A-N.1.1.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

The Number System

Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

Solve real-world and mathematical problems involving division of fractions.

► Eligible Content

A-N.1.1.1 Interpret and compute quotients of fractions (including mixed numbers), and solve word problems involving division of fractions by fractions.

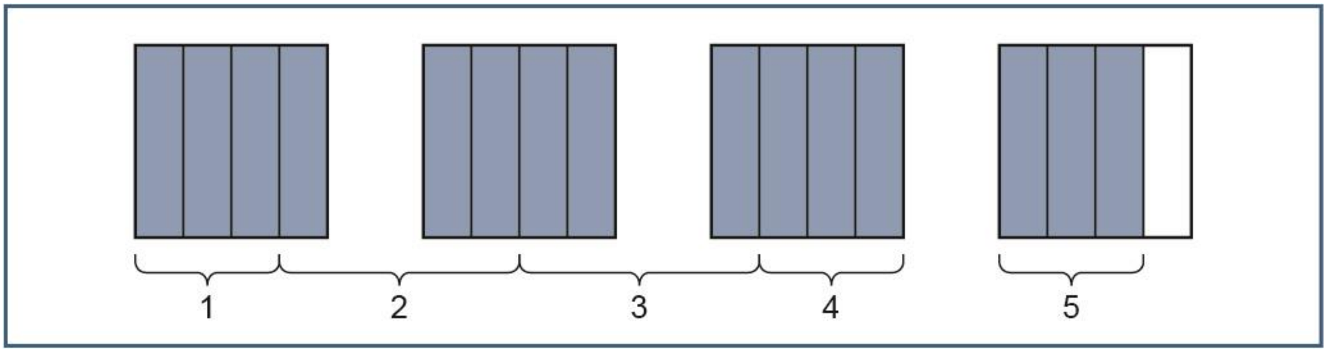
► Instructional Supports

Connections to multiplication

- Help students build on their previous understanding of operations with integers by asking them to first create a fact family comprised of integers and then create a fact family using the reciprocals of the same integers. For example, a student creates the fact family 2 • 5 = 10, 5 • 2 = 10, 10 ÷ 2 = 5, and 10 ÷ 5 = 2. The student then creates a fact family using the reciprocals: $\frac{1}{2} \bullet \frac{1}{5} = \frac{1}{10}$, $\frac{1}{5} \bullet \frac{1}{2} = \frac{1}{10}$, $\frac{1}{10} \div \frac{1}{2} = \frac{1}{5}$, and $\frac{1}{10} \div \frac{1}{5} = \frac{1}{2}$.
- Ask students to create a fraction multiplication problem with an unknown that stems from a given division problem with an unknown. Then, ask them to solve the multiplication problem and demonstrate that the solution to the multiplication problem is also the solution to the division problem. For example, the fraction division problem $9\frac{1}{3} \div ? = 2\frac{1}{3}$ can be rewritten as $2\frac{1}{3} \bullet ? = 9\frac{1}{3}$ or as $? \bullet 2\frac{1}{3} = 9\frac{1}{3}$. Have students use repeated addition to show that $4 \bullet 2\frac{1}{3} = 9\frac{1}{3}$ and repeated subtraction to show that $9\frac{1}{3} \div 4 = 2\frac{1}{3}$.

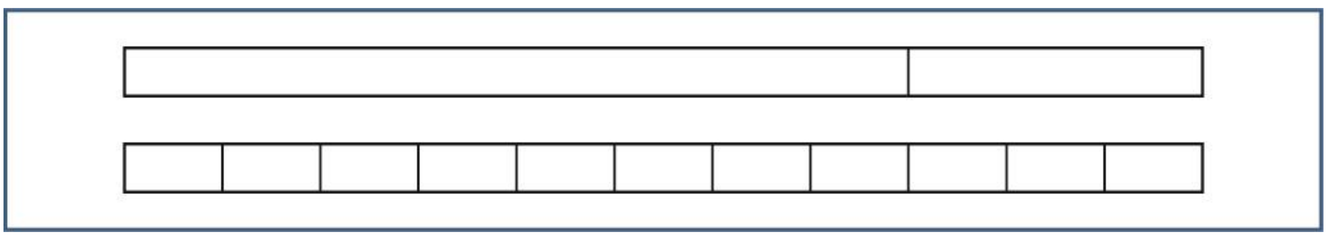
Visual models without remainders

- Ask students to use visual representations to model division with fractions. For example, give students the quotient $3\frac{3}{4} \div \frac{3}{4}$ and ask them to represent it using fraction models with shading. The dividend $3\frac{3}{4}$ can be represented by three completely shaded squares and a fourth square in which only $\frac{3}{4}$ is shaded, as shown in the following model.



Students can arrange sections to show groups of size $\frac{3}{4}$. Ask them to count the total number of groups to conclude that there are 5 groups of size $\frac{3}{4}$ in $3\frac{3}{4}$, so $3\frac{3}{4} \div \frac{3}{4} = 5$.

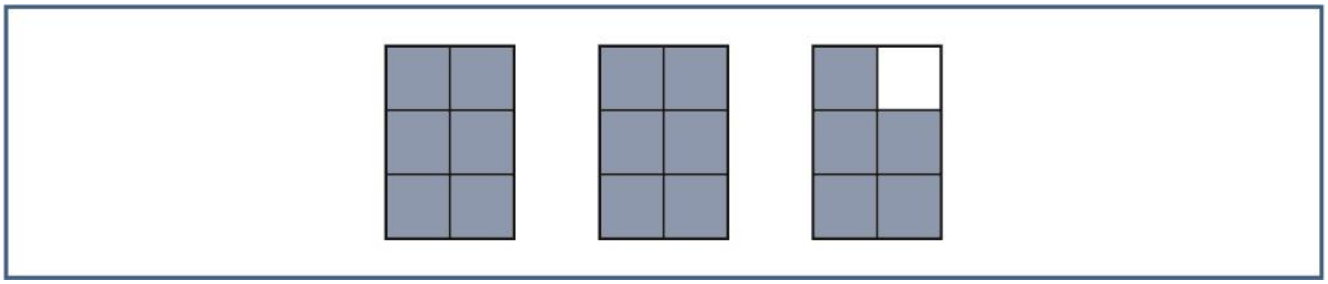
- Ask students to interpret tape diagrams and use them to model division with fractions. For example, give students the following tape diagram illustrating the quotient of two fractions.



