references the playing time, in minutes, and the second coordinate references average points scored.

- The point (0, 0) represents that Remmy's basketball team scores an average of 0 points after 0 minutes have passed.
- The point $(1, \frac{7}{4})$ represents that Remmy's basketball team scores an average of $1\frac{3}{4}$ points every 1 minute. This is the unit rate.
- The point (12, 21) represents that Remmy's basketball team scores an average of 21 points after 12 minutes have passed.

▶ Identifying and Addressing Common Misconceptions

- Some students may reverse the meanings of the coordinates. Ask students to interpret the point on the following graph showing the height of a tree over time.



Some students may interpret the plotted point as indicating that the height of the tree after 7 years is 5 feet. Ask students which coordinate is the *x*-coordinate and which coordinate is the *y*-coordinate. Then ask them to identify the labels on each axis. Remind students that the *x*-axis label is always associated with the first number and the *y*-axis label is always associated with the second number, just as “x” comes before “y” in the alphabet. Also remind students that the label associated with each axis is also associated with that coordinate.

► Enrichment

- Ask students to compare the equation for a proportional relationship with the point (1, *r*) and explain the connection between the equation and the point.
- Ask students to describe a relationship that is not proportional and to explain what the points with *x*-coordinates of 0 and 1 mean in the context of the relationship.

► Key Terms

coordinate pair, origin, proportional, unit rate

A-R.1.1.6

► Reporting Category, Assessment Anchor, Anchor Descriptor

Ratios and Proportional Relationships

Demonstrate an understanding of proportional relationships.

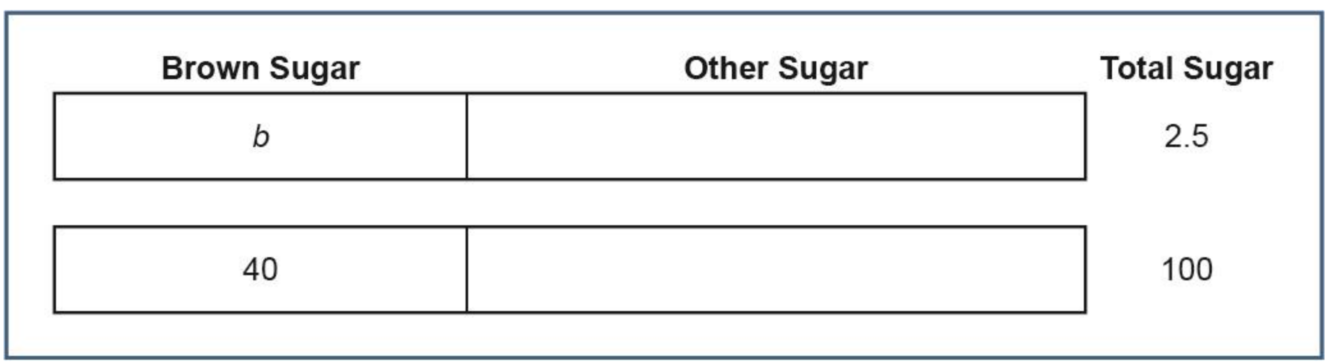
Analyze, recognize, and represent proportional relationships and use them to solve real-world and mathematical problems.

► Eligible Content

A-R.1.1.6 Use proportional relationships to solve multi-step ratio and percent problems.

► Instructional Supports

- Ask students to use proportional relationships to solve problems involving percentages. For example, ask students to find the amount of brown sugar used in a recipe when there is a total of 2.5 cups of sugar used in the recipe and 40% of it is brown sugar. Students rewrite 40% as $\frac{40}{100}$. Then, representing the amount of brown sugar as *b*, students set up the following bar model.



Students use the bar model to show the ratio of brown sugar to the total amount of sugar is $\frac{b}{2.5}$ and set up the following proportion.

$$\frac{b}{2.5} = \frac{40}{100}$$

$$100b = 100$$

$$b = 1$$

Therefore, of the 2.5 total cups of sugar in the recipe, there is 1 cup of brown sugar.

- Ask students to solve multistep problems involving proportional relationships. For example, give students the situation “Kayla spent time at a park fishing and swimming. She spent 1.5 hours fishing before going swimming. Kayla was at the park for 4 hours. What percentage of her time at the park did Kayla spend swimming?” Two possible solution paths are shown.
    - Kayla spent 1.5 of the 4 hours at the park fishing. Students can use f to represent the percentage of the time that Kayla spent fishing.

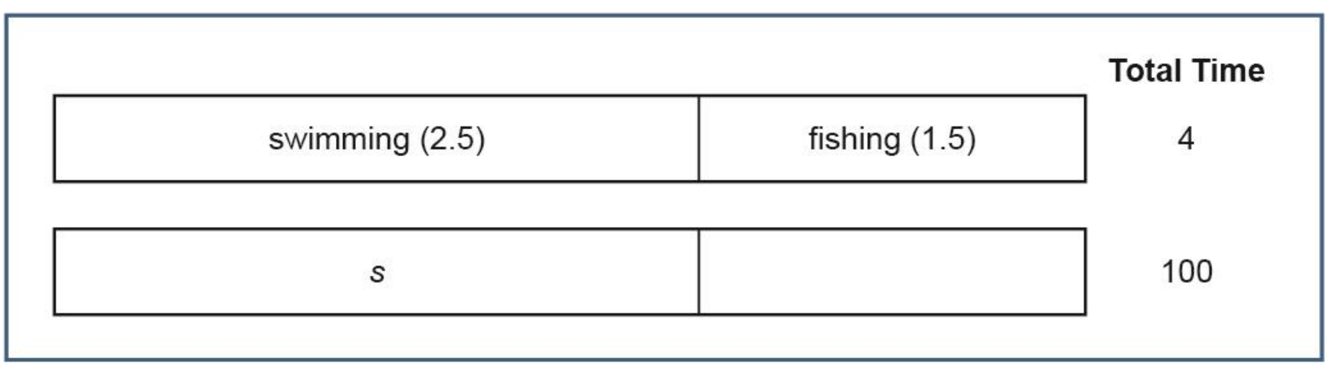
$$\frac{1.5}{4} = \frac{f}{100}$$

$$4f = 150$$

$$f = 37.5$$

Therefore, Kayla spent 37.5% of her time fishing and 62.5% (100% – 37.5%) of her time swimming.

    - Kayla spent 1.5 hours fishing. Therefore, she spent 4 – 1.5 = 2.5 hours swimming. Students can use s to represent the percentage of the time Kayla spent swimming and create the following bar model and proportion.



$$\frac{2.5}{4} = \frac{s}{100}$$

$$4s = 250$$

$$s = 62.5$$

Therefore, Kayla spent 62.5% of her time at the park swimming.

- Ask students to determine a percent increase or percent decrease. Two examples are shown.
  - Give students the situation "Brett picked 37 pounds of strawberries on Tuesday and 55 pounds of strawberries on Wednesday. By what percentage did the amount of strawberries Brett picked increase from Tuesday to Wednesday?" Students find the increase in strawberries picked by evaluating 55 – 37 = 18. Then, students set up the proportion shown to find p, the percent increase. The percent increase is based on the ratio of the difference of 18 pounds compared to the original amount on Tuesday of 37 pounds.

$$\frac{18}{37} = \frac{p}{100}$$

$$37p = 1{,}800$$

$$p = 48.65$$

Therefore, Brett picked 48.65% more strawberries on Wednesday than on Tuesday.

  - Give students the situation "Stacey earned \$60 selling cookies at last month's bake sale. At this month's bake sale, her earnings decreased by 15%. How much did she earn from this month's bake sale?" Students set up the proportion shown to find d, the amount that her earnings decreased.

$$\frac{d}{60} = \frac{15}{100}$$

$$100d = 900$$

$$d = 9$$

Therefore, her earnings decreased by \$9. This means that she earned 60 – 9 = \$51 from this month's bake sale.

- Ask students to use proportions to solve problems where the unknown value appears more than once. For example, ask students to find the original price of an item that cost \$22.75 after a 30% discount. The original price of the item (p) can be found by evaluating the proportion shown.

$$\frac{p-22.75}{p}=\frac{30}{100}$$

$$100p - 2{,}275 = 30p$$

$$70p = 2{,}275$$

$$p = 32.50$$

Therefore, the original price of the item was \$32.50.

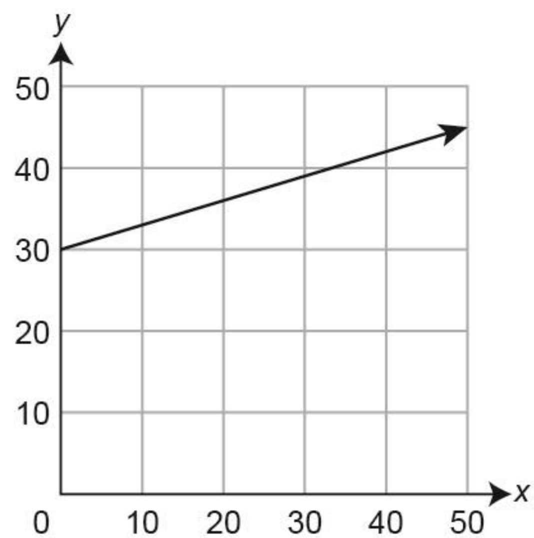
▶ Identifying and Addressing Common Misconceptions

- Some students may set up proportions incorrectly by reversing the "part" and the "whole." Ask students to solve the problem "Savannah has 64 toy cars and gives away 12.5% of them. How many toys does she give away?" Some students may set up the proportion $\frac{64}{x}=\frac{12.5}{100}$ and give an answer of 512. Ask students to explain the meaning of their answer in context. Then ask them to consider whether it makes sense for their answer to be greater than or less than 64, and whether that indicates if 64 is the part or the whole.
- Some students may incorrectly use the ending value rather than the starting value when determining percent change. Ask students to solve the problem "A car cost \$20,000 last year and now costs \$25,000. By what percentage did the cost of the car change?" Some students may set up the proportion $\frac{25{,}000-20{,}000}{25{,}000}=\frac{p}{100}$ and give an answer of 20%. Ask students to calculate 20% of \$20,000 and add that to \$20,000 and then compare their result with the new cost of \$25,000. Remind students that percent change is based off the starting value, not the ending value.
- Some students may confuse a percentage of a given value with a percent increase/decrease from a given value. Ask students to find a number that is 10% less than 80. Some students may give an answer of 8. Ask students to explain the difference between "an item that originally costs \$15

is now on sale for $6" and "an item that originally costs $15 is now on sale for $6 less." Help students draw parallels to understand the difference between a number that is "10% of 80" and "10% less than 80."

► Enrichment

- Ask students to create graphs showing percent increase and percent decrease. For example, ask students to create a graph where *y* shows the value of an *x*-percent increase from a starting value of 30. Students may create the following graph.



  Students should note that although some percent relationships are proportional, percent increase and decrease are not proportional because the starting values are not 0.

- Ask students to calculate a percent increase that is applied twice and then ask them to compare the result to a percent increase that is applied only once but with a percentage that is twice as large. For example, ask students to compare the final height of a 10-foot-tall tree whose height increases by 10% each year for the next two years to the final height of a 10-foot-tall tree whose height increases by a total of 20% over the next two years. Students should note that when multiple percent increases or decreases are applied, the base value changes after each percent increase.

- Ask students to discuss the use of language when percent increases/decreases are used to refer to values that are percentages. For example, ask students to discuss the different ways of interpreting "Last year, 40% of students wore glasses. This year, there is a 10% increase in the number of students with glasses." Students may find two different interpretations.

The percentage increased from 40% by 10 percentage points to 50%.

    - The percentage increased by 10% of 40, which is 4, to 44%.

Discuss with students the importance of clear wording when using percentages.

► Key Terms

commission, fee, gratuity, interest, markdown, markup, percent, percent decrease, percent error, percent increase, proportional, ratio, tax

B-E.1.1.1

▶ Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Represent expressions in equivalent forms.

Use properties of operations to generate equivalent expressions.

▶ Eligible Content

B-E.1.1.1 Apply properties of operations to add, subtract, factor, and expand linear expressions with rational coefficients.

▶ Instructional Supports

- Give students an expression and ask them to use the order of operations and the commutative, associative, and distributive properties of operations to generate an equivalent expression. For example, give students the expression $-8 + (-2)(8x + 7)$. The expression has a grouping symbol; however, the operations within the grouping symbol cannot be performed because the terms within the grouping symbol are not like terms. The next operation to be performed in the expression by the order of operations is multiplication. The distributive property is used to multiply both terms within the grouping symbol by (-2).

  $$-8 + (-2)(8x + 7)$$

  $$-8 + (-2)(8x) + (-2)(7)$$

  $$-8 + (-16x) + (-14)$$

  The commutative and associative properties can then be used to place the like terms together.

  $$-8 + (-14) + (-16x)$$

  The like terms can then be combined to arrive at $-22 - 16x$.

- Ask students to rewrite a given expression by factoring out the greatest common factor of two terms. For example, give students the expression $9 + 3x + 11$. The first two terms have a greatest common factor of 3. The third term does not have any factors, other than 1, in common with the other terms. Students can thus factor a 3 out of the first two terms to create the equivalent expression $3(3 + x) + 11$.

- Ask students to use integers to explain how the commutative and associative properties can be used with subtraction. For example, give students the expression $-7 - 2x + 8 - 9x$. By using the addition of opposites property to rewrite all subtraction as addition, the expression can be rewritten as $(-7) + (-2x) + 8 + (-9x)$. Now the commutative and associative properties can be used to group and combine like terms, as shown.

Expression	**Reason**
$-7 - 2x + 8 - 9x$	Given
$(-7) + (-2x) + 8 + (-9x)$	Addition of opposites
$(-7) + [(-2x) + 8] + (-9x)$	Associative property
$-7 + [8 + (-2x)] + (-9x)$	Commutative property
$(-7 + 8) + [(-2x) + (-9x)]$	Associative property
$1 + (-11x)$	Combine like terms

  Students should conclude that the commutative and associative properties can be applied to subtraction after representing subtraction as the addition of opposites.

- Ask students to explain why the distributive property needs to be applied before combining like terms in an expression of the form $a + b(c + dx)$, where a, b, c, and d are specific rational numbers. For example, give students the expression $-6 + 7(23 - 8x)$. By the order of operations, the multiplication and then subtraction within the parentheses would be performed first; however, the multiplication cannot be performed without a value for x, and the subtraction cannot be performed because the terms are not like terms. Therefore, by the order of operations, the next operation to consider in this expression is the multiplication represented by the number outside the parentheses, or the multiplication of 7 by the entire quantity $(23 - 8x)$. This multiplication can occur even without a value for x by distributing the factor of 7 to both 23 and $-8x$ and rewriting as the sum of the products: $7(23 - 8x) = 7(23) + 7(-8x) = 161 + (-56x)$. Now the -6 from the original expression can be combined with the like term of 161 to yield $155 - 56x$.

- Ask students to analyze common errors that arise when generating equivalent expressions and then to correct the errors. For example, give students the following scenario: "A student generated the expression $12 - 5x$ as an equivalent expression to $-3 - 5(-3 - x)$." Students should identify the error as likely occurring when using the distributive property. When distributing with subtraction signs, the student in the scenario did not apply the subtraction sign as a negative symbol to all terms. To make it easier to keep track of negative values when distributing and combining like terms, it is best to first rewrite all subtraction using addition of opposites. The given expression would then be rewritten as $-3 + (-5)(-3 + (-x))$, which is equivalent to $12 + 5x$.

► Identifying and Addressing Common Misconceptions

- Some students may subtract instead of divide when factoring. For example, ask students to factor the expression $21x + 14$. Students who factor the expression as $7(14x + 7)$ may have this misconception. Ask students to evaluate the given expression and the factored expression for specific values of x.

- Some students may not distribute a subtraction sign to all terms within a set of grouping symbols. For example, ask students to expand the expression $4 - 5(8x + 10)$. Students who expand the expression to $4 - 40x + 50$ may have this misconception. First encourage students to rewrite the subtraction of 5 as the addition of -5: $4 + (-5)(8x + 10)$. When written this way, it is much easier to remember that a **negative** value is being multiplied by the group and that the resulting product should therefore be opposite what it would be if the first factor were positive (in this case, $-40x + 50$ as compared to $40x - 50$). Therefore, the correct expansion to the original expression is $4 - 40x - 50$.

- Students may distribute a coefficient to both factors of one term of an expression. For example, ask students to expand the expression $2(6 + 7x)$. Students who distribute the 2 to the 6, to the 7, and again to the x to get $12 + 28x$ may have this misconception. Ask students to evaluate $2(7x)$ and to observe that the 2 does not get distributed as a multiplier. In other words, $2(7x)$ does not equal $2(7)2(x)$. Help students apply this fact to the original expression, $2(6 + 7x)$.

► Enrichment

- Ask students to expand an expression with multiple sets of grouping symbols. For example, ask students to expand the expression $4[8 - 3(7 + 4x)]$.
- Ask students to create a real-world context in which linear expressions are added or subtracted. For example, students may create the context of finding the difference in cost between two different painters for a job lasting h hours whose fees are represented by the expressions $18h + 35$ and $15h + 50$.
- Ask students to factor an expression in which the greatest common factor includes a variable. For example, give students the expression $4xy + 9x + 5xy - 3x$. Students can start by using the commutative and associative properties to combine like terms to generate the equivalent expression $9xy + 6x$. The greatest common factor of the coefficients in the two factors is 3; however, the two terms both also include x, and so $3x$ can be factored out of each term to arrive at the expression $3x(3y + 2)$.

► Key Terms

coefficient, compose, decompose, equivalent, expand, expression, factor, properties of operations

B-E.2.1.1 & B-E.2.3.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Solve real-world and mathematical problems using numerical and algebraic expressions, equations, and inequalities.

Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers.

► Eligible Content

B-E.2.1.1 Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate.

Determine the reasonableness of the answer(s) in problem solving situations.

B-E.2.3.1 Determine the reasonableness of an answer(s), or interpret the solution(s) in the context of the problem.

► Instructional Supports

- Ask students to solve real-life problems by working backward. For example, give students the following situation: "A runner is completing a route in multiple stages. The runner's goal is to finish all stages with an average time of 0.55 hours per stage. Over the first four stages the runner has averaged 0.6 hours per stage. The runner has one stage remaining. What time does the runner need on the final stage to reach the goal?" The time needed for all five stages can be found by multiplying the average by 5: $0.55 \cdot 5 = 2.75$. The total time for the first four stages can be found using the average of 0.6 multiplied by the number of completed stages: $0.6 \cdot 4 = 2.4$. The time required on the last stage can then be found by subtracting the current time from the goal: $2.75 - 2.4 = 0.35$. In order to achieve the goal, the runner needs to complete the final stage in 0.35 hours.
- Ask students to write expressions to represent real-world situations. For example, give students the following situation: "The price of a meal at a restaurant is discounted by 5%, and a coupon is used to reduce the price by an additional \$2. The discount is applied before the coupon. A tax of

10% is then added to the price of the meal. The final cost of the meal is \$24.97." Ask students to write an expression to represent the original price. The tax was calculated last, so the final cost is divided by 1.1 to find the price with both the discount and the coupon: $24.97 \div 1.1$. The coupon was applied in the step just before the tax, so 2 needs to be added to $(24.97 \div 1.1)$ to find the price of the meal with only the 5% discount: $2 + (24.97 \div 1.1)$. That discount needs to then be taken into account by dividing the whole expression by 0.95: $((24.97 \div 1.1) + 2) \div 0.95$. This can be evaluated to determine that the original price of the meal was \$26.

- Ask students to solve problems in which numbers are given in multiple forms. For example, ask students to evaluate the expression $\frac{5}{3} \bullet 1\frac{1}{2} + 2.25$. Students can rewrite $1\frac{1}{2}$ as the fraction $\frac{3}{2}$. Then, the fractions can be multiplied to find a product of $\frac{5}{2}$. The fraction $\frac{5}{2}$ can then be converted to decimal form (2.5) and added to 2.25 to arrive at a sum of 4.75.
- Ask students to explain why a particular form of a rational number may be more efficient in a specific situation.
  - Give students the expression $0.2 \bullet \frac{1}{8}$. Converting the decimal to a fraction allows for the multiplication to be performed by multiplying the numerators and the denominators, which is the most efficient method. Converting the fraction to a decimal would require using the standard algorithm to multiply a one-digit number by a three-digit number.
  - Give students the expression $\frac{7}{10} + 0.424$. Converting the fraction to a decimal allows for the addition to be performed by the standard algorithm, which is the most efficient method. Converting the decimal to a fraction would require converting the fractions to have a common denominator.
- Ask students to use estimation to assess the reasonableness of an answer. Two examples are shown.
  - Give students the product $3.07 \bullet 8.7$ and ask them to use estimation to determine whether 26.709 is a reasonable answer. The product can be estimated by approximating 3.07 as 3 and 8.7 as 9. The estimated product is then 27. The tens value is the same in both cases,

and the ones digit is off by only 1. In addition, because 7 • 7 = 49, the product should end with a 9. Therefore, 26.709 appears to be a reasonable estimate.

- Give students the following situation: "A student calculated the product of 77.8 and 87.14 to be 67.79492." Ask students to explain, without evaluating, why this product is not reasonable. The factors 77.8 and 87.14 can be approximated as 80 and 90, respectively. The product can be approximated as 80 • 90, which is equal to 72 • 100, or 7,200. Since the student's product is not close to the estimate, the product is not reasonable. Multiplying the student's product by 100 gives a value that is much closer to the estimate, so the error is likely to be one of place value.

► Identifying and Addressing Common Misconceptions

- Some students, when working backward, may reverse the operations without reversing the order. Give students the following scenario: "A customer has a \$5-off coupon applied to his purchase before also receiving a 10% discount. The customer pays \$44.46. What was the customer's bill before the coupon and discount?" Some students may recognize that the final price can be calculated from the starting price by subtracting 5 and then multiplying by 0.9, but they may try to reverse the process by adding 5 and then dividing by 0.9. Ask students to think about the process of putting on socks and shoes. Socks must be put on before shoes, but when taking them off, the shoes must come off first. Help students recognize that not only do the operations need to be reversed, but the order in which those operations are performed also needs to be reversed.
- Some students may convert numbers into forms that lead to inexact answers. Ask students to calculate $\frac{4}{3} \div 0.75$ by changing the numbers into the same form. Some students may rewrite the expression as $1.\overline{3} \div 0.75$, thus needing to round the first decimal in order to calculate the product. Ask students whether $\frac{4}{3}$ is equal to 1.3. Then ask them whether $\frac{4}{3}$ is equal to 1.33. Students should conclude that these are good approximations but not exact answers.
- Students may not consider the scale of the numbers when identifying whether an estimate is reasonable. Ask students to use estimation to check the reasonableness of the equation 689 • 4,122 = 2,840,058. Some students may round the factors to 700 and 4,000, get an estimate of 2,800,000, recognize that this is more than 40,000 off from the given product, and decide that

the estimate is not reasonable. Ask students to compare a person growing 2 centimeters with an ant growing 1 centimeter. Help them recognize that the size of the original amount affects whether a change in the original amount is considered a big change.

► Enrichment

- Ask students to explore real-world situations involving percentages in which order does not matter. For example, give students the following situation: “The price of a chair is reduced by 15%, and a customer has a coupon for 5% off one item. Which order of applying the coupon and discount gives the lowest price?”
- Ask students to explore real-world situations using percentages in which order matters. For example, give students the following situation: “The water level in a pond will be increased by 20% and also increased by 5 inches. The two operations can be performed in either order. What is the difference between the final depths when the operations are performed in each possible order?”
- Ask students to use rounding and estimation to determine a range of possible values for an answer. For example, give students the expression 170.3 • 374.9 and ask them to estimate a range of values for the product. Students can estimate that the product is greater than 150 • 300 = 45,000 and that it is less than 200 • 400 = 80,000.