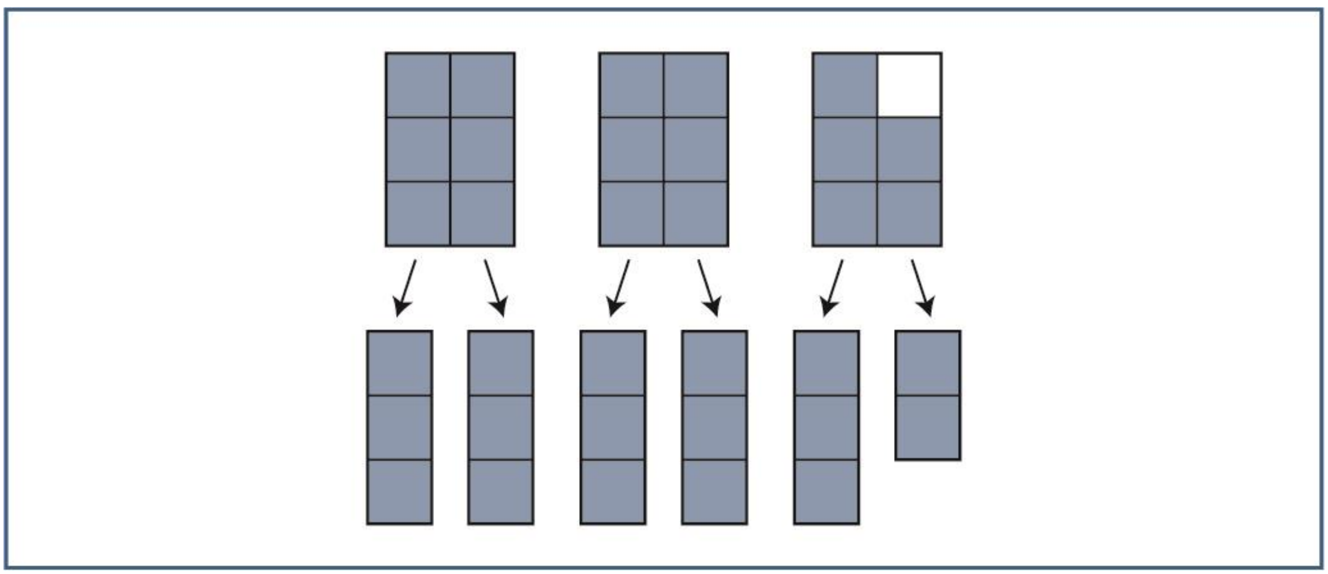
Ask students to interpret what the top row and the bottom row could represent. Here, the top row represents the fraction $1\frac{3}{8}$, while each section in the bottom row represents the fraction $\frac{1}{8}$. As such, the tape diagram shows $1\frac{3}{8}$ divided into groups of $\frac{1}{8}$. Therefore, $1\frac{3}{8} \div \frac{1}{8} = 11$.

Visual models with remainders

- Ask students to use visual representations to model division with fractions when there is a remainder. For example, give students the quotient $2\frac{5}{6} \div \frac{1}{2}$. The dividend of $2\frac{5}{6}$ is represented with the following diagram.

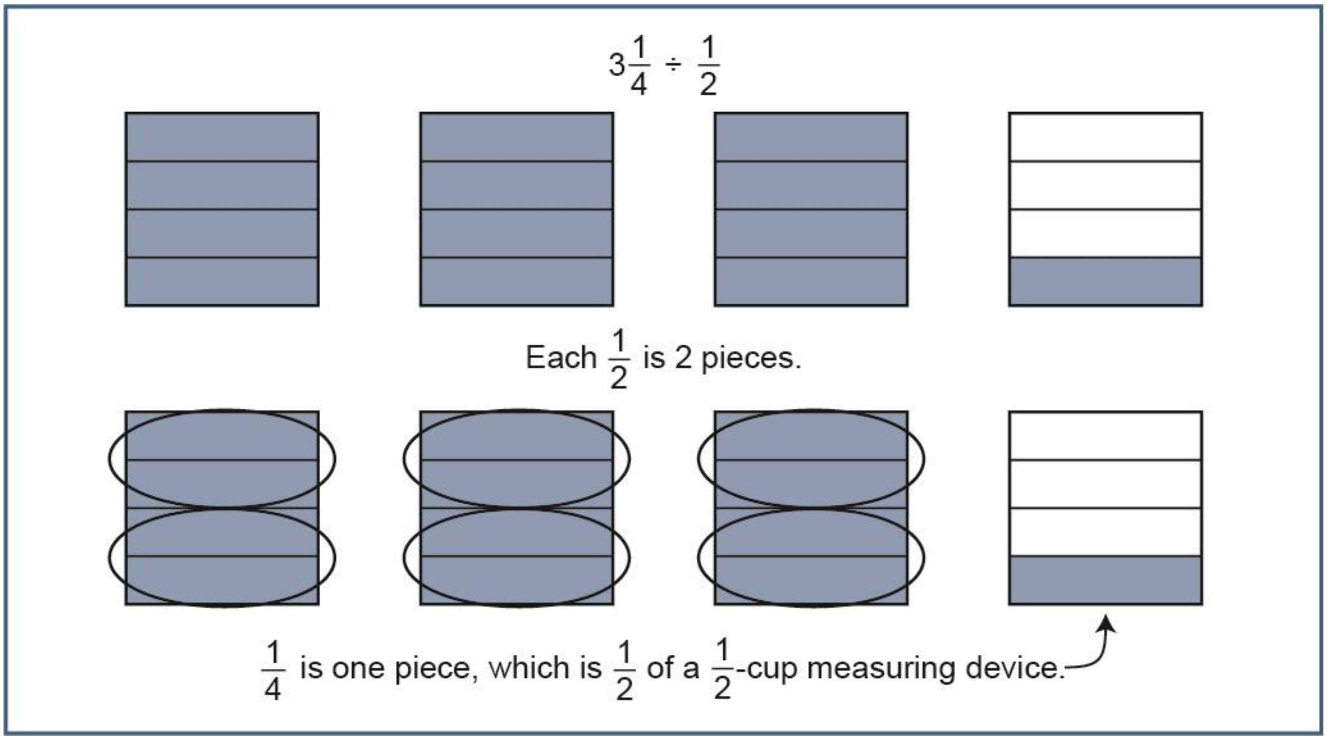


Then, ask students to represent the divisor of $\frac{1}{2}$ as many times as possible using the shaded sections from the fraction models.



The division of $2\frac{5}{6} \div \frac{1}{2}$ can be interpreted as asking how many groups of size $\frac{1}{2}$ are in $2\frac{5}{6}$. Point out to students that the unit is now $\frac{1}{2}$ of a fraction model instead of one full fraction model. Explain that there are 5 whole units of size $\frac{1}{2}$ in $2\frac{5}{6}$ with $\frac{2}{6}$ of a fraction model remaining. While the size of the remainder is $\frac{2}{6}$ of a fraction model, the $\frac{2}{6}$ is two-thirds the size of the unit of $\frac{1}{2}$, so the fraction part of the quotient is $\frac{2}{3}$. There are $5\frac{2}{3}$ units of size $\frac{1}{2}$ in $2\frac{5}{6}$. Make sure that students can identify each of these parts in their diagrams.

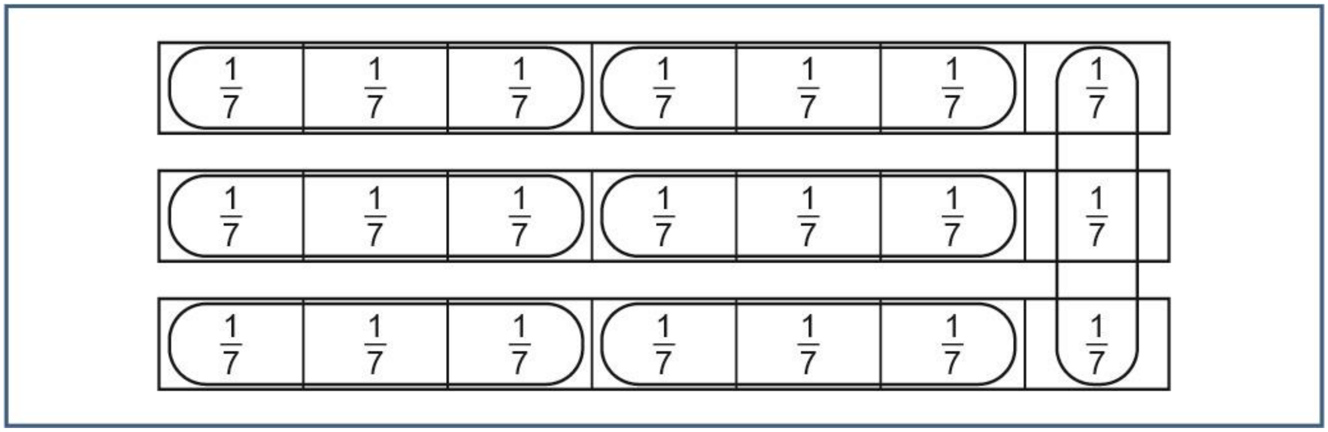
- Ask students to use concrete models to demonstrate division with fractions. For example, ask students to fill a container with $3\frac{1}{4}$ cups of water. Then, count how many times a $\frac{1}{2}$-cup measuring device can be filled completely using the water in the container.



After filling the $\frac{1}{2}$-cup measuring device 6 times, $\frac{1}{4}$ cup of water remains in the container. The fraction part of the quotient can be determined by measuring the amount of water remaining in the container in terms of the $\frac{1}{2}$-cup measuring device. Because the remaining amount of water fills up $\frac{1}{2}$ of the $\frac{1}{2}$-cup measuring device, the full quotient is $6\frac{1}{2}$.

Multiplying by the reciprocal

- Ask students to use visual representations to show the relationship between division by a fraction and multiplication by the reciprocal of the same fraction. For example, ask students to represent $3 \div \frac{3}{7}$ by dividing three fraction bars into 7 parts and circling groups of size $\frac{3}{7}$, as shown in the diagram.



There are 7 groups of size $\frac{3}{7}$, so the quotient is 7. The division can also be interpreted as dividing the fraction bars into seven parts, which multiplies the number of items being divided by 7. There are now 21 pieces to divide that are each $\frac{1}{7}$ of a fraction bar. Have students divide the pieces into groups of size $\frac{3}{7}$, which is the same as groups of 3 pieces. Dividing 21 pieces into groups of 3 pieces yields 7 groups. That is, $3 \bullet 7 \div 3 = 3 \bullet \frac{7}{3} = \frac{21}{3} = 7$. Therefore, $3 \div \frac{3}{7} = 3 \bullet \frac{7}{3}$.