► Key Terms

equivalent, estimation, expression, place value, properties of operations

B-E.2.2.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Solve real-world and mathematical problems using numerical and algebraic expressions, equations, and inequalities.

Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems.

► Eligible Content

B-E.2.2.1 Solve word problems leading to equations of the form px + q = r and p(x + q) = r, where p, q, and r are specific rational numbers.

► Instructional Supports

Solving problems

- Ask students to use an equation with a variable to represent a situation with an unknown quantity. For example, give students the following situation: “Blocks are placed in a container with a volume of 250 cubic inches. Each block is the same size and shape. After 28 blocks are placed in the container, there are 26 cubic inches of space remaining in the container. What is the volume of each block?” The unknown value in this situation is the volume of a block, which can be represented by v. There 28 blocks with this volume, so the volume of all blocks can be represented as $28v$. The total volume of the container, 250 cubic inches, is equal to the total volume of the blocks plus the volume of the remaining space in the container. Thus, the situation can be represented using the equation $28v + 26 = 250$.
- Ask students to solve problems involving an unknown quantity by writing and solving equations. For example, give students the following situation: “A store applies a sales tax on each purchase. A customer pays a total of $58.38 for a purchase that had a price of $55.60 before tax. What percentage of sales tax does the store charge?” The tax percentage can be represented by x. The total the customer pays is equal to the before-tax price plus the amount owed in tax. The amount

owed in tax can be represented as the product of the before-tax price and x. The total purchase price can thus be represented as $55.60 + 55.60x$. An equation can be set up using this information and then solved to find x.

$$55.60 + 55.60x = 58.38$$

$$55.60(1 + x) = 58.38$$

$$1 + x = \frac{58.38}{55.60}$$

$$1 + x = 1.05$$

$$x = 1.05 - 1$$

$$x = 0.05$$

Therefore, the percentage of sales tax charged is 5%.

- Ask students to solve a real-world problem using both arithmetic and algebraic methods. For example, give students the following situation: "A new bag of birdseed contains 24 pounds of birdseed. Each time the bird feeder is filled, $\frac{2}{3}$ pound of birdseed is used. How many times has the feeder been filled when 10 pounds of birdseed remains in the bag?" Some possible solutions are shown.
    - The solution can be found using an arithmetic method by starting with the initial amount and subtracting the remaining amount. This is represented by the expression $24 - 10$. This shows that 14 pounds have been used. The amount used can then be divided by the amount used each time, $\frac{2}{3}$, to arrive at a solution of 21 times.
    - The solution can be found by starting with 24 and repeatedly subtracting $\frac{2}{3}$ until reaching a difference of 10. This difference is reached after 21 repeated subtractions.
    - The solution can also be found using an algebraic method. A variable is used to represent the unknown, such as n, and an equation is set up to represent the situation: $\frac{2}{3}n + 10 = 24$. This equation can be solved to find $n = 21$.

Students should note that the operations involved in the algebraic method (subtracting 10 and dividing by $\frac{2}{3}$) are the same operations involved in the arithmetic methods.

Comparing solution methods

- Ask students to determine which solution method, arithmetic or algebraic, is more efficient for a given problem. For example, give students the following situation: "A driver on a trip has driven 150 miles so far and still has 330 miles to go. The driver's average speed is 60 miles per hour. How many hours, in total, will it take the driver to complete the trip?" Students can determine that the arithmetic method is efficient in this case since the two distances can be added, and the sum can then be divided by the average speed to find the solution.
- Ask students to compare multiple strategies to determine which method is most efficient for solving a given equation. For example, give students the equation $5(x + 9) = 60$. One way this equation can be solved is by first applying the distributive property and then solving for x.

$$\begin{aligned}
5(x + 9) &= 60 \\
5x + 45 &= 60 \\
-45\ \ &\ \ -45 \\
\hline
\frac{5x}{5} &= \frac{15}{5} \\
x &= 3
\end{aligned}$$

The equation can also be solved by first dividing each side by 5 and then solving for x.

$$\begin{aligned}
\frac{5(x + 9)}{5} &= \frac{60}{5} \\
x + 9 &= 12 \\
-9\ \ &\ \ -9 \\
\hline
x &= 3
\end{aligned}$$

Students can determine that the second method is more efficient in this case since the step of applying the distributive property is not needed.

► Identifying and Addressing Common Misconceptions

- Some students may incorrectly write equations in which multiplication should be applied to multiple terms. Ask students to write an equation representing the following situation: "Five bags each contain the same unknown number of marbles. Six marbles are added to each bag, and now there is a total of 55 marbles in the bags. How many marbles were originally in each bag?" Students who write the equation $5x + 6 = 55$ may have this misconception. Ask students to create a diagram representing the situation. Students should recognize from the diagram that because 6 marbles are added to each bag, $5(6) = 30$ marbles are being added in total, and therefore the correct equation is $5x + 5(6) = 5(x + 6) = 55$.

- Some students may perform the inverse operation of division before addition or subtraction without applying the division to both terms. Ask students to solve the equation $-6x + 2 = 24$. Students who try to divide by the -6 first and arrive at a solution of $x = -6$ may have this misconception. Ask students to show what their equation looks like when the division is applied but before it is calculated: $\frac{-6x}{-6} + 2 = \frac{24}{-6}$. Ask students to apply their understanding of fractions to find a common denominator for each term in their equation and create the equivalent equation $\frac{-6x}{-6} + \frac{-12}{-6} = \frac{24}{-6}$. Help students recognize that when the fractions are added, this resulting equation is inconsistent with the original equation because -12 is being divided by -6 instead of 2 being divided by -6.

- Some students may write equations with the constant in an incorrect location. Ask students to write an equation representing the following situation: "Cards can be purchased in boxes or individually. Each box contains the same number of cards. A customer purchases 3 boxes and 7 individual cards. The customer purchases a total of 31 cards. How many cards are in each box?" Students who write the equation $3x = 31 + 7$ may have this misconception. Ask students to estimate a solution by guessing a value for the number of cards in a box and then determining the total by multiplying by 3 and adding 7. Ask them to adjust their guess and check again until the total is 31. Then, ask them to compare their trial-and-error process with the equation $3x = 31 + 7$ to demonstrate that 7 must be added to the $3x$.

► Enrichment

- Ask students to explore multiplying equations by a constant. For example, give students the following scenario: "A special at a bakery includes 4 boxes with x muffins in each box and 6 individual muffins. A customer who buys the special receives a total of 22 muffins." Ask students to write an equation representing this situation. Students can write the equation $4x + 6 = 22$. Then add to the scenario: "A customer orders 5 of the specials. Show the steps to change the equation to match the new information." Students can multiply both sides of the equation by 5. Ask students to solve both equations for x. Students should determine that the value of x is 4 in both cases.
- Ask students to write equations in which the same unknown appears multiple times. For example, give students the following situation: "Each bottle of water holds the same amount. One person has 40 bottles of water, and another person has 6 bottles of water plus one jug of water that holds 32 ounces. Between the two people, there is a total of 860 ounces of water. How many ounces of water are in each bottle?" Students can represent this situation with the equation $40x + 6x + 32 = 860$.
- Ask students to simplify an equation to arrive at the form $px + q = r$ or $p(x + q) = r$. For example, give students the equation $–3x + 8(2 + x) = 19$. Students can start by distributing the 8 on the left side of the equation to arrive at $–3x + 16 + 8x = 19$. Students can then combine like terms to arrive at $5x + 16 = 19$.

► Key Terms

algebraic, arithmetic, equivalent, expression, inequality, properties of operations, variable

B-E.2.2.2

► Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Solve real-world and mathematical problems using numerical and algebraic expressions, equations, and inequalities.

Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems.

► Eligible Content

B-E.2.2.2 Solve word problems leading to inequalities of the form px + q > r or px + q < r, where p, q, and r are specific rational numbers, and graph the solution set of the inequality.

► Instructional Supports

Problems involving inequalities

- Ask students to explain what it means to identify the solution set of an inequality. For example, give students the inequality $0.5x - 11 < 8$ and the solution set $x < 38$. Saying that the solution set is $x < 38$ means that all numbers less than 38, when substituted into the original inequality, create a true comparison statement. In addition, all numbers not less than 38, including 38, create a false comparison statement when substituted into the original inequality.
- Ask students to identify when a situation is represented by an equation and when a situation is represented by an inequality. Two examples are shown.
    - "A driver travels for 7 hours on Monday at an average speed of 55 miles per hour. On Tuesday, the driver travels the same distance but at an average speed of 50 miles per hour." Students can determine that this situation is represented by an equation since "the same distance" indicates that the relationship between distances on the two days is one of equality.

  - "A driver has 8 hours to spend traveling on Wednesday. The driver wants to travel more than 380 miles on Wednesday. How fast should the driver drive?" This situation is represented by an inequality since "more than" indicates that the relationship is one of inequality.

- Ask students to use inequalities to represent a situation algebraically. For example, give students the following situation: "A container has 4 liters of oil in it and is not yet full. A jug with a capacity of 2.5 liters is used to add oil to the container. The container holds less than 34 liters of oil. How many times can the jug be emptied into the container?" The amount of oil in the container can be represented by the expression $4 + 2.5x$, where x is the number of times the jug has been emptied into the container. The container must hold less than 34 liters of oil, so the expression $4 + 2.5x$ must have a value that is less than 34, which can be represented by

  $4 + 2.5x < 34$.

- Ask students to extend principles used with equations to inequalities.
  - Give students the inequality $3 < 7$ and ask them to determine whether the addition property of equality applies to inequalities. Students could use the work shown to investigate this situation.

$3 < 7$	$3 < 7$	$3 < 7$
$3 + 3 \overset{?}{<} 7 + 3$	$3 + \frac{3}{4} \overset{?}{<} 7 + \frac{3}{4}$	$3 - 15 \overset{?}{<} 7 - 15$
$6 < 10$	$3\frac{3}{4} < 7\frac{3}{4}$	$-12 < -8$

Students can conclude that addition of a number to both sides of the inequality preserves the inequality.

  - Give students the inequality $8 > 3$ and ask them to determine whether the multiplication property of equality applies to inequalities. Students could use the following work to investigate this situation.

$8 > 3$	$8 > 3$	$8 > 3$
$8 \bullet 5 \overset{?}{>} 3 \bullet 5$	$8 \bullet \frac{1}{4} \overset{?}{>} 3 \bullet \frac{1}{4}$	$8 \bullet -3 \overset{?}{>} 3 \bullet -3$
$40 > 15$	$2 > \frac{3}{4}$	$-24 < -9$

Students can conclude that multiplication preserves the inequality only when multiplying by a positive number since multiplication by a negative number reverses the inequality.

Graphs of inequalities

- Ask students to explain the meaning of an open point in the graph of an inequality. For example, give students the number line shown and ask students to explain the meaning

Students explain that the number line represents a solution set that does not include the number 3 and is represented by the inequality $x < 3$.

- Ask students to represent a given situation using both an inequality and a number line. For example, give students the following situation: "A family is picking apples. Currently, 3 pounds of apples have been picked and the family picks apples at a rate of 2 pounds of apples per minute. The family wants to pick more than 30 pounds of apples so they can make and sell applesauce. How many minutes does the family need to spend picking apples in order to meet its quota?" The number of pounds of apples picked can be represented by the expression $3 + 2m$, where m is the number of minutes spent picking apples. The phrase "more than" indicates that the value of the expression must exceed the stated amount of 30. Thus, the inequality

  $3 + 2m > 30$ represents the situation. The variable can be isolated as shown.

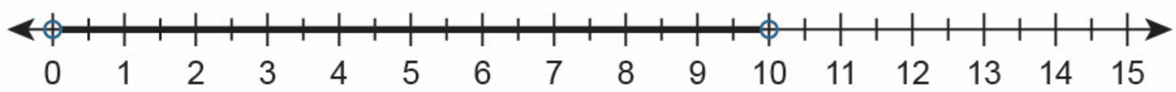
$$
\begin{array}{r}
3 + 2m > 30 \\
-3 \qquad\quad -3 \\
\hline
\dfrac{2m}{2} > \dfrac{27}{2} \\
m > 13.5
\end{array}
$$

The solution set can then be represented as shown on the number line.

Interpreting solutions of inequalities

- Ask students to interpret a solution set to an inequality in the context of a given situation. For example, give students the following situation: “The depth of water in a pool is 9 inches. Each hour the depth increases by 0.25 inches. After how many hours will the depth of water in the pool be more than 15 inches?” This situation can be represented by the inequality $9 + 0.25h > 15$, where h is the number of hours. The solution set for this inequality is $h > 24$. Students can interpret this solution set as indicating that at any time after 24 hours the depth of water in the pool will be more than 15 inches.
- Ask students to interpret a solution set to an inequality in the context of a given situation where the context has an implied restriction on the solution set. Two examples are shown.
  - “The prices of all items in a store are less than \$10.” This can be represented by the inequality $p < 10$, where p is the price of an item at the store. This solution set can be represented on a number line as shown.

The solution set of the inequality does not have an algebraic lower limit; however, in the given context, negative numbers do not make sense. The solution set, in context, can be represented by the inequality $0 < p < 10$, which can be represented on a number line as shown.



- "A naturalist is counting the number of deer in a park. The naturalist wants to find over 150 deer." This situation can be represented by the inequality $d > 150$. Algebraically, the inequality includes solutions that are not whole numbers; however, in the given context, a number that is not a whole number does not make sense. In context, the solution set contains only whole numbers greater than 150: {151, 152, 153, 154, 155, . . .}.

► Identifying and Addressing Common Misconceptions

- Some students may include an endpoint in the solution set of an exclusive inequality. Ask students to identify the least integer in the solution set of the inequality $8x + 1 > 17$. Students who identify 2 as the least integer in the solution set may have this misconception. Ask students to substitute the integer into the expression containing the variable in the given inequality and to compute the value. Then ask students to compare that result to the constant side of the given expression by using >, <, or = and to compare that symbol to the symbol used in the given inequality.
- Some students may not consider the context when interpreting a solution set. Ask students to identify the least value in the solution set of the inequality $2x – 80 > 115$ in the following context: "A store spends $80 for material to make small towels. Each towel is sold for $2. How many towels need to be sold to make a profit of more than $115?" Students who respond with 97.5 may have this misconception. Ask students to interpret the meaning of 97.5 in the context, focusing on the meaning of the part of the number to the right of the decimal point.

- Some students may not reverse the inequality symbol when multiplying or dividing by a negative number. Ask students to solve the inequality $-2x + 5 > 11$. Students who give an answer of $x > -3$ may have this misconception. With an example using –5 and 3, demonstrate that while $-5 < 3$, if both sides of the inequality are multiplied by –1, the new inequality relationship is not $5 < -3$. The inequality symbol must be reversed to $5 > -3$. Help students recognize that when signs are reversed, then comparisons between the numbers are also reversed.

► Enrichment

- Ask students to write inequalities for simple cases with two unknown quantities. For example, give students the following context: "A student is making cards. The student will make x cards today and y cards tomorrow. Between the two days, the student needs to make more than 28 cards. Represent this situation with an inequality." Students can write the inequality $x + y > 28$.
- Ask students to identify solutions to inequalities with two variables. For example, give students the following scenario: "The first two customers at a store buy chairs. The first customer buys x chairs and the second customer buys y chairs. The number of chairs the store sells to the first two customers can be represented by the inequality $x + y < 10$. What are possible numbers of chairs the customers could have bought?"
- Give students an inequality and ask them to write a real-world problem using that inequality.

► Key Terms

equivalent, expression, greater than (>), inequality, less than (<), properties of operations, solution set

B-E.2.1.1 & B-E.2.3.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Solve real-world and mathematical problems using numerical and algebraic expressions, equations, and inequalities.

Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers.

► Eligible Content

B-E.2.1.1 Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate.

Determine the reasonableness of the answer(s) in problem solving situations.

B-E.2.3.1 Determine the reasonableness of an answer(s), or interpret the solution(s) in the context of the problem.

► Instructional Supports

- Ask students to solve real-life problems by working backward. For example, give students the following situation: "A runner is completing a route in multiple stages. The runner's goal is to finish all stages with an average time of 0.55 hours per stage. Over the first four stages the runner has averaged 0.6 hours per stage. The runner has one stage remaining. What time does the runner need on the final stage to reach the goal?" The time needed for all five stages can be found by multiplying the average by 5: $0.55 \cdot 5 = 2.75$. The total time for the first four stages can be found using the average of 0.6 multiplied by the number of completed stages: $0.6 \cdot 4 = 2.4$. The time required on the last stage can then be found by subtracting the current time from the goal: $2.75 – 2.4 = 0.35$. In order to achieve the goal, the runner needs to complete the final stage in 0.35 hours.
- Ask students to write expressions to represent real-world situations. For example, give students the following situation: "The price of a meal at a restaurant is discounted by 5%, and a coupon is used to reduce the price by an additional $2. The discount is applied before the coupon. A tax of

10% is then added to the price of the meal. The final cost of the meal is \$24.97." Ask students to write an expression to represent the original price. The tax was calculated last, so the final cost is divided by 1.1 to find the price with both the discount and the coupon: 24.97 ÷ 1.1. The coupon was applied in the step just before the tax, so 2 needs to be added to (24.97 ÷ 1.1) to find the price of the meal with only the 5% discount: 2 + (24.97 ÷ 1.1). That discount needs to then be taken into account by dividing the whole expression by 0.95: ((24.97 ÷ 1.1) + 2) ÷ 0.95. This can be evaluated to determine that the original price of the meal was \$26.

- Ask students to solve problems in which numbers are given in multiple forms. For example, ask students to evaluate the expression $\frac{5}{3} \cdot 1\frac{1}{2} + 2.25$. Students can rewrite $1\frac{1}{2}$ as the fraction $\frac{3}{2}$. Then, the fractions can be multiplied to find a product of $\frac{5}{2}$. The fraction $\frac{5}{2}$ can then be converted to decimal form (2.5) and added to 2.25 to arrive at a sum of 4.75.
- Ask students to explain why a particular form of a rational number may be more efficient in a specific situation.
  - Give students the expression $0.2 \cdot \frac{1}{8}$. Converting the decimal to a fraction allows for the multiplication to be performed by multiplying the numerators and the denominators, which is the most efficient method. Converting the fraction to a decimal would require using the standard algorithm to multiply a one-digit number by a three-digit number.
  - Give students the expression $\frac{7}{10} + 0.424$. Converting the fraction to a decimal allows for the addition to be performed by the standard algorithm, which is the most efficient method. Converting the decimal to a fraction would require converting the fractions to have a common denominator.
- Ask students to use estimation to assess the reasonableness of an answer. Two examples are shown.
  - Give students the product 3.07 • 8.7 and ask them to use estimation to determine whether 26.709 is a reasonable answer. The product can be estimated by approximating 3.07 as 3 and 8.7 as 9. The estimated product is then 27. The tens value is the same in both cases,

and the ones digit is off by only 1. In addition, because 7 • 7 = 49, the product should end with a 9. Therefore, 26.709 appears to be a reasonable estimate.

- Give students the following situation: "A student calculated the product of 77.8 and 87.14 to be 67.79492." Ask students to explain, without evaluating, why this product is not reasonable. The factors 77.8 and 87.14 can be approximated as 80 and 90, respectively. The product can be approximated as 80 • 90, which is equal to 72 • 100, or 7,200. Since the student's product is not close to the estimate, the product is not reasonable. Multiplying the student's product by 100 gives a value that is much closer to the estimate, so the error is likely to be one of place value.

► Identifying and Addressing Common Misconceptions

- Some students, when working backward, may reverse the operations without reversing the order. Give students the following scenario: "A customer has a $5-off coupon applied to his purchase before also receiving a 10% discount. The customer pays $44.46. What was the customer's bill before the coupon and discount?" Some students may recognize that the final price can be calculated from the starting price by subtracting 5 and then multiplying by 0.9, but they may try to reverse the process by adding 5 and then dividing by 0.9. Ask students to think about the process of putting on socks and shoes. Socks must be put on before shoes, but when taking them off, the shoes must come off first. Help students recognize that not only do the operations need to be reversed, but the order in which those operations are performed also needs to be reversed.
- Some students may convert numbers into forms that lead to inexact answers. Ask students to calculate $\frac{4}{3} \div 0.75$ by changing the numbers into the same form. Some students may rewrite the expression as $1.\overline{3} \div 0.75$, thus needing to round the first decimal in order to calculate the product. Ask students whether $\frac{4}{3}$ is equal to 1.3. Then ask them whether $\frac{4}{3}$ is equal to 1.33. Students should conclude that these are good approximations but not exact answers.
- Students may not consider the scale of the numbers when identifying whether an estimate is reasonable. Ask students to use estimation to check the reasonableness of the equation 689 • 4,122 = 2,840,058. Some students may round the factors to 700 and 4,000, get an estimate of 2,800,000, recognize that this is more than 40,000 off from the given product, and decide that

the estimate is not reasonable. Ask students to compare a person growing 2 centimeters with an ant growing 1 centimeter. Help them recognize that the size of the original amount affects whether a change in the original amount is considered a big change.

► Enrichment

- Ask students to explore real-world situations involving percentages in which order does not matter. For example, give students the following situation: “The price of a chair is reduced by 15%, and a customer has a coupon for 5% off one item. Which order of applying the coupon and discount gives the lowest price?”
- Ask students to explore real-world situations using percentages in which order matters. For example, give students the following situation: “The water level in a pond will be increased by 20% and also increased by 5 inches. The two operations can be performed in either order. What is the difference between the final depths when the operations are performed in each possible order?”
- Ask students to use rounding and estimation to determine a range of possible values for an answer. For example, give students the expression 170.3 • 374.9 and ask them to estimate a range of values for the product. Students can estimate that the product is greater than 150 • 300 = 45,000 and that it is less than 200 • 400 = 80,000.

► Key Terms

equivalent, estimation, expression, place value, properties of operations

C-G.1.1.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

Geometry

Demonstrate an understanding of geometric figures and their properties.

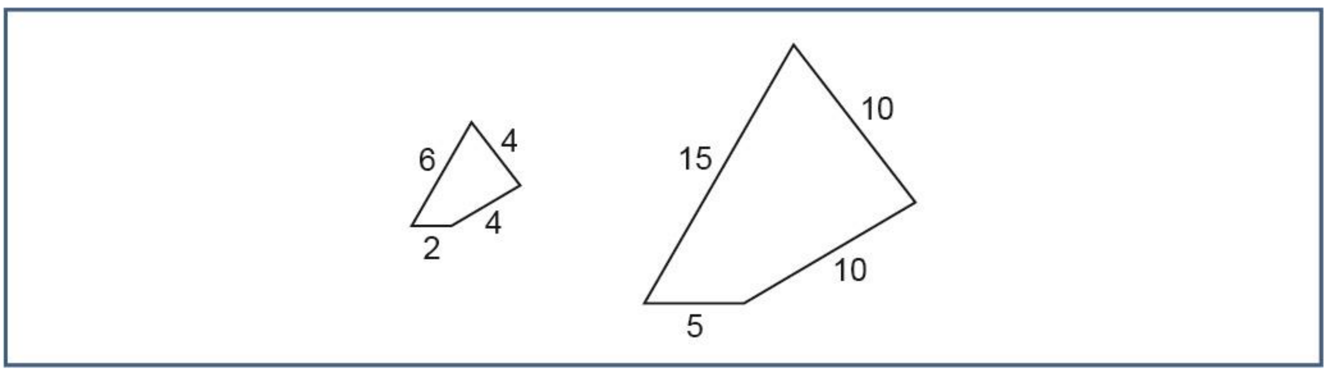
Describe and apply properties of geometric figures.