► Identifying and Addressing Common Misconceptions

- Students may apply division to the numerator and denominator separately, interpreting $\frac{a}{b} \div \frac{c}{d}$ as $\frac{a \div \frac{c}{d}}{b \div \frac{c}{d}}$. Ask students to solve $\frac{7}{8} \div \frac{1}{2}$ and explain their process. Some students may confuse the division process with the process for creating equivalent fractions and arrive at a quotient of $\frac{14}{16}$ (or $\frac{7}{8}$ if simplified). Use fraction bars to show that applying the misconception results in a quotient of $\frac{a}{b}$.
- Students who perform the division correctly may struggle with the idea that division can result in a number greater than the dividend. Ask students to solve $\frac{1}{4} \div \frac{4}{7}$ and then consider whether their quotient is reasonable. Some students may find the correct quotient of $\frac{7}{16}$ but when asked if the answer is reasonable, they may respond, due to preconceptions from whole number division, that the answer is not reasonable because the quotient is greater than the dividend. To illustrate how the quotient of a division problem can be greater than the dividend, use a $\frac{1}{2}$-cup measuring device

to measure out an amount of water and then use a $\frac{1}{4}$-cup measuring device to demonstrate that there are twice as many $\frac{1}{4}$-cup portions as $\frac{1}{2}$-cup portions in the same amount of water. This shows that $\frac{1}{2} \div \frac{1}{4} = 2$ and is an example of a situation where the quotient is greater than the dividend.

- Students may incorrectly interpret the remainder. Ask students to solve $3 \div \frac{2}{3}$. Students may find a quotient of $4\frac{1}{3}$ instead of $4\frac{1}{2}$ by first finding the whole number part and then appending the remainder of $\frac{1}{3}$. Use visual models to show that the remainder is part of the new unit which is represented by the divisor.

► Enrichment

- Ask students to explore the relationship between dividing the same quantities by different fractions, looking for patterns (e.g., the quotient when dividing by $\frac{1}{4}$ is double the quotient when dividing by $\frac{1}{2}$, or when there is no remainder when dividing by $\frac{1}{3}$, there is also no remainder when dividing by $\frac{1}{6}$).
- Have students identify real-world situations involving division of fractions by fractions and write equations to model these situations.
- Give students sets of four numbers and have them find the division problems of the form $\frac{a}{b} \div \frac{c}{d}$ that result in the greatest quotient and the least quotient. Start with whole numbers for values of *a*, *b*, *c*, and *d*. Once students have time to grow comfortable with whole numbers, start using fractions for some or all the values.
- Explore equivalent fractions and relate that concept to the algorithm of multiplying by the inverse. Ask students to create fractions equivalent to $\frac{4}{9}$ by multiplying both the numerator and denominator by a whole number. Then, ask students to create fractions that are equivalent to $\frac{4}{9}$ by multiplying both the numerator and denominator by a fraction. Explain that they have created a complex fraction that is equivalent to $\frac{4}{9}$ and that just as a simple fraction represents division of

two whole numbers, a complex fraction represents division of two fractions. Then, ask students to create fractions that are equivalent to $\frac{\frac{1}{4}}{\frac{3}{8}}$ by multiplying both the numerator and denominator by a fraction. Have students look for a multiplier that results in a simple fraction. Finally, ask students to demonstrate why the general rule $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$ is true by rewriting the expression $\frac{a}{b} \div \frac{c}{d}$ as a complex fraction. For example, the expression $\frac{a}{b} \div \frac{c}{d}$ can be rewritten as $\frac{\frac{a}{b}}{\frac{c}{d}}$. Both the numerator and denominator can be multiplied by the reciprocal of $\frac{c}{d}$, and the denominator of the complex fraction becomes 1 because $\frac{\frac{a}{b}}{\frac{c}{d}} \bullet \frac{\frac{d}{c}}{\frac{d}{c}} = \frac{\frac{ad}{bc}}{\frac{cd}{dc}} = \frac{\frac{ad}{bc}}{\frac{cd}{cd}} = \frac{\frac{ad}{bc}}{1}$. Since $\frac{\frac{ad}{bc}}{1}$ is equivalent to $\frac{ad}{bc}$, it can be concluded that the general rule $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$ is true.

► Key Terms

fraction model, reciprocal, unit, tape diagram

A-N.2.1.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

The Number System

Compute with multi-digit numbers and find common factors and multiples.

Compute with multi-digit numbers using the four arithmetic operations with or without a calculator.

► Eligible Content

A-N.2.1.1 Solve problems involving operations (+, –, ×, ÷) with whole numbers, decimals (through thousandths), straight computation, or word problems.

► Instructional Supports

- Help students relate the standard algorithm for division to partial quotients by asking them to solve the same division problem using both the standard algorithm and partial quotients and identify the corresponding steps between the methods. For example, the quotient of 1,449 and 3 can be solved using either method shown.

Partial Quotients

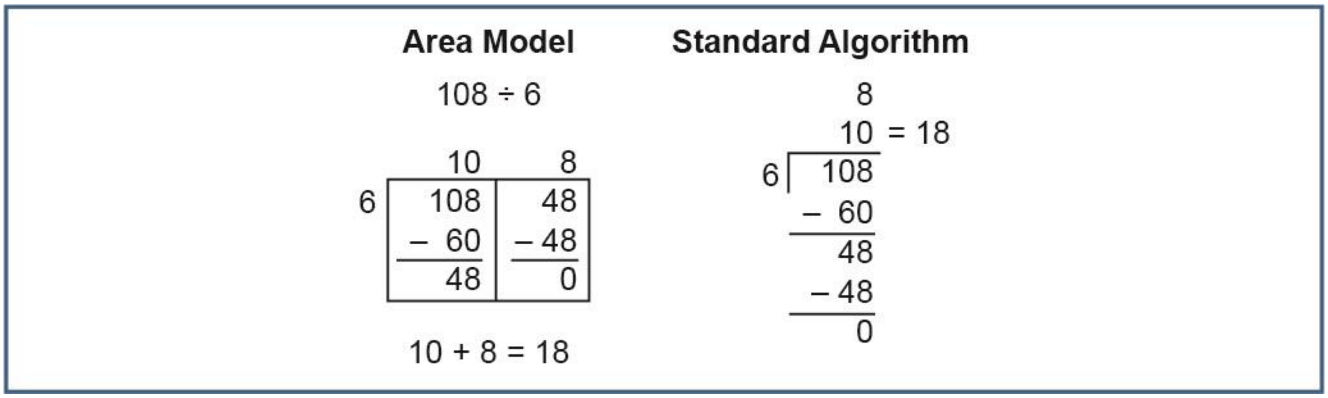
$$\begin{array}{rr|l} 3 & \overline{)\,1,449} & \\ & -1,200 & \times\ 400 \\ \hline & 249 & \\ & -240 & \times\ 80 \\ \hline & 9 & \\ & -9 & \times\ 3 \\ \hline & 0 & \end{array}$$

Standard Algorithm

$$\begin{array}{rr} & 3 \\ & 80 \\ & 400 \\ 3 & \overline{)\,1,449} \\ & -1,200 \\ \hline & 249 \\ & -240 \\ \hline & 9 \\ & -9 \\ \hline & 0 \end{array}$$

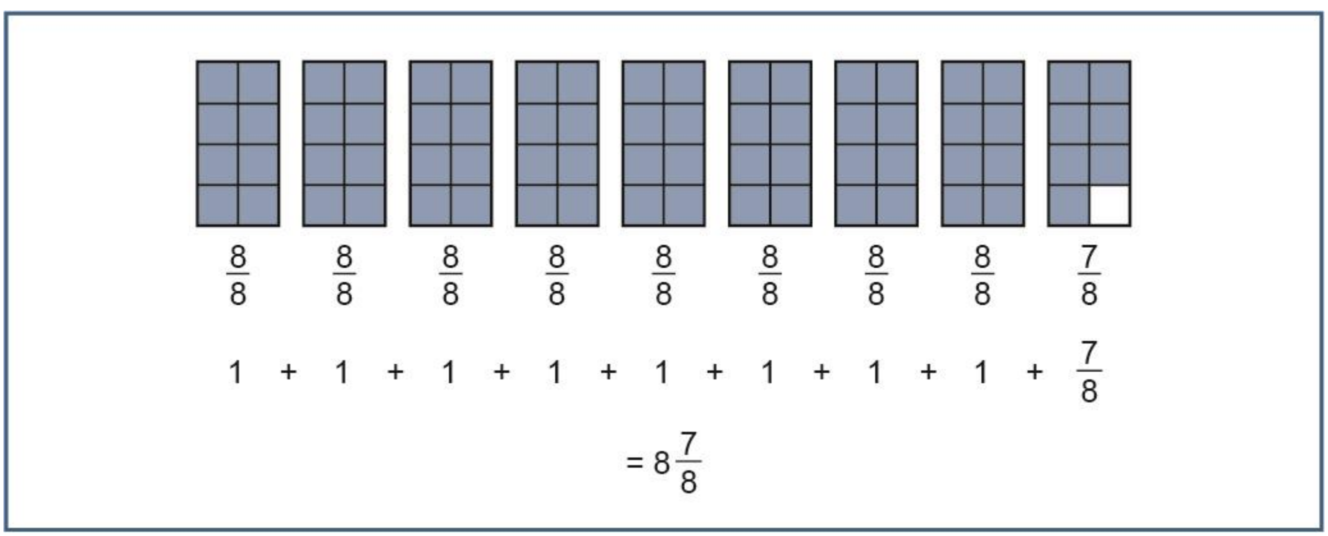
Both methods illustrate that the resulting quotient of 483 is derived from the sum of 400 + 80 + 3.

- Help students relate the standard algorithm for division to area models by asking them to solve the same division problem using both the standard algorithm and an area model and identify the corresponding steps between the methods. For example, the quotient of 108 and 6 can be found using either method shown.

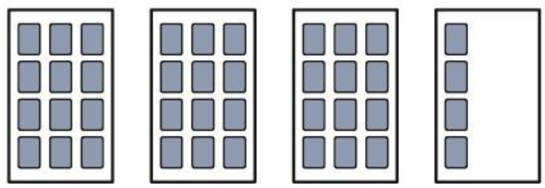


Both the area model and the standard algorithm show that there are a total of 18 groups of 6 in 108.

- Ask students to relate the standard algorithm for division to fraction models by finding a quotient with a remainder using drawings of squares. For example, ask students to find the quotient of 71 and 8 by drawing groups of 8 squares until the total number of squares is greater than or equal to 71. Then have them shade in 71 squares, filling as many complete groups as possible. Ask students to count the number of shaded squares in the group of 8 that is not fully shaded. These 7 squares represent 7 parts of a complete group of 8 squares. This indicates that the remainder of 7 represents $\frac{7}{8}$ and the quotient of 71 and 8 is $8\frac{7}{8}$.



- Ask students to create and explain an illustration that represents a real-world context involving a quotient and remainder. Then have students apply the standard algorithm to the same problem and explain the connection between the remainder and the illustration. For example, give students this problem: “Nia is placing cards in boxes. Each box holds 12 cards, and Nia has 40 cards. How many boxes can Nia fill?” Students may create the model shown.