► Eligible Content

C-G.1.1.1 Solve problems involving scale drawings of geometric figures, including finding length and area.

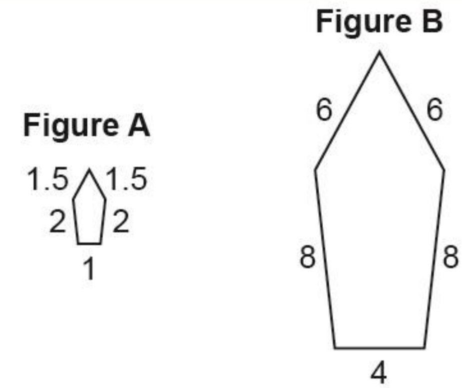
► Instructional Supports

- Ask students to determine and explain relationships between scale drawings. For example, give students the following figures and explain that the figure on the right is a scale drawing of the original figure on the left.



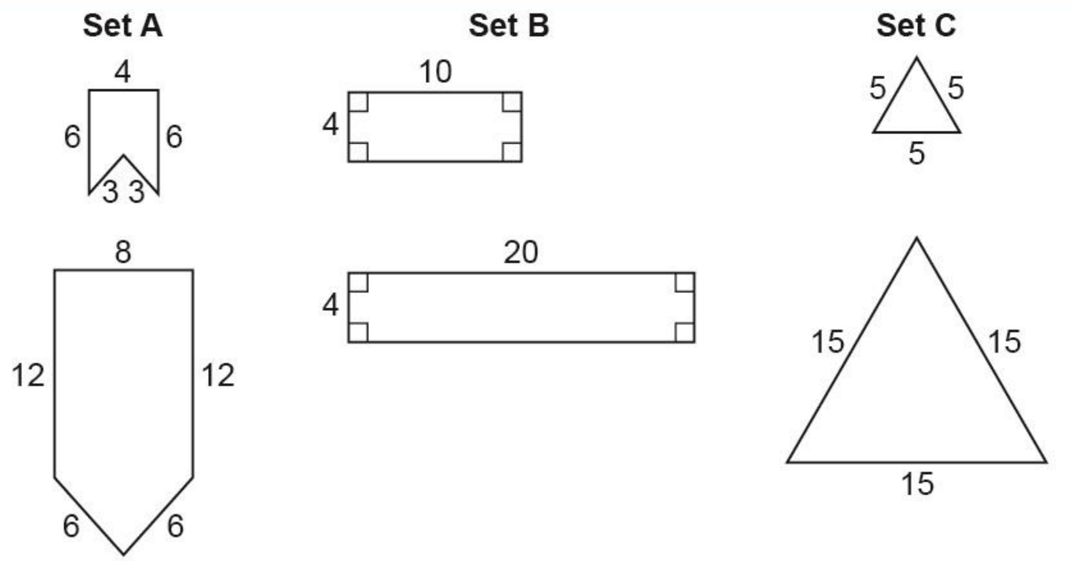
Explain to students that scale drawings are enlarged or shrunken versions of each other where the ratio of the side lengths of one figure to the corresponding side lengths of the other figure are equal and angle measures remain unchanged. Ask students to divide the corresponding side lengths of the figures to show that the length of each side of the larger figure is 2.5 times the length of the corresponding side in the original figure. Ask students to measure the angles of the figures to show that the corresponding angles in each figure are the same.

- Ask students to identify the scale factor between two figures in a scale drawing. For example, give students the following two figures and explain that each side length of one figure can be multiplied by the same scale factor to create the corresponding side length of the other figure.



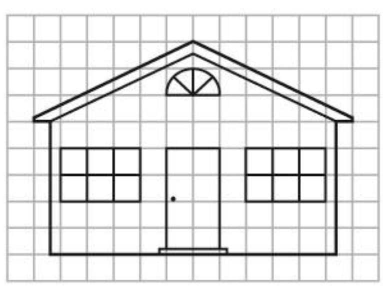
Students should find that each side length of figure B is 4 times the corresponding side length of figure A. Explain to students that this means the scale factor from figure A to figure B is 4. Similarly, the side lengths of figure A are 0.25 times the corresponding side lengths of figure B, so the scale factor from figure B to figure A is 0.25. Help students understand that there are two possible scale factors: one can be used to create the larger figure, starting with the smaller figure, and one can be used to create the smaller figure, starting with the larger figure.

- Ask students to determine whether one figure is a scale drawing of another figure. Ask students to compare the three sets of figures shown.

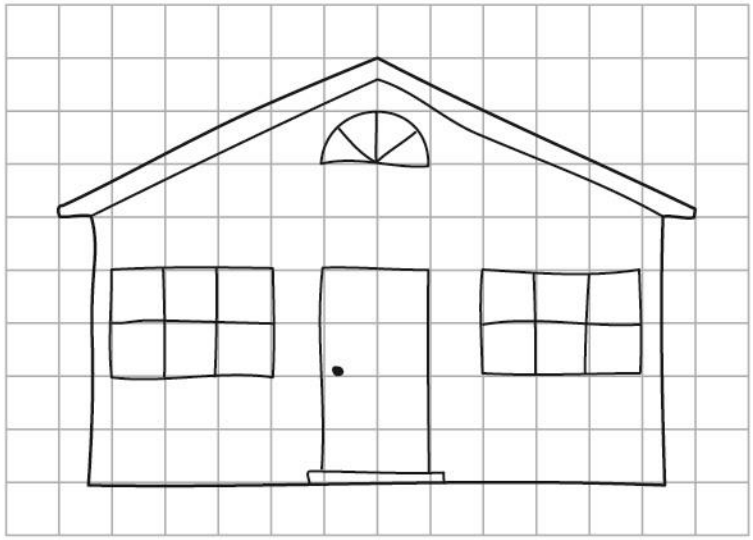


Students should determine that the figures in set A are not scale drawings of each other because the corresponding angles of the figures are not the same. Students should also find that the figures in set B are not scale drawings of each other because each of the pairs of corresponding sides are not multiplied by the same scale factor. The figures in set C have corresponding angles with equal measures as well as corresponding sides showing the same scale factor of 3. Therefore, students should conclude that the figures in set C are scale drawings of each other.

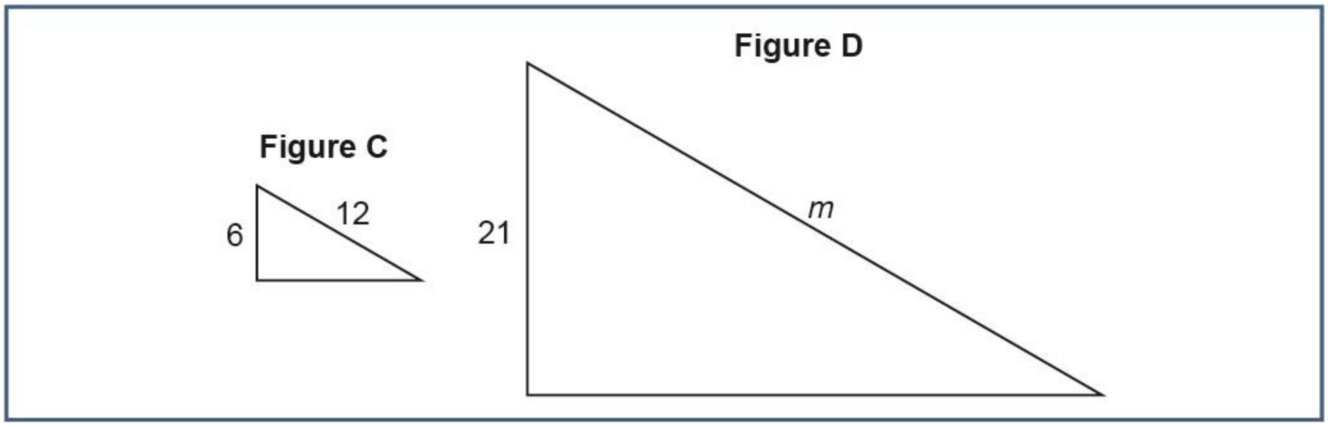
- Ask students to create scale drawings on a grid. For example, give students the drawing shown.



Then give students paper with grid spaces that are twice as tall and twice as wide as the original grid. Ask students to create a scale drawing by drawing each square from the original drawing onto the corresponding square of the new larger grid, as shown in the following example.

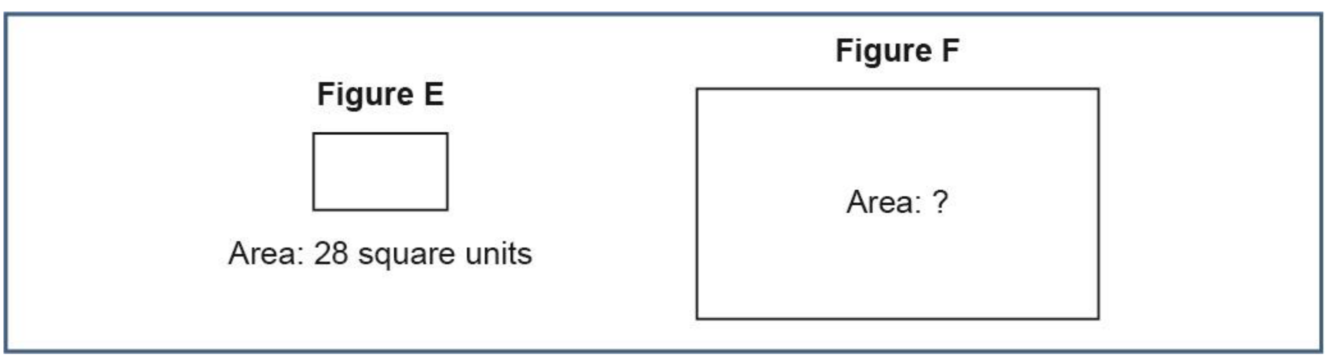


- Ask students to determine an unknown side length of a figure in a scale drawing. For example, give students the following figures and tell them that figure D is a scale drawing of figure C.



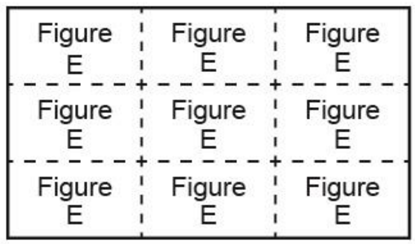
Ask students to identify the pairs of corresponding sides in the figures. Students should find that the side with a length of 6 units in figure C corresponds to the side with a length of 21 units in figure D. Therefore, the scale factor from figure C to figure D is 3.5 because 6 • 3.5 = 21. Students should then identify that the side with a length of 12 units in figure C corresponds to the side with a length of *m* units in figure D. Students can find the value of *m* by multiplying 12 by the scale factor, 3.5. Students should determine that the unknown side length, *m*, in figure D is 42 units.

- Give students a pair of scale drawings, the scale factor between them, and the area of one of the figures and ask them to find the area of the other figure. For example, give students the following figures and ask them to find the area of figure F if figure F is a scale drawing of figure E with a scale factor of 3.



Ask students to list possible dimensions for figure E and to use those dimensions to find the dimensions and the area for figure F. Then ask students to compare their answers. Students should notice that regardless of the starting dimensions, the area of figure F is 252 square units. Ask students to cut out copies of figure E and show that because the dimensions of figure F are 3 times the dimensions of figure E, then 3^2 copies of figure E can be created inside figure F. Therefore, since the area of figure E is 28 square units, the area of figure F is $28 \cdot 3^2 = 252$ square units.

Figure F



In general, for any figure that is created by a scale factor of n, the area of the figure created is n^2 times the area of the original figure.

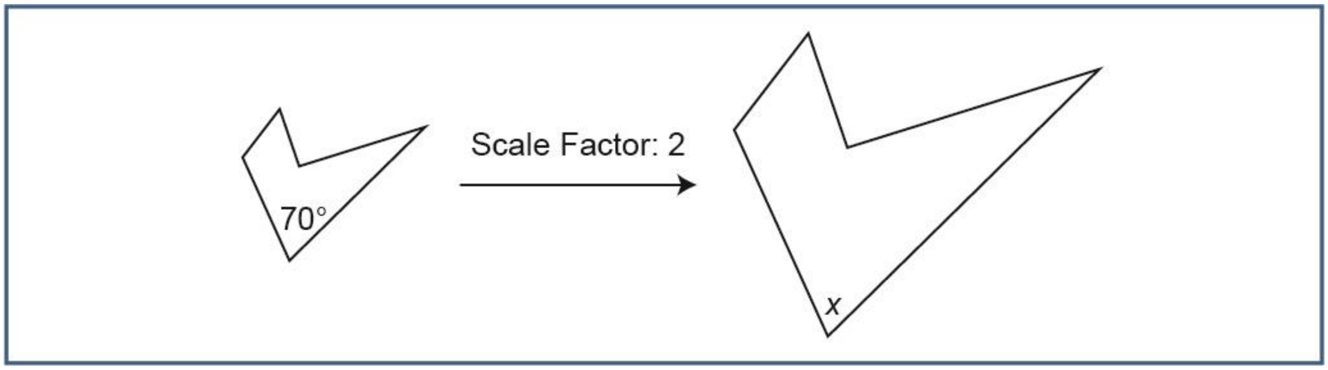
- Ask students to use scale factors to find the dimensions and areas of figures in real-world contexts. For example, ask students to determine the dimensions of a scale drawing of a rectangular park that is 4.6 miles long and 3.1 miles wide using the scale factor 0.25 miles = 3 inches. Students should find that because $4.6 \div 0.25 = 18.4$ and because $18.4 \cdot 3 = 55.2$, the scale drawing will be 55.2 inches long. Similarly, because $3.1 \div 0.25 = 12.4$ and because $12.4 \cdot 3 = 37.2$, the scale drawing will be 37.2 inches wide.

► Identifying and Addressing Common Misconceptions

- Some students may think that the scale factor between two figures is the same regardless of which figure is referred to as the original figure. Tell students that figure B is an enlargement of figure A by a scale factor of 10 and ask them to identify the scale factor from figure B to figure A. Some students may give an answer of 10. Ask students to create a figure, called figure A, and use a scale factor of 10 to create a scale drawing, called figure B. Help students understand that the scale factor is 10 because the length of each side of figure A can be multiplied by 10 to get the length of the corresponding side in figure B. Then show that creating a scale drawing by multiplying the side lengths of figure B by 10 does not result in figure A.

Help students recognize that multiplying the side lengths of figure B by 0.1 or $\frac{1}{10}$ results in a scale drawing of figure A. Therefore, the scale factor from figure B to figure A is $\frac{1}{10}$.

- Some students may think that angle measures change by the same scale factor as the sides. Give students the following figure and ask them to find the value of *x*.



Some students may give an answer of 35 degrees or 140 degrees. Ask students the measures of the angles of a square. Then ask them to draw two different-sized squares and ask them the measures of the angles. Help students understand that regardless of the scale factor, the corresponding angles of the figures are always the same.

- Some students may think that area changes by the same scale factor as the sides. Give students a square with an area of 16 square units and ask them to identify the area of the square after being enlarged by a scale factor of 5. Some students may give an answer of 80 square units. Ask students to draw the enlarged square and ask them how many copies of the original square would fit along the bottom of the larger square. Then ask them how many copies of the original square would fit along the left side of the larger square. Help them conclude that 25 copies of the original square would fit inside the larger square and that the area of the larger square is 25 times the area of the original square.

► Enrichment

- Ask students to find the scale factor used to create figure C directly from figure A when figure B is a scale drawing of figure A and figure C is a scale drawing of figure B.

- Give students a figure and ask them to determine what scale factors can be used to create scale drawings that would fit on paper of a given size. Furthermore, ask them to find the scale factor that uses up as much of the paper as possible while still fitting on the paper.
- Ask students to use modeling clay or blocks to create a pair of three-dimensional scale models.

► Key Terms

area, geometrical figure, proportion, ratio, scale drawing, scale factor

C-G.1.1.2 & C-G.1.1.3

► Reporting Category, Assessment Anchor, Anchor Descriptor

Geometry

Demonstrate an understanding of geometric figures and their properties.

Describe and apply properties of geometric figures.

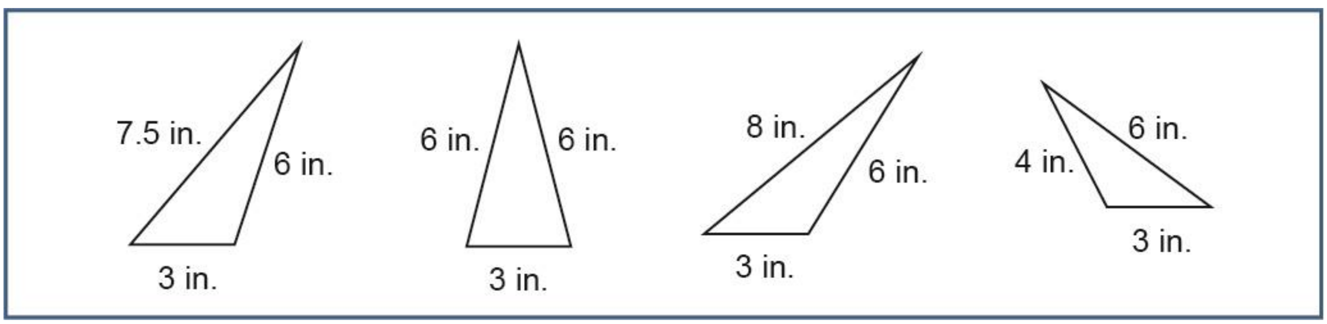
► Eligible Content

C-G.1.1.2 Identify or describe the properties of all types of triangles based on angle and side measure.

C-G.1.1.3 Use and apply the triangle inequality theorem.

► Instructional Supports

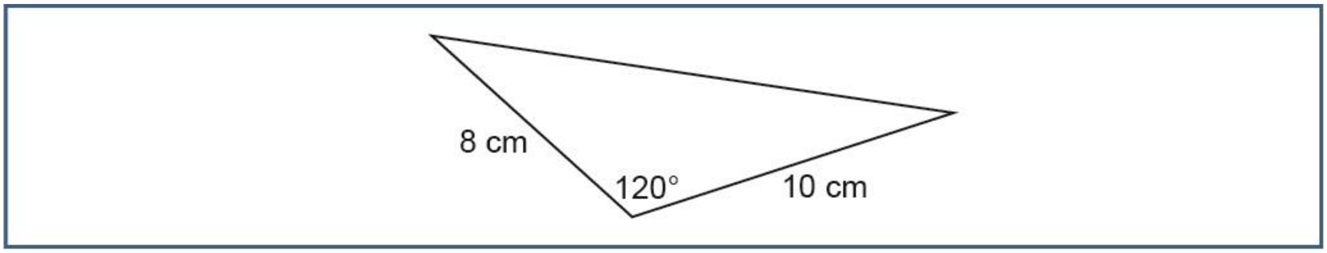
- Help students investigate the triangle inequality theorem by asking them to draw several triangles that have two sides with given lengths and to record the lengths of all three sides in a table. For example, ask students to draw several triangles with at least one side length of 3 inches and at least one side length of 6 inches and to record the lengths of all three sides in a table. Some possible figures and measurements are shown.



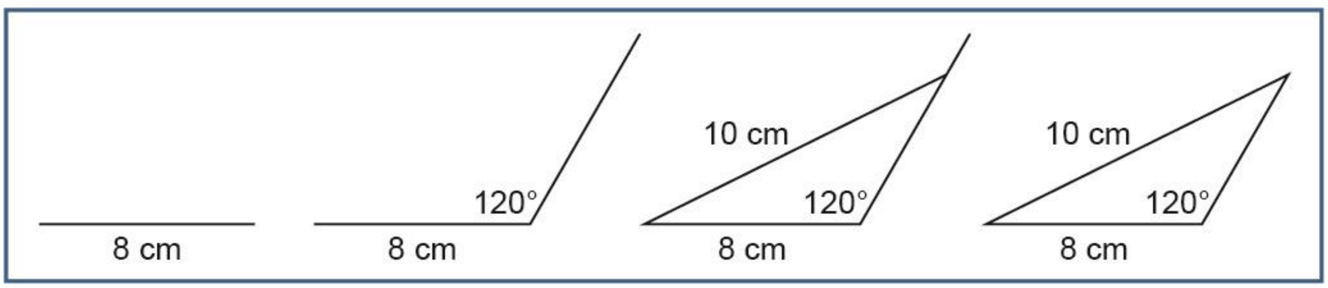
Length of First Side	Length of Second Side	Length of Third Side
3 in.	6 in.	7.5 in.
3 in.	6 in.	6 in.
3 in.	6 in.	8 in.
3 in.	6 in.	4 in.

Have students share the side lengths of the triangles they created and record the results on a class chart. Then ask students to discuss any patterns they notice in the chart. Guide students to see that the length of the third side is always less than 9 inches. Ask students to try to create a triangle where the length of the third side is 9 inches or longer, and after they try, have them discuss with partners why this is not possible. Help students conclude that a triangle with given side lengths can be constructed as long as the sum of the two shorter side lengths is greater than the length of the longest side.

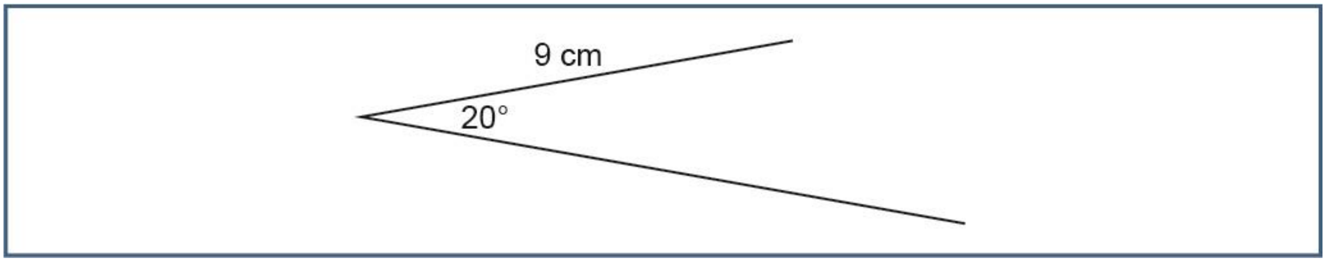
- Ask students to use manipulatives such as popsicle sticks, chopsticks, or pipe cleaners to help draw a triangle with a given set of side lengths. For example, ask students to draw a triangle with side lengths of 3 inches, 5 inches, and 7 inches. Students can use a ruler and scissors to measure and cut sticks that are 3 inches, 5 inches, and 7 inches in length. Then students can use the sticks to form a triangle on a piece of paper. Have students trace the triangles onto the piece of paper, labeling the side lengths. Ask students to complete their drawings by using a protractor to measure and label all the angles of the triangle. Then have students compare their triangles with those of other students. Help students recognize that if they know the three side lengths of a triangle, then there is a unique triangle that can be constructed from those side lengths.
- Ask students to draw two different triangles with two given side lengths and a given angle measure. For example, ask students to draw two different triangles with side lengths of 8 centimeters and 10 centimeters and an angle measure of 120 degrees. Two examples are shown.
    - Some students may draw a triangle in which the 120-degree angle is between the two given sides. Students may begin by using a ruler to draw a line segment with a length of 8 centimeters. Then students can place a protractor at one end of the line segment and use it to create an angle that measures 120 degrees. Using the ruler, students can then extend the new line segment to a length of 10 centimeters to create a triangle such as the following.