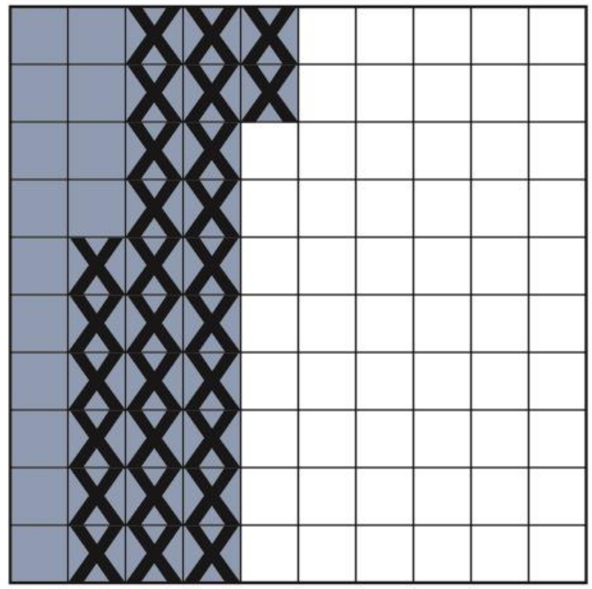


$$\begin{array}{r|l} & 3 \text{ R } 4 = 3\frac{4}{12} = 3\frac{1}{3} \\ 12 & 40 \\ & -36 \\ \hline & 4 \end{array}$$

Each box represents a group of 12 cards. There are 3 full boxes and 4 cards left over. In order to express the answer as a mixed number, the units need to be consistent. The 4 can be expressed as $\frac{4}{12}$ of a full box since there are 4 cards left over and a full box would contain 12 cards.

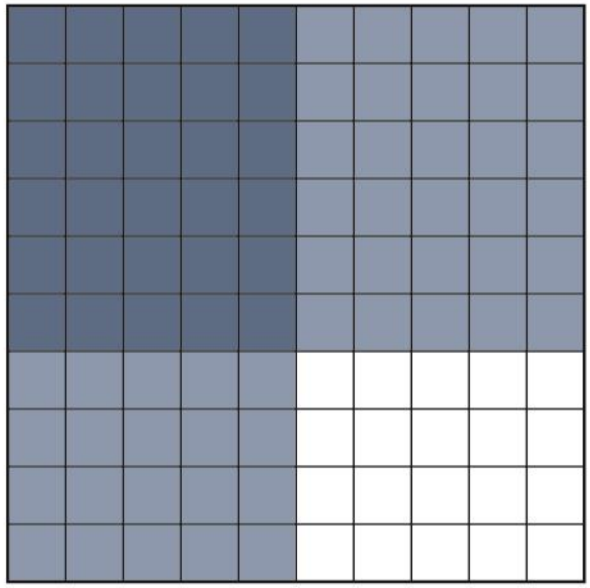
Relating subtraction of decimal numbers with subtraction of whole numbers

- Ask students to relate the subtraction of decimal numbers to the subtraction of whole numbers. For example, ask students to find the difference of 42 and 28 using a decimal grid. Then have students use the same decimal grid to find the difference of 4.2 and 2.8 and the difference of 0.42 and 0.28. Ask students to explain the meaning of each square on the grid in each of these situations.



Relating multiplication of decimal numbers to multiplication of whole numbers

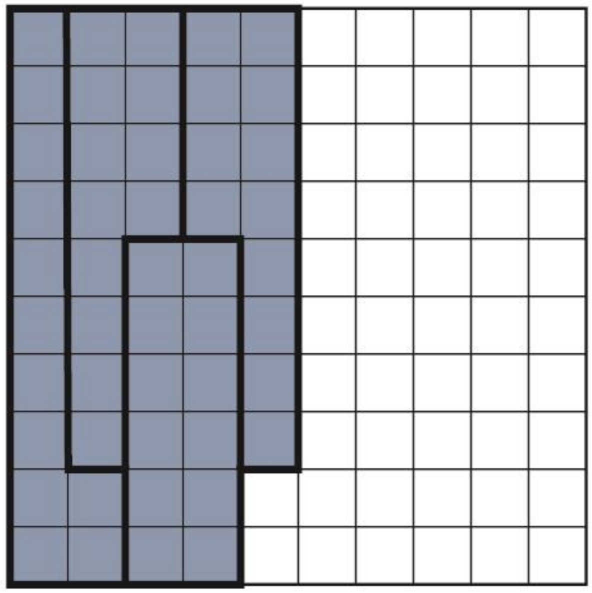
- Ask students to relate the multiplication of decimal numbers to the multiplication of whole numbers. For example, give students the problem 6 • 5. Then have students use a decimal grid to find the product of 0.6 and 0.5. Ask students to relate the decimal grid to the product of 6 and 5.



Counting the number of squares in the grid that are in both a shaded row and a shaded column gives 30. Since each square represents 0.01 of the whole, 0.6 • 0.5 = 0.30. So, in the same way that 6 • 5 = 30, 0.6 • 0.5 = 0.30.

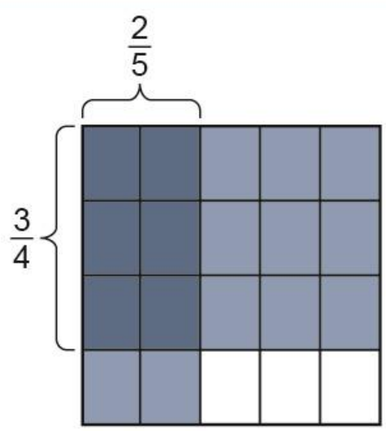
Relating division of decimal numbers to division of whole numbers

- Ask students to use decimal grids to show division of decimals and relate the grid to the standard algorithm for division. For example, give students the problem 0.48 ÷ 0.12 and have the students solve it using the standard algorithm. Then, have students represent 0.48 on a decimal grid by shading $\frac{48}{100}$ of the grid and divide the shaded region into groups representing 0.12. Students can then count the groups and compare the count to the result of the standard algorithm.



Relating multiplication of decimal numbers to multiplication of fractions

- Ask students to relate the multiplication of decimals to the multiplication of fractions. For example, give students the problem 0.75 • 0.4 and have them solve by first converting both numbers to fractions and then representing the product with an area model.



Have students determine the fraction of the rectangle with overlapping shading and compare this fraction to the result of applying the standard algorithm for multiplication to find the product.

$$\begin{array}{r} 0.75 \\ \times\ 0.40 \\ \hline 0.300 \end{array}$$

The overlapping shaded portion is $\frac{6}{20}$, which is equivalent to $\frac{3}{10}$. This is the same as the result of multiplying the decimals using the standard algorithm.