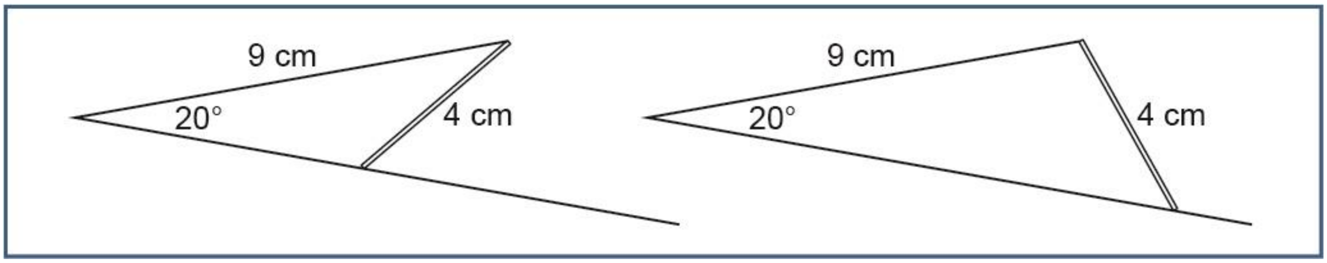
- Some students may draw a triangle in which the 120-degree angle is not between the two given sides. Students may draw such a triangle by first using a ruler to draw a line segment with a length of 8 centimeters. Then students can place a protractor at one end of the line segment and use it to draw a second segment to create an angle that measures 120 degrees. Tell students they can adjust the length of the second segment as they complete the third side of the triangle. To draw the third side of the triangle, students can use a ruler to draw a segment with a length of 10 centimeters that extends from the opposite side of the first segment and connects with the second segment. Students may need to adjust the length of the second segment at this time. These steps are shown in the following diagram.



- Help students recognize that sometimes two different triangles can be drawn when given two side lengths and an angle measure, even when the location of the angle is specified. For example, ask students to use chopsticks to help create a triangle with side lengths of 4 centimeters and 9 centimeters and a 20-degree angle that is not between the given side measures. Have students draw a 9-centimeter segment and a 20-degree angle at one end of the segment. Students can make the other side of the angle longer than is needed.

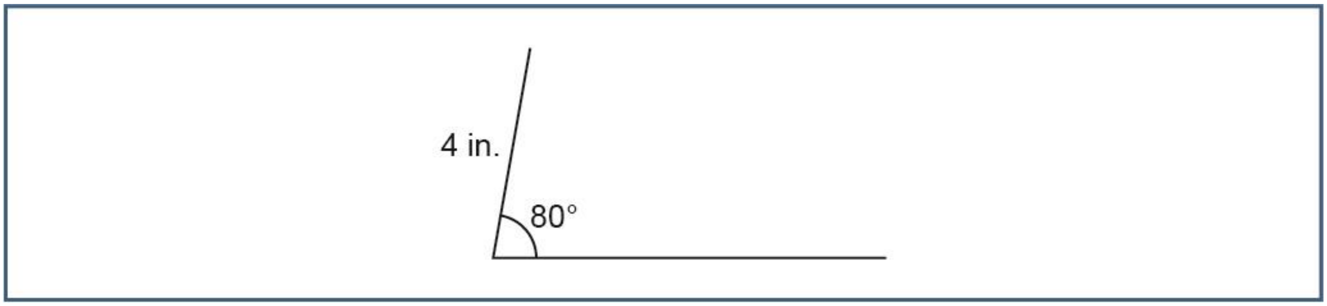


Then give students a 4-centimeter stick and ask them to place it so that one end is at the unconnected end of the 9-centimeter segment and the other end lies anywhere on the other line. Students should notice that there are two ways that the 4-centimeter stick can be placed.

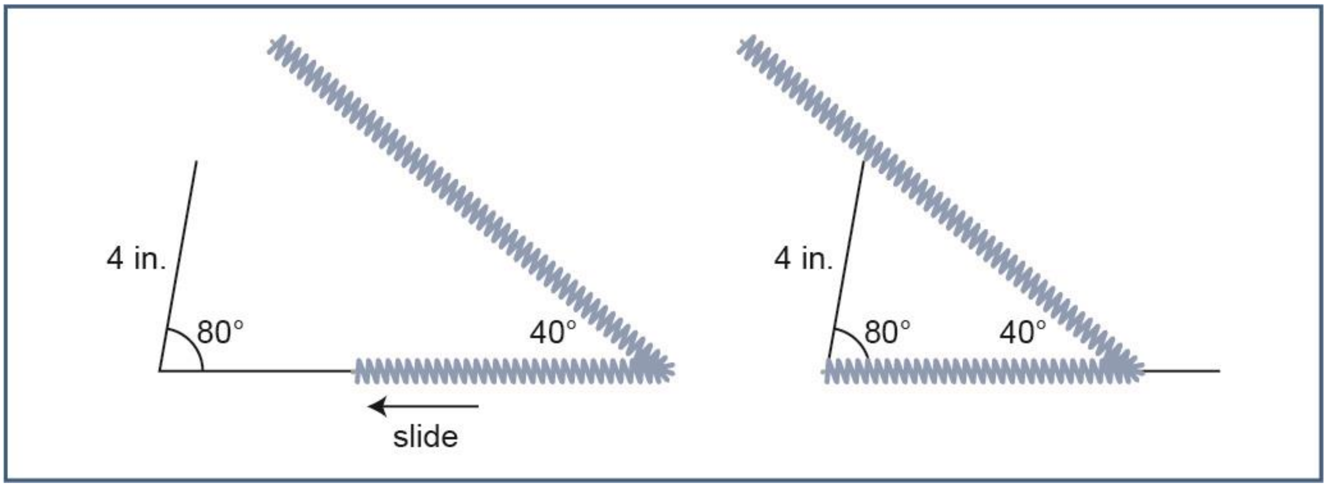


Ask students to discuss why these two triangles are different from one another.

- Ask students to investigate different triangles that can be created with a given side length and two given angle measures. For example, ask students to draw triangles with a side length of 4 inches and angle measures of 40 degrees and 80 degrees and to discuss how to create different triangles having these same specifications. The three possible triangles are described.
    - Some students may draw a triangle in which the 4-inch side is between the two given angles. Students may begin by using a ruler to draw a line segment with a length of 4 inches. Then students can place a protractor at one end of the line segment to help draw a second segment that creates an angle with a measure of 80 degrees and use the same procedure to create a 40-degree angle at the other end of the line segment.
    - Some students may draw a triangle in which the 4-inch side is not between the two given angles. Students may begin by using a ruler to draw a line segment with a length of 4 inches. Then students can place a protractor at one end of the line segment to create an angle with a measure of 80 degrees. Students should extend the second line segment farther than the anticipated length of that side of the triangle.



Students can adjust this length later. Next, ask students to bend a pipe cleaner into a 40-degree angle and slide one side of the angle along the second segment until the other side meets up with the other end of the 4-inch segment.



- The 4-inch side can also be drawn adjacent to the 40-degree angle instead of the 80-degree angle, and then the same pipe cleaner method can be used to create the 80-degree angle, as is described above.

- Ask students to explore triangles that can be drawn when given three angle measures. Two examples are described.
  - Ask students to use a ruler and a protractor to draw a triangle with angles that measure 35 degrees, 65 degrees, and 80 degrees. Next, ask students to draw a second triangle with angles of the same measures but to begin by drawing a side of the triangle that is double the length of one of the sides of the first triangle. Students should note that there are infinitely many triangles that can be drawn with the given angle measures.

  - Ask students to use a ruler and protractor to draw a triangle with angles that measure 50 degrees, 75 degrees, and 85 degrees. Students should find that it is impossible to draw a triangle with the given angle measures. Ask students to explore more combinations of angle measures and help them notice that triangles can be drawn only when the angle measures sum to 180 degrees.

- Ask students to use rulers, protractors, or technology to determine whether a given set of conditions can create one unique triangle, more than one triangle, or no triangle. For example, ask students to determine whether a unique triangle, more than one triangle, or no triangle can be created given each of the conditions listed and to explain their thinking.
  - side lengths of 14 meters, 31 meters, and 50 meters
  - side lengths of 2.5 inches, 2.5 inches, and 2.5 inches
  - side lengths of 4.2 centimeters and 7.5 centimeters and an angle measure of 55 degrees
  - a side length of 10 yards and angle measures of 100 degrees and 95 degrees
  - an acute angle, a right angle, and an obtuse angle

  Students should find that triangles can be created only with the second and third sets of criteria. The second set of criteria results in a unique equilateral triangle, and the third set of criteria can be used to create more than one triangle by changing where the given angle is in relation to the two given sides. It is impossible to create a triangle for the other three sets of criteria. For the first set of criteria, students should conclude that the sum of the two shorter lengths is not greater than the longest length, so no triangle can be created. Each of the last two sets of criteria have angle sums that are greater than 180 degrees, so no triangle can be created.

► Identifying and Addressing Common Misconceptions

- Students may think that because more than one triangle can be created with a given set of dimensions, an infinite number of triangles can be created with that set of dimensions. Ask students, “How many different triangles can be drawn with an angle that measures 37 degrees and side lengths that measure 8 inches and 3 inches?” Some students may say there are an infinite number of triangles that can be drawn with those measures. Ask students to draw as many different triangles as they can with the given measures. Have students practice with different sets of measures. Guide students to see that an infinite number of triangles can be drawn only when the given set of measures involves only angle measures.

- Students may think there is only one way to draw a triangle with a given set of dimensions. Ask students to draw a triangle with a side length of 5 centimeters and angles that measure 48 degrees and 72 degrees. Some students may believe this is the only triangle that can be created with the given measures. Ask students to examine the relationship between the 5-centimeter side and the two given angles. Guide them to find another way to position the 5-centimeter side and the two given angles to create a different triangle.

► Enrichment

- Ask students to determine all the possible lengths of the third side, *x*, of a triangle when given the other two side lengths and write an inequality to represent those side lengths.
- Ask students to compare the side lengths of a triangle when the angle measures are known and vice versa. Students should find that the sides opposite the largest angles are the longest sides. They should also conclude that when two sides of a triangle are the same length, the angles opposite of those sides must also have the same measure and vice versa.

► Key Terms

angle, condition, construct, geometric shape, protractor, triangle

C-G.1.1.2 & C-G.1.1.3

▶ Reporting Category, Assessment Anchor, Anchor Descriptor

Geometry

Demonstrate an understanding of geometric figures and their properties.

Describe and apply properties of geometric figures.

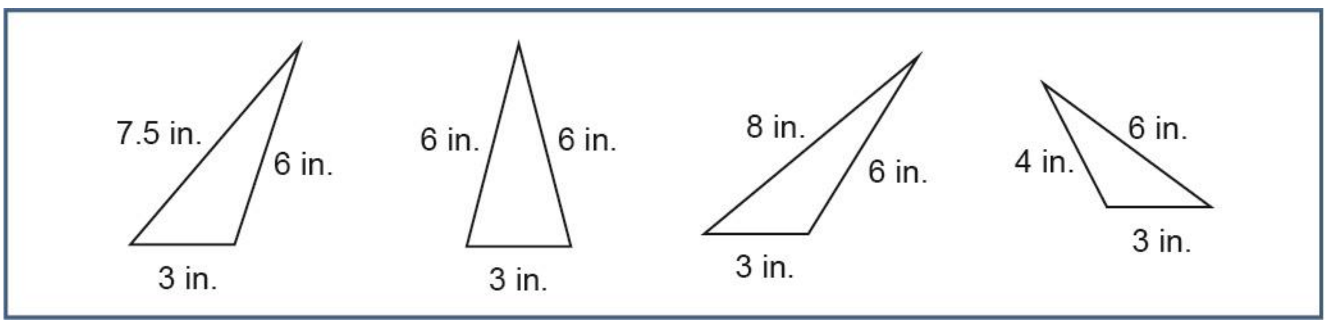
▶ Eligible Content

C-G.1.1.2 Identify or describe the properties of all types of triangles based on angle and side measure.

C-G.1.1.3 Use and apply the triangle inequality theorem.

▶ Instructional Supports

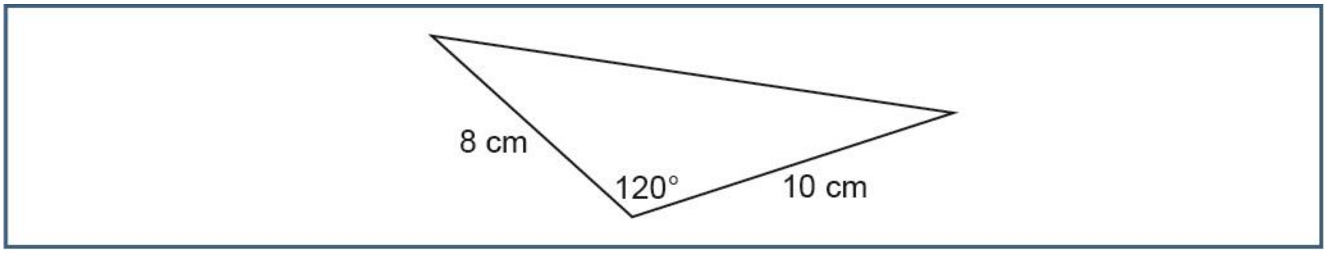
- Help students investigate the triangle inequality theorem by asking them to draw several triangles that have two sides with given lengths and to record the lengths of all three sides in a table. For example, ask students to draw several triangles with at least one side length of 3 inches and at least one side length of 6 inches and to record the lengths of all three sides in a table. Some possible figures and measurements are shown.



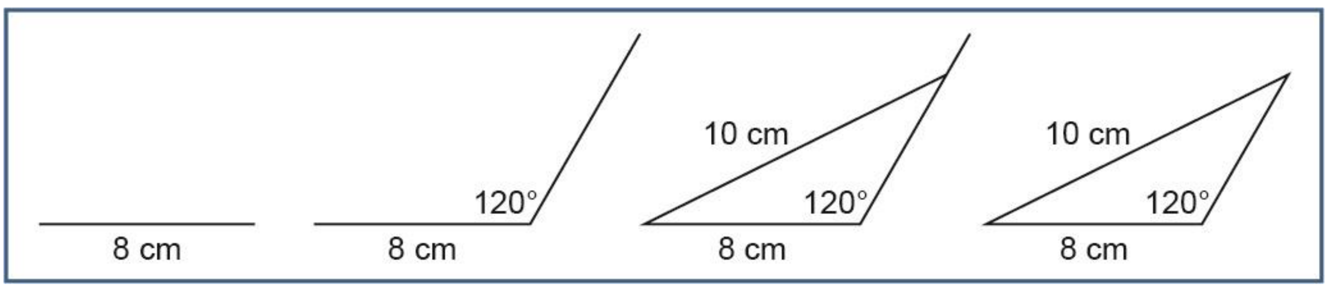
Length of First Side	Length of Second Side	Length of Third Side
3 in.	6 in.	7.5 in.
3 in.	6 in.	6 in.
3 in.	6 in.	8 in.
3 in.	6 in.	4 in.

Have students share the side lengths of the triangles they created and record the results on a class chart. Then ask students to discuss any patterns they notice in the chart. Guide students to see that the length of the third side is always less than 9 inches. Ask students to try to create a triangle where the length of the third side is 9 inches or longer, and after they try, have them discuss with partners why this is not possible. Help students conclude that a triangle with given side lengths can be constructed as long as the sum of the two shorter side lengths is greater than the length of the longest side.

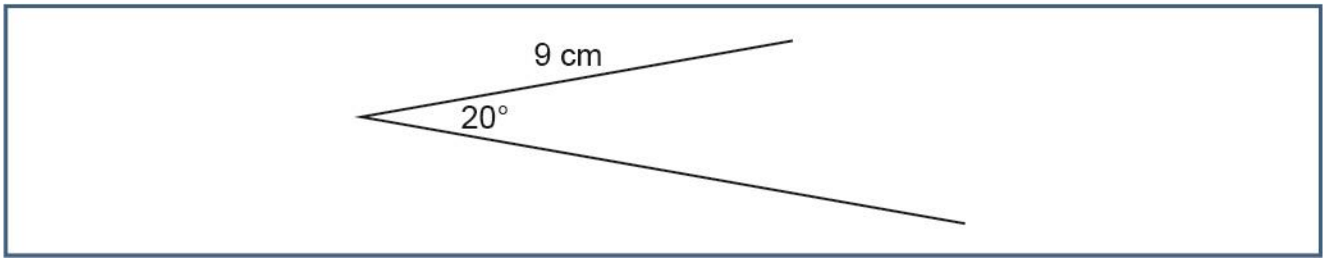
- Ask students to use manipulatives such as popsicle sticks, chopsticks, or pipe cleaners to help draw a triangle with a given set of side lengths. For example, ask students to draw a triangle with side lengths of 3 inches, 5 inches, and 7 inches. Students can use a ruler and scissors to measure and cut sticks that are 3 inches, 5 inches, and 7 inches in length. Then students can use the sticks to form a triangle on a piece of paper. Have students trace the triangles onto the piece of paper, labeling the side lengths. Ask students to complete their drawings by using a protractor to measure and label all the angles of the triangle. Then have students compare their triangles with those of other students. Help students recognize that if they know the three side lengths of a triangle, then there is a unique triangle that can be constructed from those side lengths.
- Ask students to draw two different triangles with two given side lengths and a given angle measure. For example, ask students to draw two different triangles with side lengths of 8 centimeters and 10 centimeters and an angle measure of 120 degrees. Two examples are shown.
    - Some students may draw a triangle in which the 120-degree angle is between the two given sides. Students may begin by using a ruler to draw a line segment with a length of 8 centimeters. Then students can place a protractor at one end of the line segment and use it to create an angle that measures 120 degrees. Using the ruler, students can then extend the new line segment to a length of 10 centimeters to create a triangle such as the following.



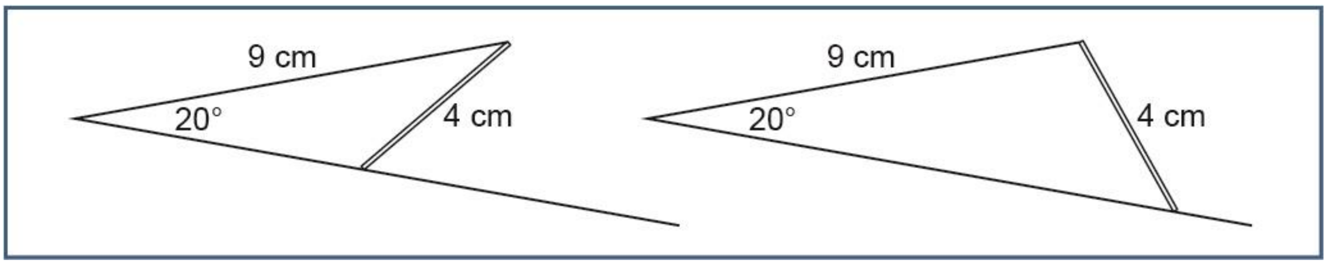
- Some students may draw a triangle in which the 120-degree angle is not between the two given sides. Students may draw such a triangle by first using a ruler to draw a line segment with a length of 8 centimeters. Then students can place a protractor at one end of the line segment and use it to draw a second segment to create an angle that measures 120 degrees. Tell students they can adjust the length of the second segment as they complete the third side of the triangle. To draw the third side of the triangle, students can use a ruler to draw a segment with a length of 10 centimeters that extends from the opposite side of the first segment and connects with the second segment. Students may need to adjust the length of the second segment at this time. These steps are shown in the following diagram.



- Help students recognize that sometimes two different triangles can be drawn when given two side lengths and an angle measure, even when the location of the angle is specified. For example, ask students to use chopsticks to help create a triangle with side lengths of 4 centimeters and 9 centimeters and a 20-degree angle that is not between the given side measures. Have students draw a 9-centimeter segment and a 20-degree angle at one end of the segment. Students can make the other side of the angle longer than is needed.

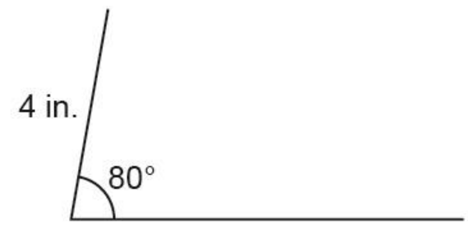


Then give students a 4-centimeter stick and ask them to place it so that one end is at the unconnected end of the 9-centimeter segment and the other end lies anywhere on the other line. Students should notice that there are two ways that the 4-centimeter stick can be placed.

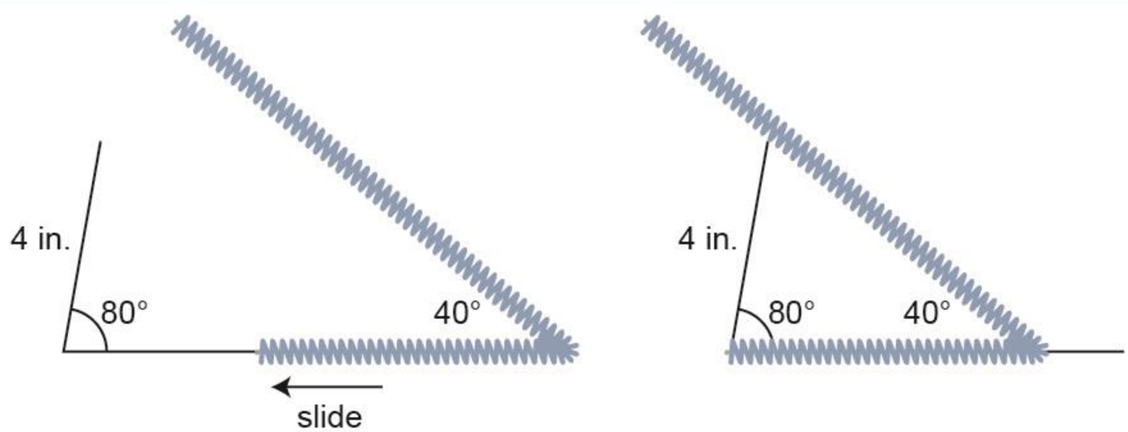


Ask students to discuss why these two triangles are different from one another.

- Ask students to investigate different triangles that can be created with a given side length and two given angle measures. For example, ask students to draw triangles with a side length of 4 inches and angle measures of 40 degrees and 80 degrees and to discuss how to create different triangles having these same specifications. The three possible triangles are described.
    - Some students may draw a triangle in which the 4-inch side is between the two given angles. Students may begin by using a ruler to draw a line segment with a length of 4 inches. Then students can place a protractor at one end of the line segment to help draw a second segment that creates an angle with a measure of 80 degrees and use the same procedure to create a 40-degree angle at the other end of the line segment.
    - Some students may draw a triangle in which the 4-inch side is not between the two given angles. Students may begin by using a ruler to draw a line segment with a length of 4 inches. Then students can place a protractor at one end of the line segment to create an angle with a measure of 80 degrees. Students should extend the second line segment farther than the anticipated length of that side of the triangle.



Students can adjust this length later. Next, ask students to bend a pipe cleaner into a 40-degree angle and slide one side of the angle along the second segment until the other side meets up with the other end of the 4-inch segment.



- The 4-inch side can also be drawn adjacent to the 40-degree angle instead of the 80-degree angle, and then the same pipe cleaner method can be used to create the 80-degree angle, as is described above.

- Ask students to explore triangles that can be drawn when given three angle measures. Two examples are described.
  - Ask students to use a ruler and a protractor to draw a triangle with angles that measure 35 degrees, 65 degrees, and 80 degrees. Next, ask students to draw a second triangle with angles of the same measures but to begin by drawing a side of the triangle that is double the length of one of the sides of the first triangle. Students should note that there are infinitely many triangles that can be drawn with the given angle measures.