► Identifying and Addressing Common Misconceptions

- Students may assign place values incorrectly when using the standard algorithm. Ask students to calculate 26,000 ÷ 20 using the standard algorithm. Some students may misunderstand the role of place value in the standard algorithm and calculate a quotient of 13 or 130. Use the partial quotients method and the standard algorithm to solve the same division problem and have students find the relationship between each step in the two methods.
- Students may attempt to place one digit in the quotient for each digit in the dividend. Ask students to calculate 208 ÷ 4 using the standard algorithm. When trying to divide 2 by 4, students may be unable to complete the algorithm or feel obligated to use a nonzero digit in the quotient. Use base-ten blocks to demonstrate that 0 can be placed in the standard algorithm and the value can be realigned. It may be helpful to demonstrate that the "skipped" digits, such as the 2 in 208 ÷ 4, are equivalent to placing a 0 in the corresponding locations while performing the standard algorithm.
- Students may confuse the placement and role of the dividend and divisor when using the standard algorithm. Ask students to calculate 89 ÷ 445 using the standard algorithm. Some students may calculate a quotient of 5 instead of 0.5. Use an area model and the standard algorithm to solve the same problem and have students find the relationship between each part of the area model and the standard algorithm.
- Students may not align decimals when using the standard algorithm for addition or subtraction. Ask students to solve 11.3 – 0.53. Some students may align the right digit as would be done in addition or subtraction of whole numbers and find a difference of 6. Suggest that students use

estimation to check the reasonableness of their answer. An approximation of the problem as $11 - \frac{1}{2}$ can show that the answer should be near 10.5 and that 6 is not reasonable.

- Students may place the decimal point incorrectly when multiplying. Ask students to solve 2.95 • 4.8. Students who misplace the decimal point may calculate a product of either 1.416 because they ignored the 0 at the end or 141.6 because they moved the decimal point from the left instead of from the right. Suggest that students use estimation to check the reasonableness of their answer. An approximation of the problem as 5 times 3 can show that the answer should be near 15 and that 1.416 and 141.6 are not reasonable.
- Students may attempt to divide the greater number by the lesser number regardless of the actual presentation of the division problem. Ask students to solve 1.28 ÷ 2.56. Students who divide the greater number by the lesser number will find a quotient of 2 instead of 0.5. Help students use decimal grids to represent the division problem, relating each step in the process to the numbers in the standard algorithm.

► Enrichment

- Ask students to create a diagram relating each part of the standard algorithm to the corresponding part of other methods the student has learned. Students should recognize the relationships between partial quotients and the standard algorithm as well as the relationship between the area model and the distributive property.
- Ask students to identify and illustrate how the standard algorithm for division connects to the inverse relationship between division and multiplication. An area model can be used to show students this connection in a concrete application.
- Ask students to use the standard algorithm and visual models to explore the relationship between remainders and the fractional part of a mixed number quotient. Have students explicitly state, in context, the connection between the remainder and the fractional part of a mixed number quotient.
- Ask students to explore the effect of multiplying one or both numbers in a two-number addition or subtraction problem by a power of 10. Have students use place value to explain the relationship between multiplying both numbers by 10 and the original problem.

- Ask students to explore the effect of multiplying one or both numbers in a two-number multiplication problem by a power of 10. Have students use the commutative property of multiplication to explain the relationship.
- Ask students to explore the effect of multiplying one or both numbers in a two-number division problem by a power of 10. Have students use the commutative property of multiplication to explain the relationship.
- Ask students to identify and explain the inverse relationships between operations.

► Key Terms

dividend, divisor, factors, fluency, fraction, multiples, remainder, standard algorithm, decimals, operations

A-N.2.2.1 & A-N.2.2.2

► Reporting Category, Assessment Anchor, Anchor Descriptor

The Number System

Compute with multi-digit numbers and find common factors and multiples.

Apply number theory concepts (specifically, factors and multiples).

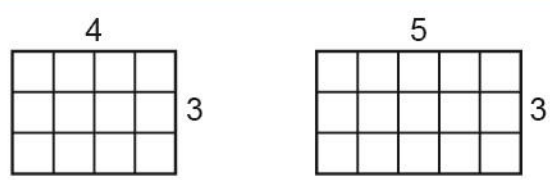
► Eligible Content

A-N.2.2.1 Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12.

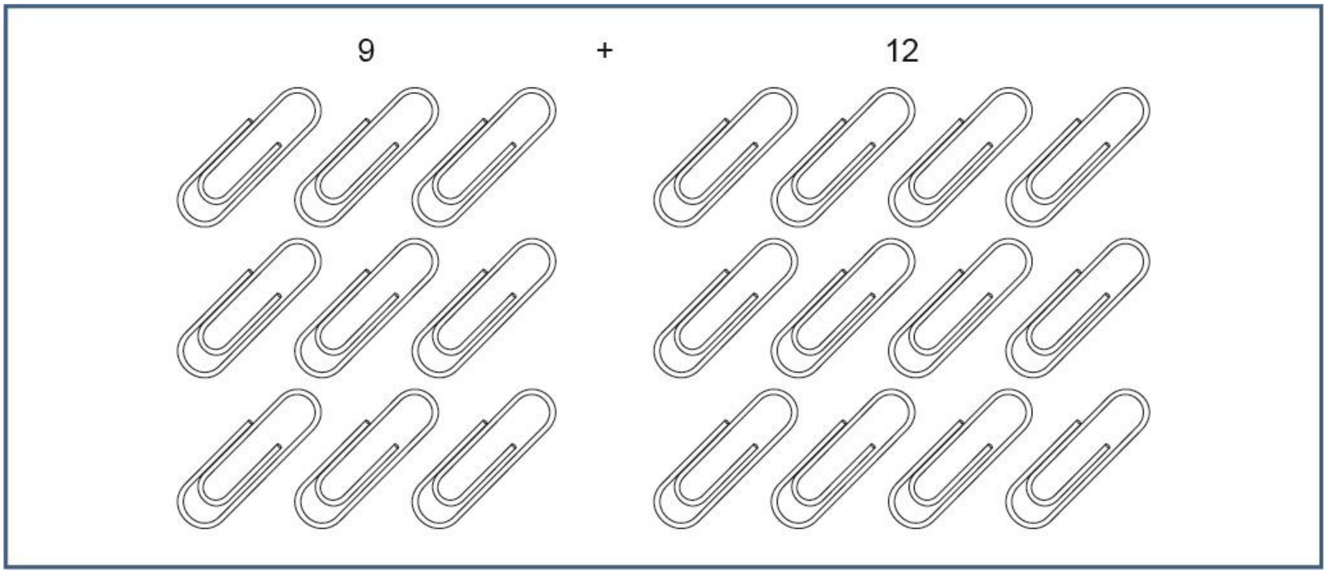
A-N.2.2.2 Apply the distributive property to express a sum of two whole numbers, 1 through 100, with a common factor as a multiple of a sum of two whole numbers with no common factor.