- Ask students to use a ruler and protractor to draw a triangle with angles that measure 50 degrees, 75 degrees, and 85 degrees. Students should find that it is impossible to draw a triangle with the given angle measures. Ask students to explore more combinations of angle measures and help them notice that triangles can be drawn only when the angle measures sum to 180 degrees.

- Ask students to use rulers, protractors, or technology to determine whether a given set of conditions can create one unique triangle, more than one triangle, or no triangle. For example, ask students to determine whether a unique triangle, more than one triangle, or no triangle can be created given each of the conditions listed and to explain their thinking.
    - side lengths of 14 meters, 31 meters, and 50 meters
    - side lengths of 2.5 inches, 2.5 inches, and 2.5 inches
    - side lengths of 4.2 centimeters and 7.5 centimeters and an angle measure of 55 degrees
    - a side length of 10 yards and angle measures of 100 degrees and 95 degrees
    - an acute angle, a right angle, and an obtuse angle

    Students should find that triangles can be created only with the second and third sets of criteria. The second set of criteria results in a unique equilateral triangle, and the third set of criteria can be used to create more than one triangle by changing where the given angle is in relation to the two given sides. It is impossible to create a triangle for the other three sets of criteria. For the first set of criteria, students should conclude that the sum of the two shorter lengths is not greater than the longest length, so no triangle can be created. Each of the last two sets of criteria have angle sums that are greater than 180 degrees, so no triangle can be created.

▶ Identifying and Addressing Common Misconceptions

- Students may think that because more than one triangle can be created with a given set of dimensions, an infinite number of triangles can be created with that set of dimensions. Ask students, “How many different triangles can be drawn with an angle that measures 37 degrees and side lengths that measure 8 inches and 3 inches?” Some students may say there are an infinite number of triangles that can be drawn with those measures. Ask students to draw as many different triangles as they can with the given measures. Have students practice with different sets of measures. Guide students to see that an infinite number of triangles can be drawn only when the given set of measures involves only angle measures.

- Students may think there is only one way to draw a triangle with a given set of dimensions. Ask students to draw a triangle with a side length of 5 centimeters and angles that measure 48 degrees and 72 degrees. Some students may believe this is the only triangle that can be created with the given measures. Ask students to examine the relationship between the 5-centimeter side and the two given angles. Guide them to find another way to position the 5-centimeter side and the two given angles to create a different triangle.

► Enrichment

- Ask students to determine all the possible lengths of the third side, *x*, of a triangle when given the other two side lengths and write an inequality to represent those side lengths.
- Ask students to compare the side lengths of a triangle when the angle measures are known and vice versa. Students should find that the sides opposite the largest angles are the longest sides. They should also conclude that when two sides of a triangle are the same length, the angles opposite of those sides must also have the same measure and vice versa.

► Key Terms

angle, condition, construct, geometric shape, protractor, triangle

C-G.1.1.4

► Reporting Category, Assessment Anchor, Anchor Descriptor

Geometry

Demonstrate an understanding of geometric figures and their properties.

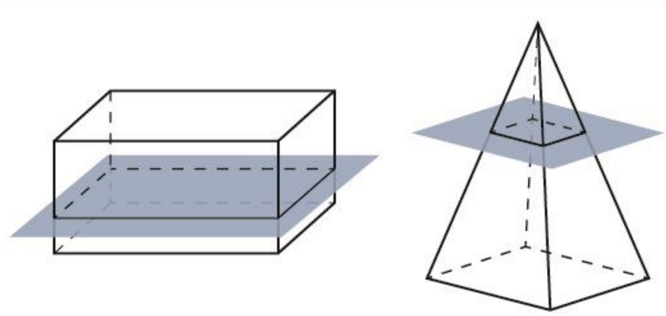
Describe and apply properties of geometric figures.

► Eligible Content

C-G.1.1.4 Describe the two-dimensional figures that result from slicing three-dimensional figures.

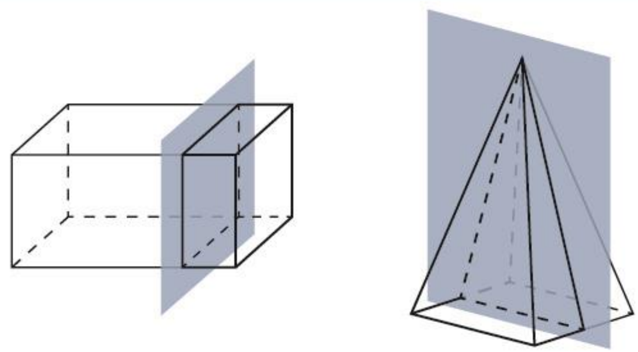
► Instructional Supports

- Give students a rectangular prism that can be sliced (e.g., a block of cheese or modeling clay) and ask them to make predictions about the two-dimensional shapes that can be created by slicing the prism. Then have them test their predictions by slicing the prism. Discuss which shapes are possible and which are not.
- Help students recognize that right rectangular prisms and right rectangular pyramids are composed of stacks of rectangles. For example, a stack of printer paper can illustrate horizontal cross sections of a right rectangular prism.
- Ask students to identify the shape of a cross section of a prism or pyramid that is cut parallel to its base. For example, give students a rectangular prism and a rectangular pyramid and have them determine the shape that results from slicing each figure parallel to its base.

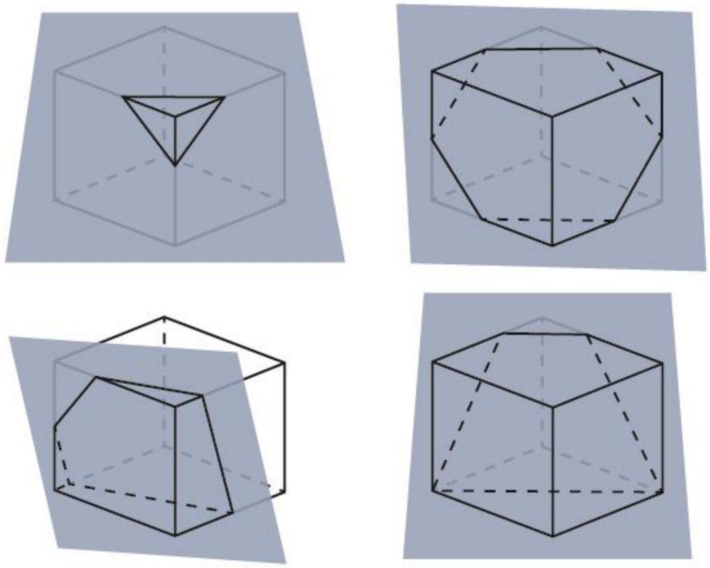


Students should recognize that the cross section of the prism is the same size and shape as the base of the prism and the cross section of the pyramid is the same shape as the base of the pyramid but smaller.

- Ask students to identify the shape of a cross section of a pyramid or prism that is cut perpendicular to its base. For example, give students a rectangular prism and a pyramid and have them determine the shapes that result from slicing each figure perpendicular to its base. Help students recognize that the cross section of the prism is a rectangle and the cross section of the pyramid could be either a triangle or a trapezoid. Two possible cross sections are shown.



- Give students a three-dimensional figure and ask them to identify possible shapes of its cross sections. For example, give students a cube and have them identify all the different cross sections that can be created. Some possible cross sections are shown.



A cross section of a cube could be a triangle, quadrilateral, pentagon, or hexagon depending on how the cube is cut.

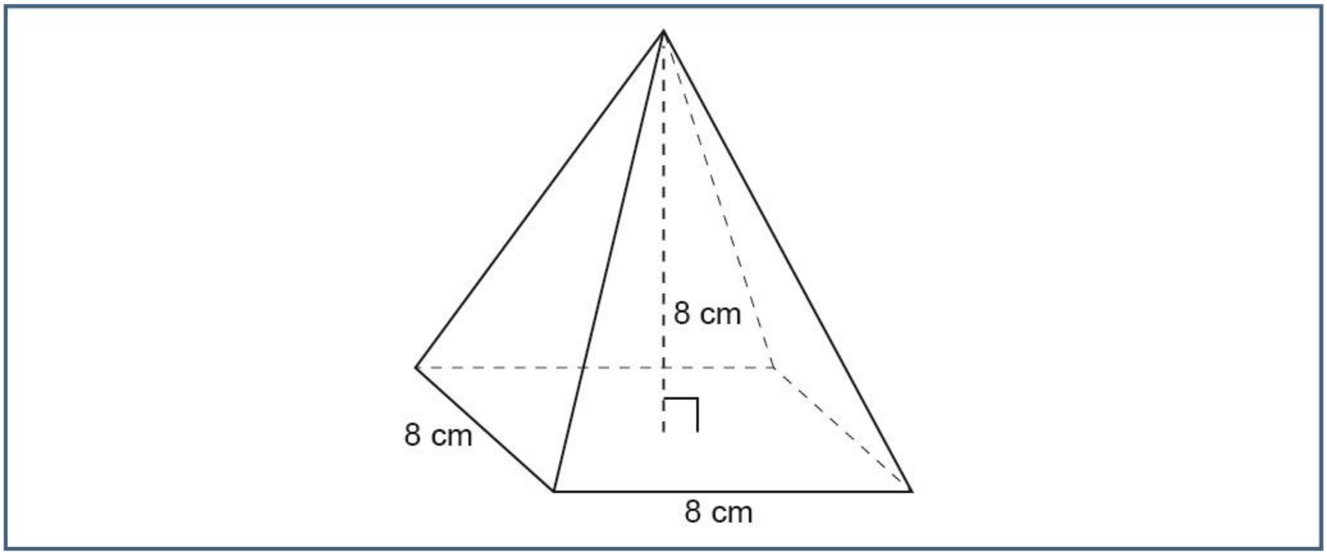
► Identifying and Addressing Common Misconceptions

- Some students may think that a cross section that is perpendicular to the base of a pyramid is always a triangle. Ask students to identify all the shapes that can be formed by slicing a pyramid perpendicular to its base. Some students may not consider slices that do not include the vertex. Use a physical model of a pyramid and show students how perpendicular slices that do not contain the vertex may result in other shapes.

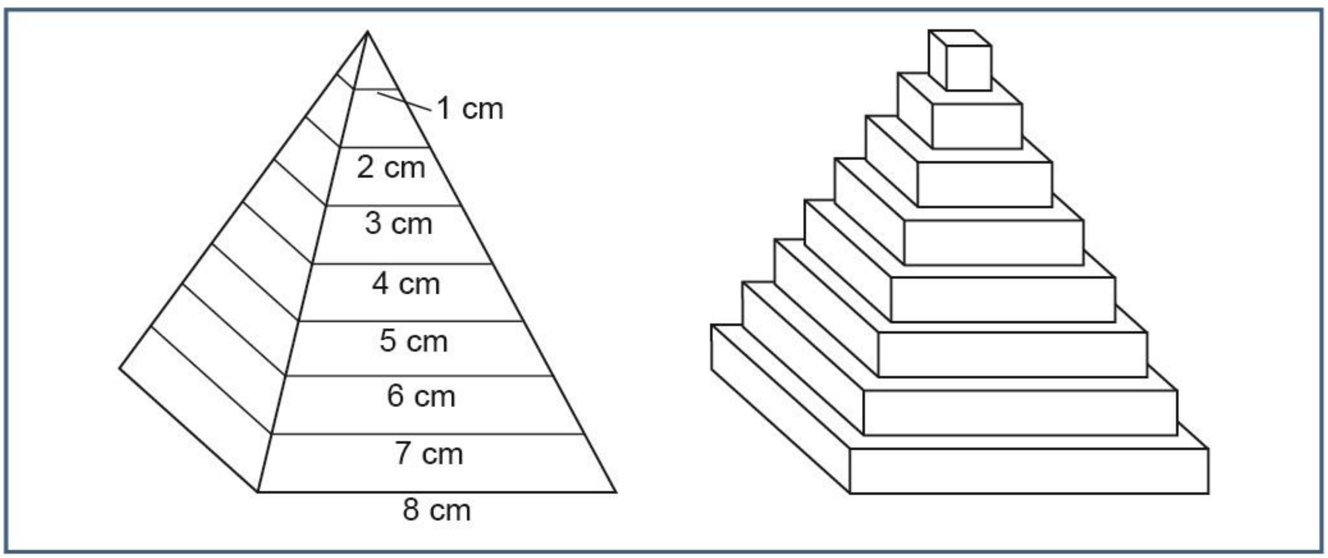
► Enrichment

- Ask students to identify the shapes of cross sections of other three-dimensional figures (e.g., pentagonal pyramids, non-right prisms, cylinders, composite figures).
- Ask students to recognize that cross sections of pyramids that are parallel to the base create shapes that are scale images of the base, and ask them to find the scale factor between the base and the cross section. In addition, ask them to connect the scale factor with the distance from the cross section to the vertex.

- Ask students to approximate the volume of a square pyramid by creating cross-sectional layers of rectangular prisms. For example, give students the following square pyramid.



Ask students to use 7 cross sections to cut the pyramid into 8 layers. Because the height of the pyramid is 8 centimeters, the thickness of each layer is 1 centimeter. Ask students to replace each layer of the pyramid with a rectangular prism having a base that matches the cross section and a height of 1 centimeter. Cross sections of the pyramid can be changed into rectangular prisms as shown.



The prisms have volumes of 1, 4, 9, 16, 25, 36, 49, and 64 cubic centimeters. Therefore, the volume of the stack of prisms is 204 cubic centimeters. Ask students to determine, without calculating, whether this estimate is greater than or less than the actual volume of the pyramid, and then ask them to use the formula for the volume of a pyramid to compare the estimate to the actual volume. As a further investigation, ask students to hypothesize what would happen if more cross sections were used.

► Key Terms

base, construct, cross section, horizontal, parallel, perpendicular, plane section, right rectangular prism, right rectangular pyramid, slice, three-dimensional, two-dimensional, vertical

C-G.2.1.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

Geometry

Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

Identify, use and describe properties of angles and their measures.

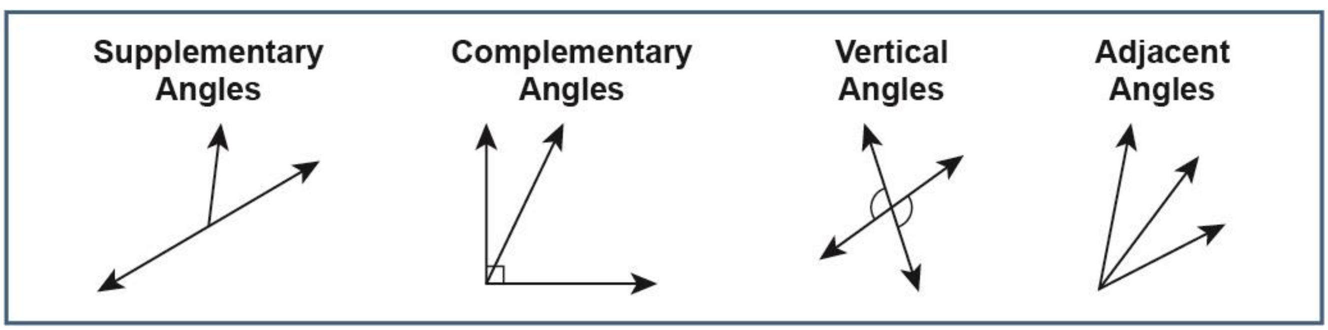
► Eligible Content

C-G.2.1.1 Identify and use properties of supplementary, complementary, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure.

► Instructional Supports

Types of angles

- Ask students to define and show examples of supplementary, complementary, vertical, and adjacent angles. For example, ask students to draw a pair of supplementary angles, a pair of complementary angles, a pair of vertical angles, and a pair of adjacent angles and explain how each drawing fits the given term.

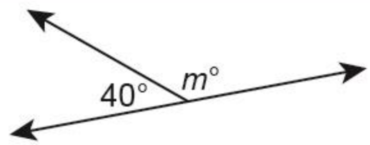


Students should explain that

  - supplementary angles are two angles whose measures add to 180 degrees.
  - complementary angles are two angles whose measures add to 90 degrees.

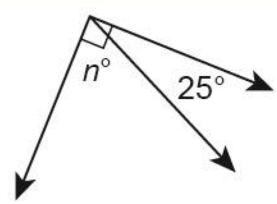
- o vertical angles are two opposite angles formed by two intersecting lines where the point of intersection is a shared vertex of the two angles.
- o adjacent angles are two angles that share the same vertex and have a common ray between them.

- Ask students to determine an unknown angle measure when two angles are supplementary. For example, ask students to determine the value of m in the diagram shown.



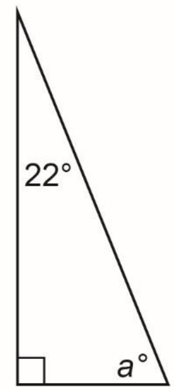
Because the angles are supplementary, the sum of their measures equals 180 degrees. Therefore, $40 + m = 180$, and $m = 140$.

- Ask students to determine an unknown angle measure when two angles are complementary. For example, ask students to determine the value of n in the diagram shown.



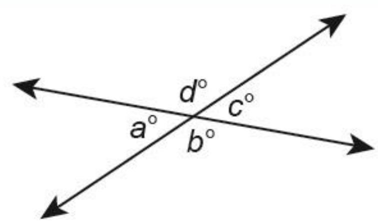
Because the angles are complementary, the sum of their measures equals 90 degrees. Therefore, $25 + n = 90$, and $n = 65$.

- Ask students to explain why acute angles in a right triangle are complementary and to use that fact to determine missing angles in right triangles. For example, give students the figure shown.



Because the measures of the angles of a triangle add to 180 degrees and the measure of a right angle is 90 degrees, the sum of the measures of the other two angles is 180 – 90 = 90 degrees. One of the acute angles measures 22 degrees. Therefore, the measure of the other acute angle is $a = 90 - 22 = 68$ degrees.

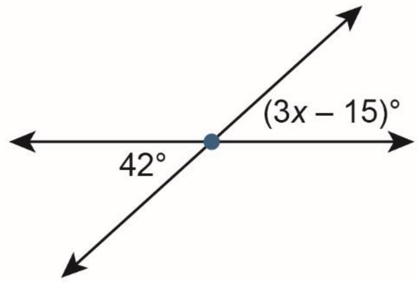
- Ask students to show why the measures of vertical angles are equal. For example, show students the following diagram and ask them to explain why *b* must equal *d*.



Students should recognize that $b + c = 180$ because *b* and *c* are supplementary angles. Likewise, $d + c = 180$. Both *b* and *d* can be added to *c* to get 180; therefore, $b = d$.

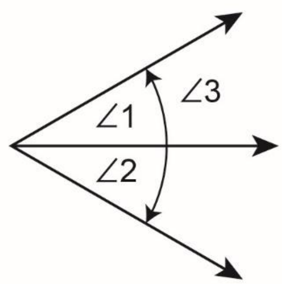
Working with angles

- Ask students to write and solve an equation involving angles. For example, ask students to determine the value of *x* in the diagram shown.



The labeled angles are vertical angles and therefore have the same measure. Students use this fact to write the equation $3x - 15 = 42$. Students add 15 to both sides of the equation to get $3x = 57$ and then solve for x by dividing both sides of the equation by 3 to get $x = 19$.

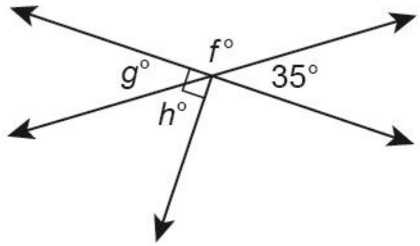
- Ask students to write fact families to show the relationships between adjacent angles and larger angles formed by pairs of adjacent angles. For example, give students the following diagram.



Students write the following equations.

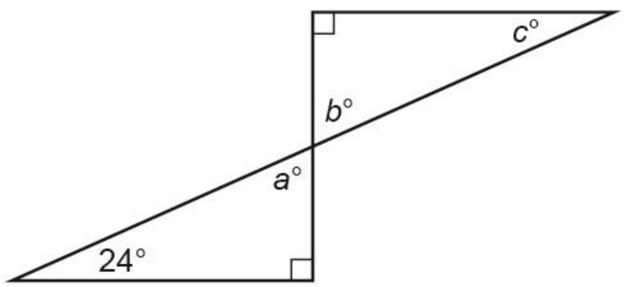
- $m\angle 1 + m\angle 2 = m\angle 3$
- $m\angle 2 + m\angle 1 = m\angle 3$
- $m\angle 3 - m\angle 2 = m\angle 1$
- $m\angle 3 - m\angle 1 = m\angle 2$

- Ask students to determine multiple unknown angle measures in a diagram.
  - Give students the following diagram.



The value of *f* must be 145 because angle *f* and the 35-degree angle are supplementary. The value of *g* must be 35 because angle *g* and the 35-degree angle are vertical angles. The value of *h* must be 55 because angles *h* and *g* are complementary.

- Give students the following diagram.



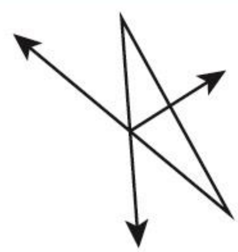
Because the angles of a triangle sum to 180 degrees, *a* must be 66 ($180 = 90 + 24 + a$). Angles *a* and *b* are vertical angles; therefore, *b* is also 66. Since *b*, *c*, and the right angle are part of a triangle with a sum of 180 degrees, *c* is 24.

► Identifying and Addressing Common Misconceptions

- Some students may confuse the meanings of supplementary and complementary angles. Ask students to draw an example of supplementary angles. Some students may show complementary angles. Help students review their terminology. Some students may find it helpful to have mnemonics, such as "Supplementary angles form straight lines; complementary angles form corners."
- Some students may not recognize the acute angles of a right triangle as being complementary. Show students a right triangle and ask them to describe the relationship between the acute angles.

Some students may not recognize that their measures must sum to 90 degrees. Ask students the sum of the measures of the angles of a triangle and ask them the measure of a right angle. Help students recognize that 90 degrees remain for the sum of the acute angles after subtracting the 90 degrees of the right angle.

- Some students may not recognize supplementary angles that do not form a horizontal line. Give students a diagram with supplementary angles forming a non-horizontal line and ask them to identify any supplementary angle pairs. Some students may say there are none. Have students rotate their paper until the line is horizontal and show them the angle measures have not changed. Therefore, the angles are still supplementary even if the line they form is not horizontal.
- Some students may not recognize that an “X” figure makes vertical angles even when it’s embedded in a larger diagram. Give students the following diagram and ask them to identify pairs of vertical angles.



Some students may not be able to identify any. Ask students to use tracing paper to trace two intersecting lines. With the lines isolated, ask students to identify vertical angles in the simpler diagram. Then help them find and identify those same angles in the original diagram.

► Enrichment

- Ask students to determine unknown angle measures in larger diagrams with more unknowns using facts about triangle angle sums and other known equations.

► Key Terms

adjacent angles, complementary angles, supplementary angles, vertical angles

C-G.2.1.2

▶ Reporting Category, Assessment Anchor, Anchor Descriptor

Geometry

Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

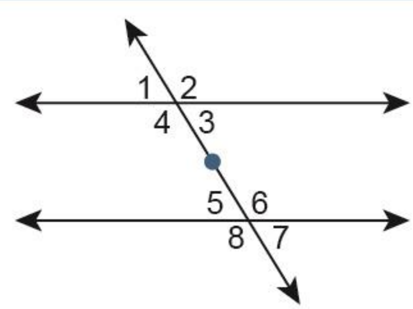
Identify, use and describe properties of angles and their measures.

▶ Eligible Content

C-G.2.1.2 Identify and use properties of angles formed when two parallel lines are cut by a transversal (e.g., angles may include alternate interior, alternate exterior, vertical, corresponding).

▶ Instructional Supports

- Ask students to identify, without measuring, pairs or groups of congruent angles formed when a pair of parallel lines is cut by a transversal. For example, give students the following figure.



Give students a piece of tracing paper and ask them to trace the figure.

  - Ask students to translate the tracing paper along the transversal so that one set of four angles lies on top of the other set. Help students use this translation to identify congruent angles.

  - Ask students to rotate the paper around the midpoint of the transversal so that one set of four angles lies on top of the other set. Help students use this rotation to identify congruent angles.

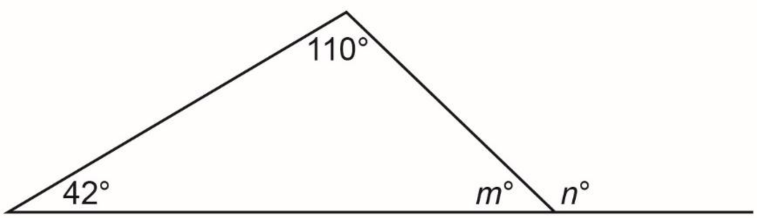
Help students conclude that all the odd-numbered angles are congruent to each other and all the even-numbered angles are congruent to each other.

► Identifying and Addressing Common Misconceptions

- Some students may think that corresponding angles and alternate interior angles formed by a pair of lines cut by a transversal are always congruent. Show students two nonparallel lines cut by a transversal and ask them to identify angles that are congruent. Some students may give pairs other than vertical angles. Ask them to measure the angles to confirm that corresponding and alternate interior angles are not congruent. Remind them that these angles are congruent only if the lines are parallel.

► Enrichment

- Ask students to determine the measure of an interior angle of a triangle and the measure of its corresponding exterior angle when the measures of the other two interior angles are given. For example, ask students to find the missing angle measurements in the following triangle.



- Students should identify that the sum of the measures of the interior angles of a triangle is 180 degrees and write the equation $42 + 110 + m = 180$ to represent this. Students should then find that $m = 180 - 110 - 42 = 28$. Students should also identify that a straight angle measures 180 degrees and write the equation $n + 28 = 180$ to represent this. Students should find that $n = 152$. Help students observe that this is equal to the sum of the other two interior angles and explain why this is always the case. Students should notice that the exterior angle and its interior angle always add to 180 degrees and that the two remote interior angles and the interior angle also always add to 180 degrees. Therefore, students can conclude that the exterior angle is equal to the sum of the two remote interior angles.

- Ask students to add additional parallel lines and transversals to a drawing of two parallel lines cut by a transversal and justify which angles are congruent.

► Key Terms

alternate exterior angles, alternate interior angles, corresponding, parallel lines, supplementary angles, transversal, vertical angles

C-G.2.2.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

Geometry

Solve real-world and mathematical problems involving angle measure, circumference, area, surface area, and volume.

Determine circumference, area, surface area, and volume.

► Eligible Content

C-G.2.2.1 Find the area and circumference of a circle. Solve problems involving area and circumference of a circle(s). Formulas will be provided.

► Instructional Supports

Circumference

- Ask students to explore the circumference of a circle. For example, give students several objects that have circular faces (e.g., plates or cups), string, and a ruler and ask them to measure the objects and record the measurements in the first three columns of the table shown.

Object	Diameter (d)	Circumference (C)	Ratio of C to d

  Ask students to use their measurements to complete the table. Help them recognize that the ratio of the circumference to the diameter of any circle is always π and that the circumference can be calculated using the formula $C = \pi \bullet d$.

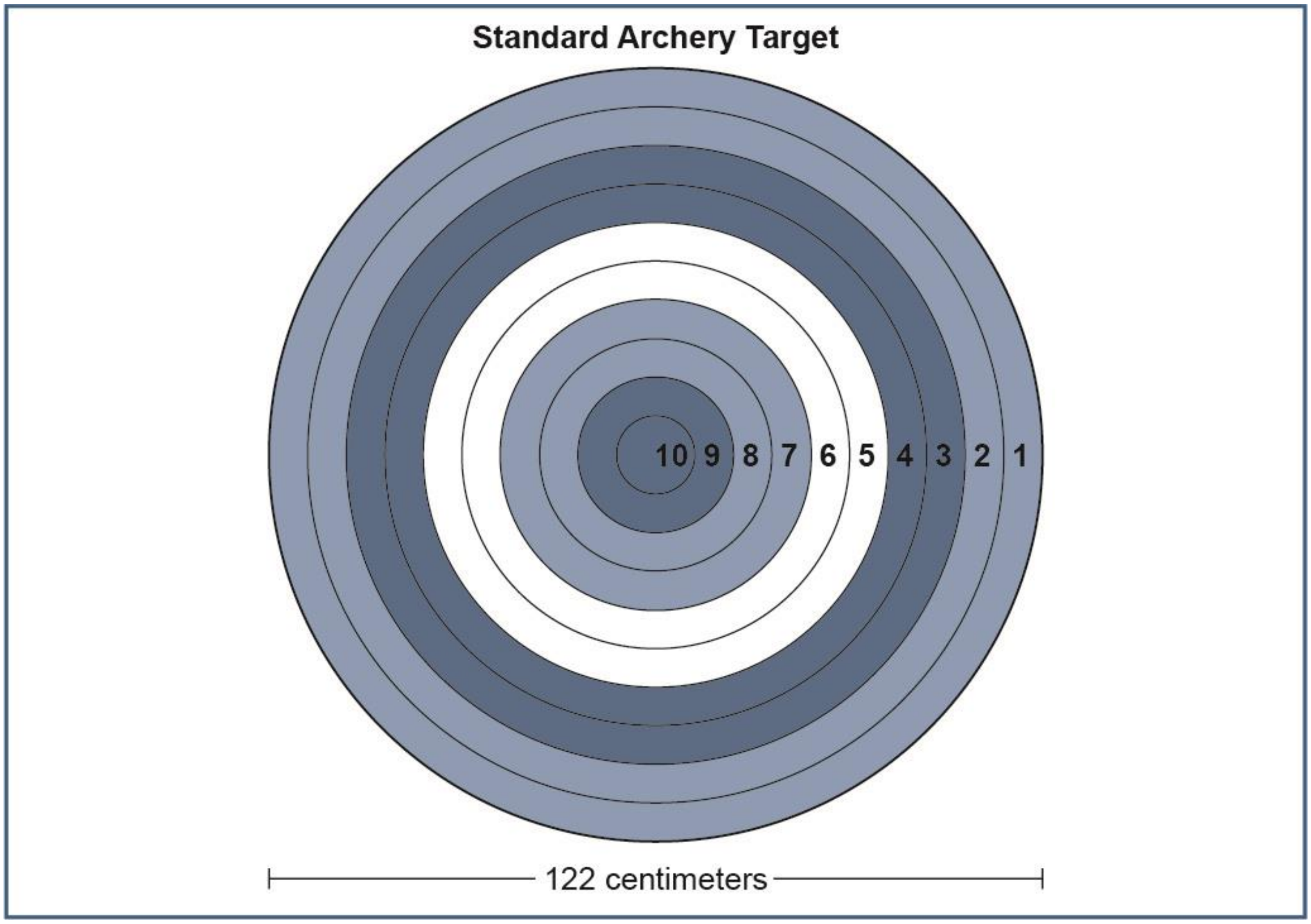
- Ask students to use the circumference formula to calculate the distance around a circle. For example, ask students to determine the circumference of a circle with a diameter of 4.6 inches. In general, students can use either the formula $C = \pi \bullet d$ or the formula $C = 2 \bullet \pi \bullet r$ to calculate the circumference of a circle. For this example, because the diameter is given, students can use the formula $C = \pi \cdot 4.6$. Therefore, the circumference of the circle is 4.6π inches (about 14.4 inches when approximating π with 3.14).

Area

- Ask students to use the area formula to calculate the area of a circle. For example, ask students to determine the area of a circle with a radius of 5.9 centimeters. Students can use the formula $A = \pi \cdot r^2$ to calculate the area of a circle when given the radius. For this circle, students can calculate the area as $\pi \cdot 5.9^2$. Therefore, the area of the circle is 34.81π square centimeters (about 109.3 square centimeters when approximating π with 3.14).

Area and circumference

- Ask students to calculate the area of a circle given the circumference and vice versa. For example, ask students to determine the area of a circle with a circumference of 10 feet. Students start by using the circumference of the circle to determine the radius. The circumference formula becomes $10 = 2\pi \cdot r$, so $r = \frac{5}{\pi}$. Students then use the radius to determine the area. The area formula becomes $A = \pi \cdot \left(\frac{5}{\pi}\right)^2$, so the area of the circle is $\frac{25}{\pi}$ square feet (approximately 7.96 square feet).
- Ask students to use the circumference and area formulas for circles in real-world situations. For example, ask students to calculate the area of the 6-point ring on a standard archery target. The layout of a standard archery target is shown along with the point values of each ring.



The area of the 6-point ring can be determined by subtracting the area of the whole circle enclosing the 7-point ring (or the 7-point circle) from the area of the whole circle enclosing the 6-point ring (or the 6-point circle). Because each ring has the same width, the width of each ring can be calculated by dividing the radius of the entire target (61 centimeters) by 10; therefore, each circle on the target has a radius that is 6.1 centimeters greater than that of the next smallest circle.

Circle	Radius (cm)
10-point	6.1
9-point	12.2
8-point	18.3
7-point	24.4
6-point	30.5

Since the 7-point circle has a radius of 24.4 centimeters, its area can be calculated as $\pi \cdot (24.4)^2 = 595.36\pi$ square centimeters. Similarly, the area of the 6-point circle can be calculated as $\pi \cdot (30.5)^2 = 930.25\pi$ square centimeters. Therefore, the area of the 6-point *ring* is $930.25\pi - 595.36\pi = 334.89\pi$ square centimeters (approximately 1,051.55 square centimeters).

► Identifying and Addressing Common Misconceptions

- Some students may confuse the radius and the diameter of a circle. Give students a circle and ask them to draw a diameter. Some students may show a radius. Help students build on previous vocabulary and think of the radius as "radiating" out from the center of the circle.
- Some students may think the area of a circle should always be a greater number than the circumference. Ask students to determine which of the two values would be greater for a circle with a radius of 0.75 inches. Students who think the area is the greater value may have this misconception. Ask students what happens to a number when it is multiplied by a fraction greater than 1 compared to when it is multiplied by a fraction less than 1. Use this concept to show them that the formulas for area and circumference both contain $(\pi \cdot r)$ being multiplied by one other value. For the area formula, that value is the radius of 0.75, which is less than 1; for the circumference formula, that value is 2, which is greater than 1. Therefore, the area calculation would be smaller than $\pi \cdot r$, and the circumference calculation would be greater than $\pi \cdot r$. (In this case, the area calculation is $\pi \cdot (0.75)^2 \approx 1.76$ square inches, and the circumference calculation is $\pi \cdot 0.75 \cdot 2 \approx 4.71$ inches.)

► Enrichment

- Ask students to combine their understanding of the area and circumference of circles with their understanding of prisms to determine the volume and surface area of a cylinder.
- Ask students to calculate the areas of different sizes of circular pizzas sold at a restaurant and compare the costs based on the areas rather than the diameters to determine the best deals.

► Key Terms

area, circle, circumference, derive, diameter, pi (π), radius, square unit, unit

C-G.2.2.2

► Reporting Category, Assessment Anchor, Anchor Descriptor

Geometry

Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

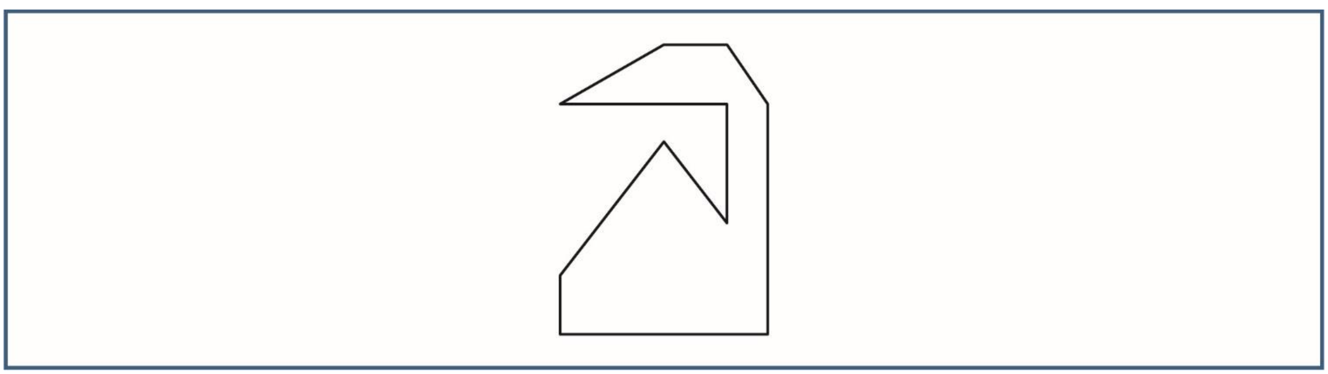
Determine circumference, area, surface area, and volume.

► Eligible Content

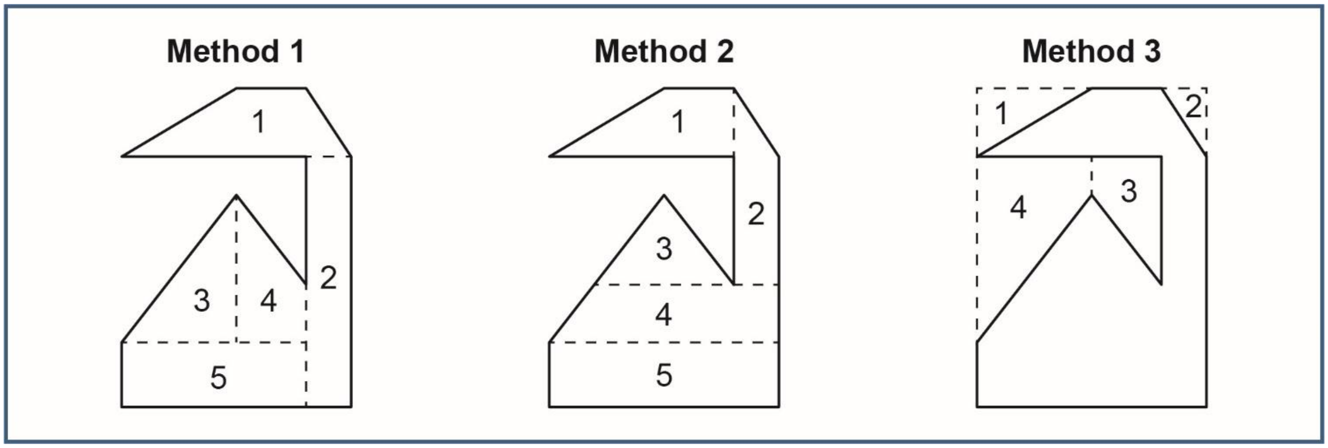
C-G.2.2.2 Solve real-world and mathematical problems involving area, volume, and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms. Formulas will be provided.

► Instructional Supports

- Ask students to determine multiple methods of calculating the area of a given polygon. For example, give students the figure shown.

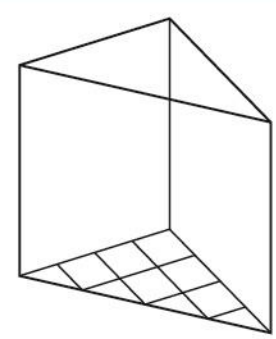


Some possible methods are shown.

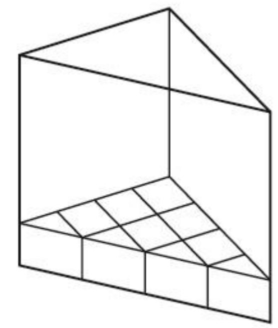


- Using method 1, students can calculate the area by adding the areas of trapezoid 1, rectangle 2, triangle 3, trapezoid 4, and rectangle 5.
- Using method 2, students can calculate the area by adding the areas of trapezoid 1, trapezoid 2, triangle 3, trapezoid 4, and rectangle 5.
- Using method 3, students can calculate the area by determining the area of the larger rectangle and subtracting the areas of triangle 1, triangle 2, trapezoid 3, and trapezoid 4.

- Ask students to extend their understanding of rectangular prisms to determine volumes of other prisms. For example, ask students to calculate the volume of a triangular prism with a height of 5 centimeters and a base that is a right triangle with legs of 4 centimeters and 4 centimeters.

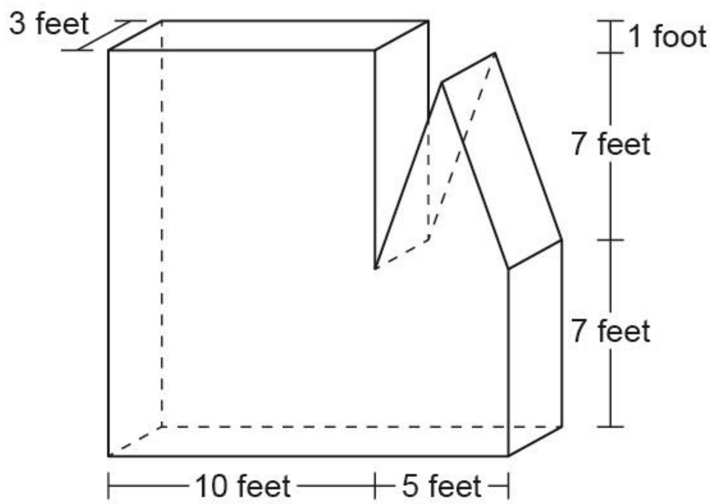


The area of the base triangle is $\frac{1}{2}(4)(4) = 8$ square centimeters. Therefore, it can be covered with 8 unit squares that have been cut appropriately. Additionally, 8 appropriately cut unit cubes can be placed on the base to form a layer with a volume of 8 cubic centimeters.



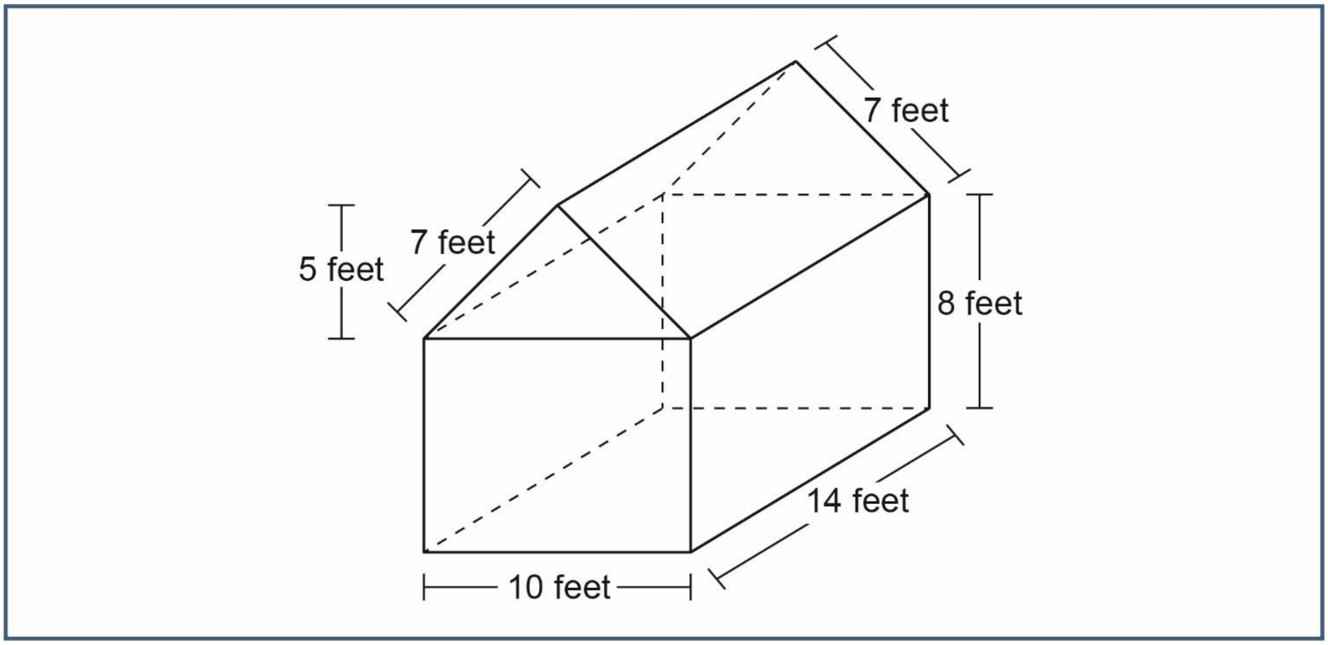
Because the height of the prism is 5 centimeters, 5 of these layers can be stacked to fill the prism. Therefore, the prism has a volume of 8 • 5 = 40 cubic centimeters. In general, the volume of any prism with a base that has an area of B square units and a height of h units is given by the formula $B \cdot h$.

- Ask students to calculate volumes of composite prisms. For example, give students the figure shown.

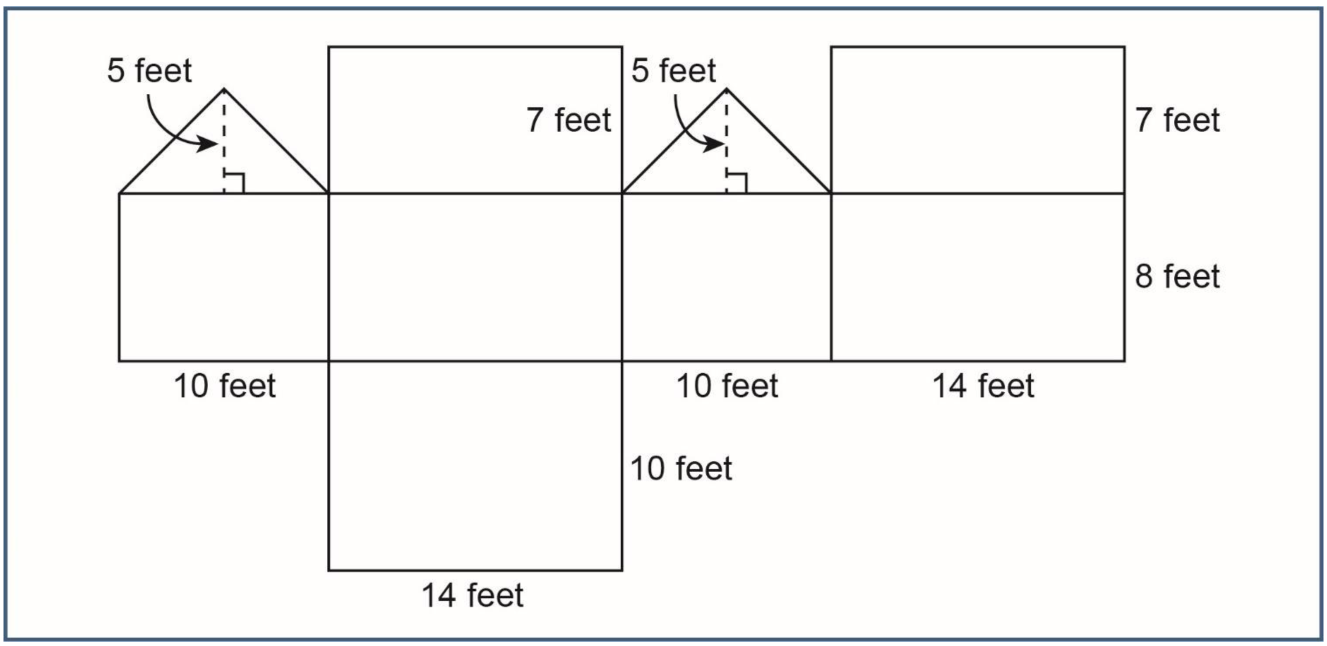


The figure is made up of two rectangular prisms and one triangular prism. The larger rectangular prism at the bottom of the figure has a volume of 15 • 3 • 7 = 315 cubic feet, and the smaller rectangular prism at the top left of the figure has a volume of 10 • 3 • 8 = 240 cubic feet. The triangular prism has a volume of (0.5 • 5 • 7) • 3 = 52.5 cubic feet. Therefore, the volume of the figure is 315 + 240 + 52.5 = 607.5 cubic feet.

- Ask students to calculate surface areas of composite prisms. For example, ask students to determine the number of minutes it takes Jesse to paint the figure shown when she paints at a rate of 21 square feet per minute.



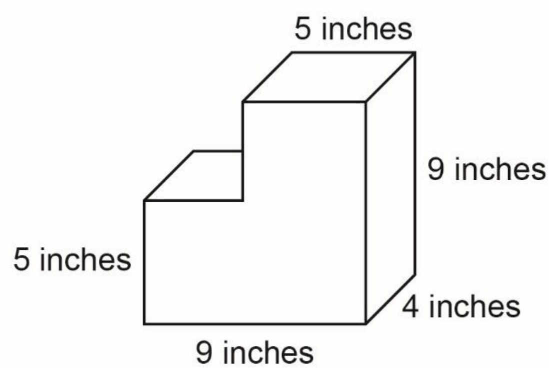
Ask students to draw the net of the figure to help visualize and calculate its surface area.



Ask students to calculate the area of each of the nine shapes of the net. The sum of the areas is shown by the equation $2(0.5 \cdot 10 \cdot 5) + 2(7 \cdot 14) + 2(8 \cdot 10) + 2(8 \cdot 14) + (10 \cdot 14) = 770$ square feet. Since Jesse paints at a rate of 21 square feet per minute, students divide the surface area by 21 to determine the number of minutes it takes Jesse to paint the figure. Therefore, it takes Jesse $770 \div 21 = 36\frac{2}{3}$ minutes (36 minutes, 40 seconds) to paint the figure.

▶ Identifying and Addressing Common Misconceptions

- Some students may confuse square units and cubic units. Ask students to calculate the volume of a given figure. Some students may give an answer in square units. Ask students to explain the meaning of a square unit and a cubic unit. Then ask students to create a square unit and a cubic unit out of materials like paper and clay. Use the visual illustrations of square and cubic units to help them understand their connections to area and volume.
- Some students may calculate the surface area of a composite figure by decomposing it and adding the surface areas of the decomposed parts. Ask students to calculate the surface area of the figure shown.



Some students may decompose the figure into an upper and lower prism, determine their surface areas to be 112 square inches and 202 square inches, respectively, and add them to get a total of 314 square inches. Help students draw the two prisms separately. Then point to each face of each prism and ask students to identify the corresponding faces of the original figure. Help students recognize that some parts of the prisms are not part of the exterior of the original figure and therefore should not be included in the surface area.

▶ Enrichment

- Ask students to calculate the areas, volumes, and surface areas of composite figures involving pyramids.
- Ask students to use a tape measure to calculate the volumes and surface areas of rooms, closets, or hallways in the school.

▶ Key Terms

area, base, prism, surface area, three-dimensional, two-dimensional, volume

D-S.1.1.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

Statistics and Probability

Use random sampling to draw inferences about a population.

Use random samples.

► Eligible Content

D-S.1.1.1 Determine whether a sample is a random sample given a real-world situation.

► Instructional Supports

- Ask students to discuss the advantages and disadvantages of using a sample to gain information about a population. For example, suppose the mayor of a town of 35,000 wants to determine which day of the week the residents prefer to have the garbage collected. Students may note that it is advantageous to survey a sample of the population because it is generally less expensive, less time consuming, and more practical than surveying the entire population. Students may also note that using a sample from the population carries the disadvantage of omitting some people's opinions and possibly not reaching an accurate conclusion as a result.
- Give students a context in which a sample population is being surveyed and ask them to identify factors among individuals in the population that may affect responses. For example, tell students a city planner wants to know how satisfied residents are with the traffic and parking in the city. Students may identify some factors that may affect responses including job hours (some employees may often drive during rush hour while others may work a night shift), car size (larger vehicles are harder to park than smaller vehicles), foresight and organization (people who leave earlier and allocate more time for commutes may find traffic and parking to be less frustrating), and location within the city (some parts of the city have more traffic than others).
- Ask students to explain what it means for a sample to be representative of a population and why having a representative sample is important for gaining information. Students should explain that a representative sample is one whose proportions are the same as the population's proportions in

as many ways as possible. Some examples include the following: if a population is 30% senior citizens, then the sample should also be 30% senior citizens; if $\frac{1}{10}$ of a population is left-handed people, then the sample should also be $\frac{1}{10}$ left-handed people; if a population has twice as many musicians as athletes, then the sample should also have twice as many musicians as athletes. It is important to have a representative sample of the population because the more representative a sample is in all aspects, the more likely the sample is to be representative of the population with respect to the aspects of the survey.

- Provide students with a population and ask them to design a random selection method to create a representative sample of the population. For example, tell students that a person wants to create a random sample from all the players on a sports team. Students might suggest a method of writing each player's jersey number on a separate piece of paper and then putting it in a helmet before randomly selecting some of the numbers out of the helmet to create the random sample.
- Provide students with a sampling method and ask them to identify potential biases. For example, suppose a city health official surveys all the customers in a sporting goods store to determine how often the city's residents exercise. Students should recognize that such a sampling method is biased because customers in a sporting goods store are likely to enjoy some form of exercise. This sampling method is unlikely to result in a sample that is representative of the broader population of the city's residents.
- Ask students to explain why random samples are important for gaining information about a population. Students should explain that random samples are important because they are more likely to create a representative sample than other methods. Some systematic methods usually have some bias in the selection method that may result in proportions in the sample that do not match proportions in the population.

► Identifying and Addressing Common Misconceptions

- Some students may think that when choosing between two different sampling methods, the method that generates a larger sample automatically provides a better generalization of the population as a whole. Ask students which of the following sampling methods would provide a better representation of the entire student body.

- Ask 250 eighth graders what their favorite lunch menu item is.
- Ask 50 sixth graders, 50 seventh graders, and 50 eighth graders what their favorite lunch menu item is.

Some students may choose the first method because it includes more students. Help students recognize that although the first sampling method involves a larger number of students, it does not provide the best generalization of the entire student body because it only involves students from one grade rather than multiple grades.

- Some students may think that any sampling method that involves an element of randomness is automatically unbiased. Ask students whether a survey that randomly selects half the names from a list of seventh-grade band members produces a sample that is representative of all seventh-grade students. Some students may think that the sample would be representative because the names are randomly selected from a list. Help students recognize that even though students are randomly selected from a list, the list itself is not representative of the entire student body and is therefore biased.
- Some students may think that a random sampling method will always result in a sample that is representative of the population. Ask students whether a random sample always gives better results than a non-random sample. Some students may think that a random sample is always better. Ask students if it is possible to randomly select 5 left-handed students from a classroom of 6 left-handed students and 24 right-handed students. Help them to conclude that while it is extremely unlikely, it is possible for a random method to result in a sample that is not representative of the entire population.
- Some students may think that a sample involving a specific group of people can never be representative of a population. Ask students whether a random sample of girls at a school is representative of the population. Some students may answer without knowing the population in question. Help students understand that what is considered to be a representative sample depends on the population being studied. If only girls are being studied, then a sample of only girls may be representative.

► Enrichment

- Provide students with a list of all the first names in the class (or other population) and ask them to find or create a method for randomly selecting 10 of the names using a spreadsheet.
- Ask students to create a survey question and to administer the survey to several samples of people, including a random sample, a convenience sample, and a biased sample of a population. Then compare the results and discuss the discrepancies.
- Ask one group of students to pose a survey question to an entire (and relatively small) population and another group to pose the same question to a random sample of the population. Compare the results and discuss the validity of the random sample given that the results may not be identical.

► Key Terms

draw inferences, population, random sampling, representative, sample, valid

D-S.1.1.2

► Reporting Category, Assessment Anchor, Anchor Descriptor

Statistics and Probability

Use random sampling to draw inferences about a population.

Use random samples.

► Eligible Content

D-S.1.1.2 Use data from a random sample to draw inferences about a population with an unknown characteristic of interest.

► Instructional Supports

- Give students a random sample and ask them to make a prediction or inference about the population that the sample was taken from. For example, "An art teacher wants to know, on average, how many different paint colors his students used for an assignment. The art teacher randomly selects 15 of his 95 students and asks how many different paint colors they used for the assignment. The 15 student responses are shown in the following data set. What is a reasonable estimate of the average number of paint colors used by the 95 students?"

  $$\{4, 5, 5, 6, 8, 6, 6, 4, 4, 5, 5, 5, 9, 6, 4\}$$

  The mean number of colors used by the 15 students in the sample is $5\frac{7}{15}$, or about 5.5. Because the 15 students are a random sample of the class, the 15 students are likely to be representative of the entire class. Therefore, it is reasonable to predict that the mean number of paint colors used for the assignment by all 95 students is also about 5.5.

- Ask students to explore how the size of a sample affects the accuracy of inferences made from sample data. For example, place 15 red tiles and 35 non-red tiles into an empty bag and ask students to use samples to predict the proportion of red tiles in the bag. As a class, randomly select repeated samples from the bag (after replacing the previous sample) and record the results in the following table.

Sample Size	Number of Red Tiles in Sample	Number of Non-red Tiles in Sample	Proportion of Red Tiles in Sample
2			
5			
10			
15			
25			
40			

Show students the contents of the bag and compare the proportions in the samples to the actual proportion of red tiles. Students should observe that, in general, the larger samples tend to give more accurate predictions.

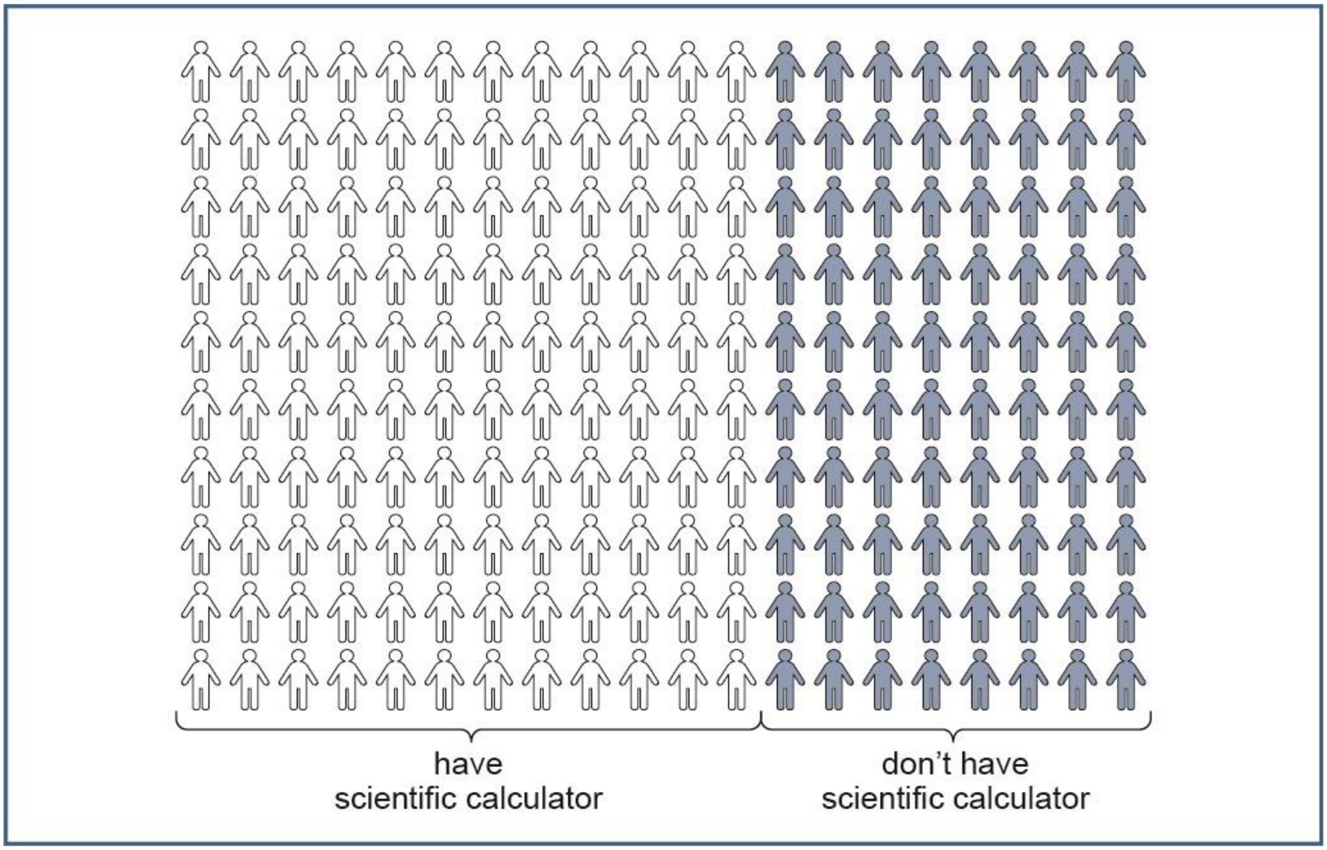
- Ask students to compare multiple random samples of the same size and then to compare the inferences suggested by those samples. For example, put 50 pennies into an empty container and ask each student to select a random sample of 8 pennies, record the years on the pennies, and put the pennies back. Repeat until each student has taken a sample. Then ask each student to calculate the mean of the 8-number sample. Create a line plot of the students' means and compare them to the actual mean year of all 50 pennies. Help students observe that using random samples to make a prediction does not always give one consistent prediction, and there will be some variability in the accuracy of the predictions. Be sure to note that most samples do create good estimates.
- Ask students to investigate variation in estimates based on samples. For example, ask a class of students to write their heights, in inches, on cards and shuffle the cards together. Then have each student select a random sample of 5 cards and use the mean of the heights on those cards to estimate the mean height of the class. Then ask them to put the cards back, shuffle the cards, take another random sample of 5 cards, and use the mean of those 5 heights to make a new estimate of the mean height of the class. This process should be repeated until the class has 8 estimates. A possible set of height estimates is shown.

{56, 56, 59, 60, 60, 61, 61, 63}

The range of the estimates is 7 inches. This range gives an idea of how much variation there can be in the inferences made using samples of 5 heights. Most estimates of the mean height that are obtained by taking a random sample of 5 heights will fall within a 7-inch interval. In addition, students can estimate the class's mean height using random samples of 10 heights instead of 5 heights, again repeating 8 times. In general, increasing the size of the random sample results in a set of estimates with less variation.

► Identifying and Addressing Common Misconceptions

- Some students may think that a larger sample always provides a more accurate inference than a smaller sample. Give the students the following situation: "A town wants to determine, on average, how often its citizens eat out." Ask them whether a random sample of 100 citizens or a survey of 200 citizens at the mall food court is more likely to give a better inference. Some students may think the larger sample may give a better estimate. Help students recognize that while the survey at the food court is a larger sample, it is probably not as representative of the population and, therefore, is unlikely to give an accurate estimate.
- Some students may think that a statistic derived from a random sample of a population is always identical to the corresponding population measure. Give students the following situation: "Someone randomly surveyed 40 of the 200 seventh graders in a school, and 20 reported that they own a scientific calculator." Ask them to explain what conclusions can be drawn from the sample. Some students may conclude that since exactly half of the sample owns a scientific calculator (20 out of 40), then exactly half of all seventh graders must own a scientific calculator (100 out of 200). Show students the following hypothetical population of 200 students, 120 of whom own scientific calculators.



Ask students whether it is possible to randomly choose, from this hypothetical population, 20 students who have scientific calculators and 20 students who do not. Students should recognize that it *is* possible to have a sample that does not match the population. Although the random sample suggests that about half of the seventh graders own a scientific calculator, it does not guarantee that exactly half of the seventh graders own a scientific calculator.

- Some students may think that any statistic from a random sample can be used to make inferences about the population. Ask students to explain what inferences can be made about a population from a sample with a range of 10. Some students may think that 10 is a reasonable estimate of the range of the population. Give students the set {11, 13, 15, 17, 19, 21, 23, 25, 27, 29}, which has a mean of 20 and a range of 18, and ask them to find a sample of four values with a mean less than 20 and another sample of four values with a mean greater than 20. Help students understand that inferences can be made about the mean because some samples underestimate the mean, some samples overestimate the mean, and, overall, they balance out so that the mean of the sample is often close to the mean of the population. On the other hand, ask students to find a sample of four values with a range less than 18 and another sample of four values with a range greater than 18. Students should find the latter is impossible. Because the sample always has a range that is less than or equal to the range of the population, it is not helpful to use the range of a sample to make inferences about the range of the population.

► Enrichment

- Ask students to hypothesize how the spread of a population affects the accuracy of estimates made from a sample. In general, inferences made from a sample of a population with a small amount of variation are more accurate than inferences made from a sample of a population with a large amount of variation. For example, a sample of the weights of 20 people is taken from a group of 200 people and the mean weight of the group of people is calculated to be 165 pounds. A sample of 20 animals is taken from a group of 200 animals of different species at a zoo, and the mean weight of the group of animals is also calculated to be 165 pounds. The 165 pounds is most likely a better estimate of the mean weight of the 200 people than it is for the 200 animals because the weights of the people are likely to have less variation.
- Give students a sample taken from a population and ask them to determine an interval around the mean of the sample that the mean of the population will likely fall within. For example, a random sample of 20 people results in a mean hair length of 8 inches. The mean hair length of all people is unlikely to be exactly 8 inches. Ask students to discuss how confident they are that the mean hair length is between 7.9 and 8.1 inches. Between 7.5 and 8.5 inches? Between 7 inches and 9 inches? Between 6 inches and 10 inches? At what point do students feel 50% confident? 90% confident?
- Give students a population characteristic and a sample. Then ask students to explain if they think the sample was taken randomly or not.
- Ask students to discuss whether there is a point where increasing the sample size is not practical even though the results might be more accurate. Students may mention situations dealing with money, time, geographical restrictions, or a logistical inability to sample randomly when the sample size gets too large. While increasing sample size will give more accurate results (provided the sample is random), the gain in accuracy might not be worth the cost of other practical factors.

- Ask students to find real-life examples of population sampling and break down how the methods are used. For example, students may look at voting surveys, television ratings, market research, etc.

▶ Key Terms

gauge, inference, multiple, population, prediction, random sampling, variability

D-S.2.1.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

Statistics and Probability

Draw comparative inferences about populations.

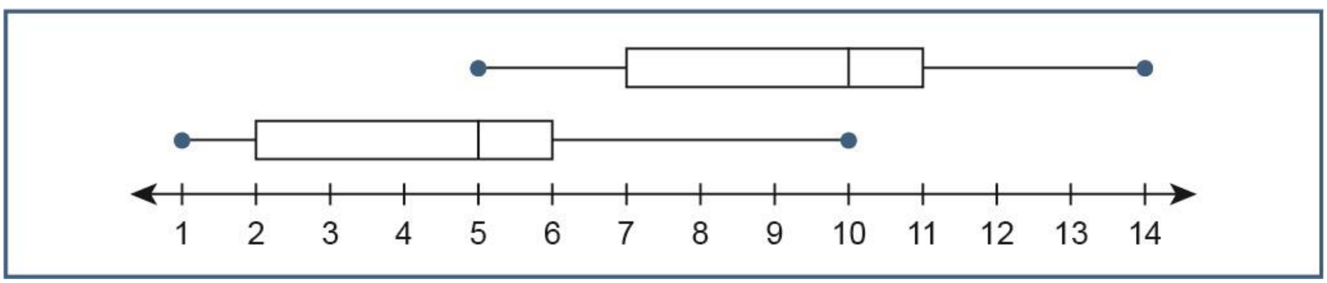
Use statistical measures to compare two numerical data distributions.

► Eligible Content

D-S.2.1.1 Compare two numerical data distributions using measures of center and variability.

► Instructional Supports

- Show students two box plots that have the same interquartile range and ask them to describe the distance between the medians in terms of the interquartile range. For example, give students the following box plots.



  Students should determine that the top box plot has a median of 10 and the bottom box plot has a median of 5. They should also find that both box plots have an interquartile range of 4. Students can describe the distance between the two medians as 1.25 times the interquartile range of either box plot.

- Provide students with two data sets and ask them to compare the centers of the data sets in terms of the variabilities of the data sets. For example, give students the following data sets showing the number of haircuts 15 seventh-grade boys had last year and the number of haircuts 15 seventh-grade girls had last year. Ask students to determine the means and mean absolute deviations and to describe the distance between the means in terms of the mean absolute deviation.

Seventh-Grade Boy Haircuts: {5, 6, 7, 7, 7, 7, 7, 8, 8, 8, 9, 9, 10, 11, 11}

Seventh-Grade Girl Haircuts: {1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 6, 6, 6, 7}

Students should determine that the mean number of haircuts for seventh-grade boys is 8 and the mean number of haircuts for seventh-grade girls is 4. They should also find that both data sets have a mean absolute deviation of $\frac{4}{3}$. Students can describe the difference between the means as 3 times the mean absolute deviation of either data set.

- Ask students to predict the amount of overlap between two data sets based on the centers and variabilities of the data sets and explain their reasoning. For example, give students the following descriptions and ask them to make and explain their predictions.
    - Two data sets both have an interquartile range of 4. The first data set has a median of 20 and the second data set has a median of 32. Students should be able to explain that the two data sets are unlikely to have significant overlap because the distance between the two medians is three times the distance of the interquartile ranges.
    - Two sets of data both have an interquartile range of 6. The first data set has a median of 22, and the second data set has a median of 19. Students should be able to explain that these two data sets are likely to have significant overlap because the distance between the two medians is half the distance of the interquartile ranges.
- Ask students to predict which of two pairs of data sets has greater overlap based on the centers and variabilities of the data sets and explain their reasoning. For example, one pair of data sets has means of 10 and 18 and a common mean absolute deviation of 2. The second pair of data sets has means of 100 and 200 and a common mean absolute deviation of 75. Students should recognize that the second pair of data sets is likely to have more overlap because the distance between the two centers is only $1\frac{1}{3}$ times the variability whereas the distance between the centers of the first pair of data sets is 4 times the variability.

► Identifying and Addressing Common Misconceptions

- Some students may think that overlap between data sets can be estimated while only knowing measures of variability. Give students the scenario “Matt and Karen play a video game. The range of Matt's scores is 20, and the range of Karen's scores is 30," and ask them to estimate the degree of overlap between their scores. Some students may answer based on whether they perceive ranges of 20 and 30 to be large ranges or small ranges. Ask students to create box plots with ranges of 20 and 30 and medians of 100 and 200 and help them recognize that it is impossible for the data sets to have any overlap in that situation. Then ask students to create box plots with ranges of 20 and 30 and medians of 40 and 40 and help them recognize that the box plots will usually have a significant degree of overlap.

- Some students may think that knowing the measures of center is all that is needed to estimate the overlap between data sets. Give students the problem "Company A makes a light bulb that has an average life span of 1,247 hours, and Company B makes a light bulb that has an average life span of 1,245 hours. Will any light bulbs from Company B have a longer life span than any light bulbs from Company A? Explain your reasoning." Some students may make predictions based on their perceptions of the two-hour difference in life spans. Help students understand that such a prediction cannot reasonably be made without knowing the variability of the life span of the light bulbs. Present the following two scenarios to help illustrate this.
    - If the life spans of the light bulbs made by each company have a range of 20 hours, then the light bulbs from Company A could last between 1,237 and 1,257 hours, while the light bulbs from Company B could last between 1,235 and 1,255 hours. In this case, there is significant overlap between the life spans, and the chances of finding a light bulb by Company B that lasts longer than a light bulb by Company A are high.
    - If the life spans of the light bulbs made by each company have a range of 0.5 hours, then the light bulbs from Company A could last between 1,246.75 and 1,247.25 hours, while the light bulbs from Company B could last between 1,244.75 and 1,245.25 hours. In this case, even the shortest-lasting light bulbs from Company A would last longer than the longest-lasting light bulbs from Company B.

► Enrichment

- Ask students to describe a value of a data set by stating its distance from the center in terms of a measure of variability. Two examples are shown.
    - The data set {1, 2, 2, 3, 5, 8, 11, 13, 16} has a median of 5 and an interquartile range (IQR) of 10. Therefore, the value of 3 is $\frac{1}{5}$ of the IQR below the median and the value of 13 is $\frac{4}{5}$ of the IQR above the median.
    - The data set {4, 5, 6, 7, 8, 9, 9, 10, 10, 12} has a mean of 8 and a mean absolute deviation (MAD) of 2. Therefore, the value of 4 is 2 MADs below the mean and the value of 10 is 1 MAD above the mean.
- Ask students to compare two data sets using the medians and interquartile ranges, and then to compare the same two data sets again using the means and mean absolute deviations. Observe whether the two methods suggest the same amount of overlap. Students should note that the interpretation of the relationships depends on the measure of variability that is being used. For example, a distance between two centers that is twice the range indicates that the data sets do not overlap but a distance that is twice the interquartile range or twice the mean absolute deviation still allows for overlap.

► Key Terms

data distribution, deviation, interquartile range, mean absolute deviation , measure of central tendency, measure of variability, range, variability

D-S.3.1.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

Statistics and Probability

Investigate chance processes and develop, use, and evaluate probability models.

Predict or determine the likelihood of outcomes.

► Eligible Content

D-S.3.1.1 Predict or determine whether some outcomes are certain, more likely, less likely, equally likely, or impossible (i.e., a probability near 0 indicates an unlikely event, a probability around 1/2 indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event).

► Instructional Supports

- Ask students to identify whether the probability of a given event is closest to 0, $\frac{1}{2}$, or 1. For example, students should identify that the probability of randomly selecting a date in January that is odd is closest to $\frac{1}{2}$, that the probability of randomly selecting the number 45 from a list containing the numbers 1 through 365 is closest to 0, and that the probability the current month has more than 28 days is closest to 1.
- Provide students with the following probability categories and ask them to generate events that belong to each category.

  certain

  very likely

  somewhat likely

  neither likely nor unlikely

  somewhat unlikely

  very unlikely

impossible

For example, students may identify that rolling an even number on a number cube is neither likely nor unlikely because the probability is $\frac{1}{2}$; that randomly selecting a month of the year whose name, written in English, contains the letter "r" is somewhat likely because the probability is $\frac{2}{3}$; or that randomly selecting a green marble from a bag containing only red marbles is impossible.

- Ask students to explain what it means for an event to have a probability of 0 or a probability of 1 and to give an example of each. Some possible responses are shown.
  - *A probability of 0 means that an event is impossible. An example of this is randomly selecting a person with a negative height.*
  - *A probability of 1 means that an event will always occur. An example of this is randomly selecting a square with four 90-degree angles.*
- Give students an event that has a probability of $\frac{1}{2}$, and ask them to determine a related event with a probability that is greater than $\frac{1}{2}$ or less than $\frac{1}{2}$. For example, tell students that a spinner is divided into eight equal-size sections labeled 1–8 and the event of the arrow on the spinner landing on an even number has a probability of $\frac{1}{2}$. Students may give a related event of the arrow on the spinner landing on the number 4, where the probability is less than $\frac{1}{2}$. Students may also give a related event of the arrow on the spinner landing on either an even number or a prime number, where the probability is greater than $\frac{1}{2}$.

▶ Identifying and Addressing Common Misconceptions

- Some students may think that an event is impossible because it has never happened or is difficult to imagine happening. Ask students whether flipping heads on a fair coin 500 times in a row is certain, very likely, somewhat likely, neither likely nor unlikely, somewhat unlikely, very unlikely, or impossible. Some students may think it is impossible. Ask students if it is possible to flip two heads in a row, three heads in a row, etc. Ask them to determine the point at which point flipping consecutive heads becomes impossible. Help students recognize that flipping any finite number of consecutive heads is always possible, even though it may be very rare. Ask them to articulate that such an event is very unlikely rather than impossible. Its probability is close to 0 but not actually 0.

- Some students may assume that an event is certain because, in their experience, it has always happened. Ask students whether having rain somewhere in the U.S. at some time during the month of June is certain, very likely, somewhat likely, neither likely nor unlikely, somewhat unlikely, very unlikely, or impossible. Some students may describe this as certain. Ask them if it is possible to go for one day in June without rain. Then ask if it is possible to go for 2 days in June without rain. Then ask about 3 days, 4, days, etc. Ask them to determine if there is a number of days where having no rain becomes impossible. Help them recognize that while it is very likely to rain at some time in June, it is still not certain. The probability may be very close to 1, but it is not actually 1.
- Some students may assume that the likelihood of an event will change if the event has recently happened. Ask students to estimate the probability of a fair coin flip landing on heads if the previous flip landed on heads. Some students may think that the probability is less than $\frac{1}{2}$. Ask students to examine the coin and determine whether anything has changed about the coin itself. Help them recognize that the likelihood of landing on heads is still $\frac{1}{2}$ regardless of which side it landed on during the previous flip.

► Enrichment

- Give students an event that has a probability close to 1. Ask them to describe the situation of the event not happening and to estimate the probability of the event not happening.
- Ask students to describe an event that, 30 years ago, had a probability less than $\frac{1}{2}$ but today has a probability greater than $\frac{1}{2}$.
- Give students two related events with probabilities greater than $\frac{1}{2}$, and ask them to demonstrate that the probability of at least one of the two events occurring is still less than 1. For example,

give students the event of rolling a 1, 2, 3, or 4 on a standard number cube and the event of rolling a 1, 2, 3, or 5 on a standard number cube.

► Key Terms

chance event, experimental probability, likelihood, probability, theoretical probability

D-S.3.2.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

Statistics and Probability

Investigate chance processes and develop, use, and evaluate probability models.

Use probability to predict outcomes.

► Eligible Content

D-S.3.2.1 Determine the probability of a chance event given relative frequency. Predict the approximate relative frequency given the probability.

► Instructional Supports

- Ask students to explain the meaning of a probability. For example, ask students to explain what it means to say that the probability of a flipped cup landing upside-down is $\frac{2}{5}$. The probability of $\frac{2}{5}$ indicates that in the long run, the ratio of the number of times the flipped cup lands upside-down to the number of times the cup is flipped in total will be $\frac{2}{5}$.
- Ask students to approximate an unknown probability by using repeated trials. For example, give students a container with seven slips of paper labeled "1" and three slips of paper labeled "2" without telling them the ratio. Ask students to estimate the probability of randomly selecting a slip labeled "2" by using repeated trials. Students can select a slip, record the result, return the slip to the container, and repeat the process. Students can estimate the probability by using the number of times a "2" is selected as the numerator and the total number of selections as the denominator. Have students compare their results with others in the class and then check their results with the actual ratio.
- Ask students to compare an estimated probability to a calculated probability and explain any differences they observe. For example, ask students to find the probability of rolling a 2 on a standard number cube by rolling a number cube ten times and comparing this result to the expected probability of $\frac{1}{6}$. Students should recognize that the probability calculated from data

observations is an estimate and not always equal to the actual theoretical probability, though it is usually close to it.

- Ask students to explore how estimates of a probability change as the number of trials increases. For example, ask students to roll a standard number cube ten times and record the observed probability of rolling a 4. Then ask students to roll the number cube forty more times and record the observed probability of rolling a 4 over all fifty trials. Students can then combine results with other students to further increase sample size. Students may note that the observed probability tends to get closer to the calculated probability as the number of trials increases. In general, increasing the number of trials increases the validity of the estimate.
- Ask students to use a given probability to predict how often an event will occur in a given number of repetitions. For example, give students the probability 30% for randomly selecting a yellow ball out of a bag and ask students to predict how many times a yellow ball will be selected in fifty repetitions of randomly selecting a ball, recording the color, and returning the ball to the bag. The ratio of yellow balls selected to total balls selected is expected to be about 30%, or 0.3. Students should determine that the ratio 15 to 50 is equal to 0.3. Therefore, the expected number of yellow balls selected is 15.
- Ask students to explain any discrepancies between a predicted frequency and an observed frequency. For example, "Mario predicts that because the probability of rolling a 3 on a number cube is $\frac{1}{6}$, after rolling a number cube 60 times, a 3 will be rolled $\frac{1}{6}(60) = 10$ times. Mario rolls a number cube 60 times, and a 3 is rolled 12 times." Students can observe that Mario's prediction is the most reasonable prediction, but because chance is involved, it is not always exact. However, even if the prediction does not match the outcome, it is usually close to the observed frequency.

► Identifying and Addressing Common Misconceptions

- Some students may define the approximate probability as the number of times an event occurs divided by the number of times the event does not occur. Ask students to approximate the probability of randomly selecting a green marble from a bag containing 16 green marbles and 24 red marbles. Students who find a probability of $\frac{16}{24}$ or $\frac{2}{3}$ may have this misconception. Ask these students to use the same data to approximate the probability of randomly selecting a red marble from the bag. Students with the misconception will likely respond with $\frac{24}{16}$ or $\frac{3}{2}$. Help them recognize that their answer is greater than 1 and, therefore, not a valid probability.

- Some students may extend an observed frequency from a small sample to a larger sample. Ask students to solve the problem, "A fair number cube is rolled ten times and a result of 6 is obtained three times. How many times is a result of 6 expected when the number cube is rolled one hundred times?" Students who respond with thirty may have this misconception. Ask students to explain the meaning of the ratio $\frac{3}{10}$ in context. Help students recognize that $\frac{3}{10}$ approximates the probability of rolling a 6, and that the probability of rolling a 6 is $\frac{1}{6}$. Because $\frac{3}{10}$ is just an approximation, $\frac{1}{6}$ is still the more valid probability even though it does not match what happened in practice.

► Enrichment

- Ask students to use an observed frequency to estimate the underlying quantities in an experiment. For example, give students a container with twenty-five slips of paper that are labeled with either a "1," "2," or "3" in varying amounts. Ask students to randomly select a slip from the container, record the labeled number, then return the slip. Have students repeat this process twenty times. Then ask students to use their data to determine an estimate of the number of slips labeled with each number.
- Ask students to explore how increasing the number of trials changes the probabilities of exactly matching the expected results and being close to the expected results. For example, ask each student to flip a coin twice. Record the percentage of students who have an observed result of 50% heads. Also record the percentage of students who have observed results within the range of 40% heads to 60% heads. Then ask students to flip the coin eight more times and record the percentage of students with exactly 50% heads and the percentage of students with between 40% heads and 60% heads. Students may observe that when the number of trials increases, the percentage of observed results of exactly 50% heads decreases, while the percentage of observed results between 40% and 60% heads increases.

► Key Terms

approximate, data, frequency, probability, relative frequency

D-S.3.2.2

► Reporting Category, Assessment Anchor, Anchor Descriptor

Statistics and Probability

Investigate chance processes and develop, use, and evaluate probability models.

Use probability to predict outcomes.

► Eligible Content

D-S.3.2.2 Find the probability of a simple event, including the probability of a simple event not occurring.

► Instructional Supports

- Ask students to create a uniform probability model for an experiment. For example, ask students to create a uniform probability model for randomly selecting a day of the week. Students should determine that there are 7 days of the week that all have an equally likely chance of being selected. Students should then create a probability model such as the following example.

Day	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
Probability	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$

- Ask students to create a uniform probability model and use it to determine the probability of an event. For example, give students the following problem: "A class randomly selects 1 of its 22 students to be a student council representative. What is the probability of selecting 1 of the 3 students who sing in the choir?" Remind students that the uniform probability model assigns each student a $\frac{1}{22}$ probability of being chosen. Students should also note that each of the 3 students who sing in the choir has a $\frac{1}{22}$ probability of being chosen, and the probability of selecting any 1 of those 3 students is $\frac{3}{22}$.

- Ask students to create a uniform probability model for a given experiment and use the theoretical probability of an event of the experiment to predict how many times the event will occur when the experiment is conducted multiple times. For example, give students the following experiment and event: The experiment involves spinning a spinner with five equal-size sections labeled 1−5. The event involves the arrow on the spinner landing on an even number. An example of a uniform probability model that students may create is shown.

Result	**Probability**
1	$\frac{1}{5}$
2	$\frac{1}{5}$
3	$\frac{1}{5}$
4	$\frac{1}{5}$
5	$\frac{1}{5}$

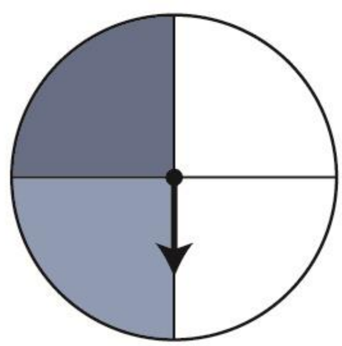
  Students should note that two of the possible results are even numbers. As such, the theoretical probability of the arrow on the spinner landing on an even number is $\frac{2}{5}$. Ask students to use this theoretical probability to determine how many times they would expect the arrow on the spinner to land on an even number in 25 trials. Students should determine that, theoretically, they would expect the arrow on the spinner to land on an even number 10 times because $\frac{2}{5} \cdot 25 = 10$. Finally, ask students to perform the experiment 25 times, record the results, and compare the outcomes to the predicted results.

- Ask students to account for discrepancies between the actual number of times an event occurs and the expected number of times an event occurs. For example, a fair coin is expected to land on heads about 25 times in 50 flips, but that does not mean the coin will always land on heads exactly 25 times. Discrepancies between the expected results and the actual results are because of the unpredictable nature of processes that involve chance. Discrepancies may also be the result of an incorrect model (e.g., the assumption of a uniform probability model will be incorrect if the coin is weighted).

- Ask students to carry out a chance process, record the results, and then develop a probability model. For example, ask students to flip a disposable plastic cup 50 times, record for each flip whether the cup lands on its side, top, or bottom, and create a probability model from the results. One possible result is that the cup lands on its side 35 times, its top 10 times, and its bottom 5 times. These results can be used to create the probability model shown.

Cup Result	Side	Top	Bottom
Probability	$\frac{35}{50}$	$\frac{10}{50}$	$\frac{5}{50}$

- Ask students to conduct an experiment and compare the results with a theoretical probability model for the experiment. For example, give students the following spinner and ask them to spin the arrow 40 times and record the results.



Potential results may include the arrow on the spinner landing on the white section 22 times, the light gray section 7 times, and the dark gray section 11 times. A probability model for these results is shown.

Region	White	Light Gray	Dark Gray
Probability	$\frac{22}{40}$	$\frac{7}{40}$	$\frac{11}{40}$

Because the four sections of the spinner are equal, the following theoretical probability model can be made.

Region	White	Light Gray	Dark Gray
Probability	$\frac{2}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Ask students to compare the two models and explain possible causes of discrepancies in the models.

► Identifying and Addressing Common Misconceptions

- Some students may incorrectly calculate probabilities of events from a uniform probability model. Give students the following probability model for a 6-section spinner and ask students to find the probability of the arrow on the spinner landing on red.

Result	blue	blue	red	red	red	green
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

  Some students may conclude that the probability is $\frac{1}{3}$ because red is 1 of 3 possible colors. Help students understand that even though there are three colors, not all colors are equally likely. On the other hand, each of the 6 sections of the spinner is an equally likely result.

- Some students may think that all experiments have uniform probability models. Ask students to create a uniform probability model for the results of each of them competing in a swim race against an Olympic swimmer. Some students may think that both winning and losing have a probability of $\frac{1}{2}$ because there are only two different possible outcomes. Ask students how many races they think they would win out of 100. Help them recognize that not all outcomes are equally likely.

- Some students may think that a probability model is invalid if it does not match the results of an experiment. Give students a probability model and observed data that does not match the model and ask students to explain the discrepancy. Some students may assume that the fault is with the probability model. Ask students whether a coin always lands heads exactly half the time. Help students understand that small discrepancies frequently occur due to chance and that even large discrepancies can occasionally occur due to chance.

- Some students may think that probability models constructed from data must match a theoretical probability model exactly. Give students the following scenario: "A bag has 5 gray cubes and 5 white cubes in it. All the cubes are the same size. The cubes in the bag are mixed up, a random cube is selected, its color is recorded, and then it is returned to the bag. This is repeated many times." Ask students to explain the differences in the probability models shown.

Probability Model Based on Experimental Data

Cube	White	Gray
Probability	$\frac{6}{10}$	$\frac{4}{10}$

Theoretical Probability Model

Cube	White	Gray
Probability	$\frac{5}{10}$	$\frac{5}{10}$

Some students may think there is a mistake in one of the models or in the experiment. Help students recognize that such a result is possible because a chance process may produce outcomes that are different from a theoretical probability model. Ask students to perform the experiment several times until they have selected a white cube 6 times and a gray cube 4 times. Observe with students that sometimes the models match and sometimes they do not. Discuss that despite the accuracy of a theoretical probability model, there are factors that may make the experimental probability differ. For example, if one of the cubes is repeatedly placed on top, it could be chosen repeatedly despite the theoretical probability not changing throughout the experiment.

- Some students may think there is a single correct probability model derived from experiments. Give students the following scenario: "Two students both ran the same experiment and came up with two different probability models based on the results of their experiments." Ask students how to determine whether one model is correct. Some students may not recognize that, unlike theoretical models, there is no single correct observed probability model. Help students recognize that when probability models are created from experimental results, two people may run the same experiment, get different results, and arrive at different probability models. As long as the experiment was run correctly, there is no wrong probability model. However, models based on larger samples may be considered to be more reliable.

► Enrichment

- Ask students to create or use a uniform probability model to order the probabilities of different events from most likely to least likely.
- Ask students to create or use a uniform probability model to compare the probabilities of events happening and the probabilities of the same events not happening.
- Ask students to find examples of experiments in which a uniform probability model would not be appropriate.
- Give students a probability model that is not uniform and ask students to find probabilities of events. For example, an experiment has the following probability model.

Result	cat	dog	fish	bird
Probability	$\frac{2}{5}$	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{3}$

  Ask students to find the probability of randomly selecting a mammal from the group of animals this model represents.
- Ask students to repeatedly update an experimental probability model by running the experiment more times and incorporating the new data into the original data. Ask students to make observations about how their probability models change as more data are gathered.
- Discuss with students how different a theoretical and an experimental probability model would need to be to convince them that there is a mistake in either one of the models or in the experiment. For example, out of 100 flips, how often would a coin need to land on heads to convince them that the coin is unfair?

► Key Terms

discrepancy, frequencies, outcome, probability model, uniform

D-S.3.2.3

► Reporting Category, Assessment Anchor, Anchor Descriptor

Statistics and Probability

Investigate chance processes and develop, use, and evaluate probability models.

Use probability to predict outcomes.

► Eligible Content

D-S.3.2.3 Find probabilities of independent compound events using organized lists, tables, tree diagrams, and simulation.

► Instructional Supports

- Give students an experiment and ask them to identify the sample space. For example, give students the following experiments: flipping a coin, rolling a standard number cube, and spinning the arrow on a spinner divided into 5 equal sections labeled red, blue, green, yellow, and orange. The sample spaces are {heads, tails}, {1, 2, 3, 4, 5, 6}, and {red, blue, green, yellow, orange}.
- Give students an experiment and an event and ask them to identify the sample space of the experiment and the outcomes of the sample space that are included in the event. For example, give students the experiment of randomly selecting one letter from the alphabet and the event of selecting a vowel. The sample space consists of the 26 letters of the alphabet, while the event consists of the outcomes {a, e, i, o, u}.
- Ask students to distinguish between simple events and compound events. For example, rolling a number cube that lands on 6 is a simple event. Rolling the number cube twice and getting a sum of 11 is a compound event because it consists of either rolling the cube and getting a 6 followed by getting a 5 or rolling the cube and getting a 5 followed by getting a 6.

- Give students some simple events, ask them to use the simple events to create compound events, and ask them to find the probabilities of the compound events. Some examples are shown.
  - Some compound events combine the sets of outcomes for simple events with "and." Give students a number cube and the following simple events: rolling an even number, rolling a prime number, and rolling a number that is greater than 3. One possible compound event consists of rolling a number that is even **and** greater than 3 (i.e., {4, 6}). Such a compound event has a probability of $\frac{2}{6}$, or $\frac{1}{3}$, because it contains 2 out of the 6 equally likely outcomes.
  - Some compound events combine the sets of outcomes for simple events with "or." Give students a number cube, a coin, and the following simple events: rolling a 3, flipping a coin that lands heads, and rolling an odd number. The sample space of this experiment is {1H, 2H, 3H, 4H, 5H, 6H, 1T, 2T, 3T, 4T, 5T, 6T}. One possible compound event consists of rolling an odd number **or** flipping a head (i.e., {1H, 2H, 3H, 4H, 5H, 6H, 1T, 3T, 5T}). Such a compound event has a probability of $\frac{9}{12}$, or $\frac{3}{4}$, because it contains 9 out of the 12 equally likely outcomes.

► Identifying and Addressing Common Misconceptions

- Some students may not recognize two very similar events as being distinct events. Give students the experiment of rolling two number cubes along with the event that both land on 6 and ask, "Is this a simple event or a compound event?" Students should be able to recognize that this scenario represents a compound event because rolling a 6 on the first number cube is a simple event that is distinct from rolling a 6 on the second number cube.
- Some students may try to determine the probability of a compound event by adding together the probability of each simple event. Give students the experiment of rolling a number cube and ask them to find the probability that the number cube will land on an even number that is greater than 5. Some students may add the probability of an even number, $\frac{3}{6}$, and the probability of a number greater than 5, $\frac{1}{6}$, and give a response of $\frac{4}{6}$ or $\frac{2}{3}$. Ask students to consider whether it is more likely to roll an even number or to roll a number that is both even **and** greater than 5. Help students

recognize that including the additional requirement must make the probability of the compound event less than or equal to the likelihood of the simple event.

- Some students may assume that the probability of a compound event cannot be 0 because the probability of each simple event is greater than 0. Give students a probability experiment of rolling a number cube and ask them to find the probability that the result is an odd number greater than 5. Some students may give a nonzero response. Help students recognize that even though both simple events that make up the compound event are possible, both simple events cannot happen at the same time.

► Enrichment

- Ask students to find probabilities of compound events that are made up of 3 or more simple events.
- Give students an experiment and a probability and ask them to determine a compound event that has the given probability.
- Ask students to investigate situations where the probability of a compound event is the same as the probabilities of the simple events that it is created from.

► Key Terms

compound event, sample space, simple event, simulation