► Instructional Supports

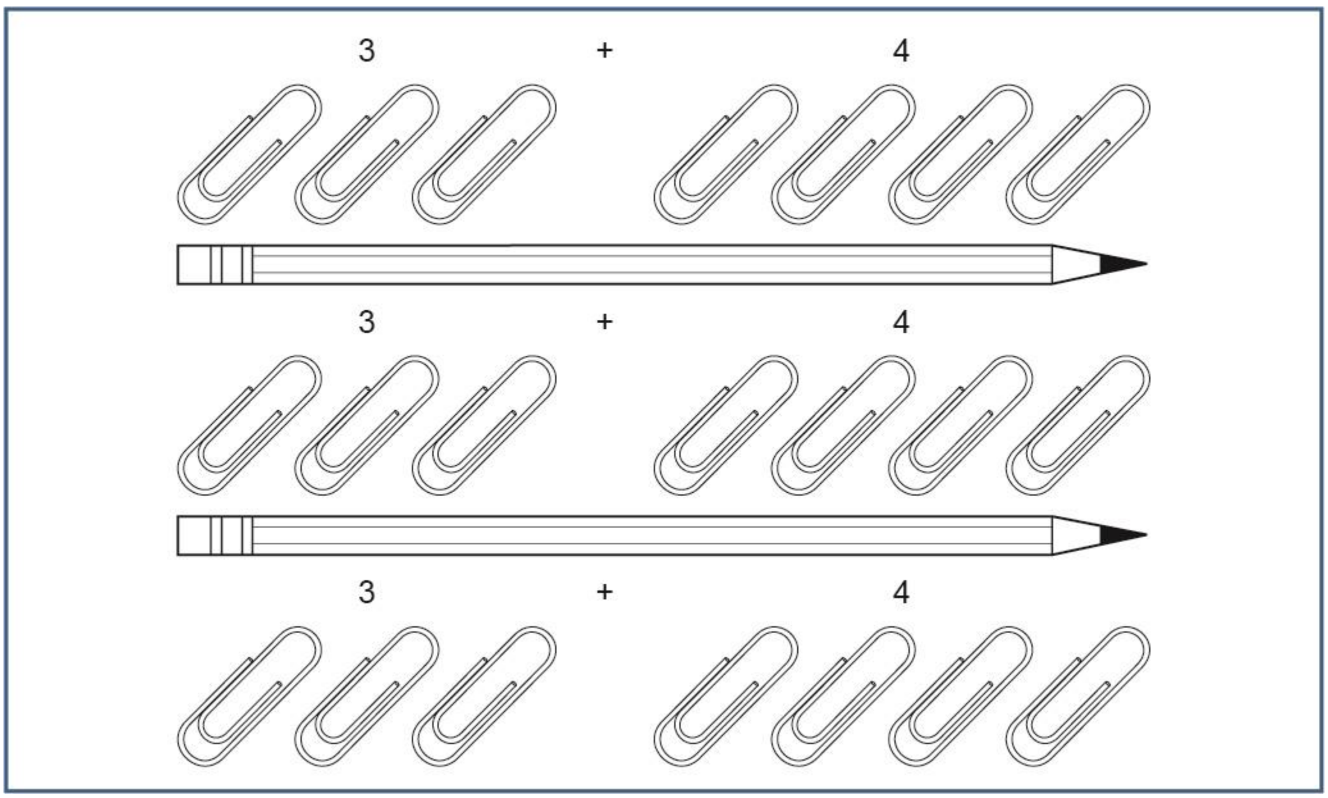
- Ask students to use area models to represent the relationship between two numbers and their common factors. For example, ask students to use models to find the common factors of 12 and 15. Have students create rectangular area models with the same height to represent each number. Ask students to discuss why this can only be done if the shared height is a common factor of the two numbers.



- Ask students to use objects to show how to rewrite a sum as a multiple of another sum. For example, give students the expression 9 + 12 and have them model the expression using paperclips. The paperclips used to represent each number can be arranged into rectangular arrays with a height equal to the greatest common factor of the two numbers, which in this case is 3.



Ask students to use objects to divide each array into equal groups.



The arrangement of the objects now shows that 9 + 12 is the same as 3 groups of 3 + 4, or 3(3 + 4).

- Ask students to use ladder diagrams to find the greatest common factor of two numbers. For example, give students the numbers 45 and 60. Students can create a ladder diagram to find the greatest common factor of the numbers by first dividing both numbers by a common factor of 5. The resulting quotients can then be divided by the common factor of 3. At this point, the resulting quotients, 3 and 4, have no common factors. The greatest common factor of 45 and 60 is found with the equation 5 • 3 = 15.

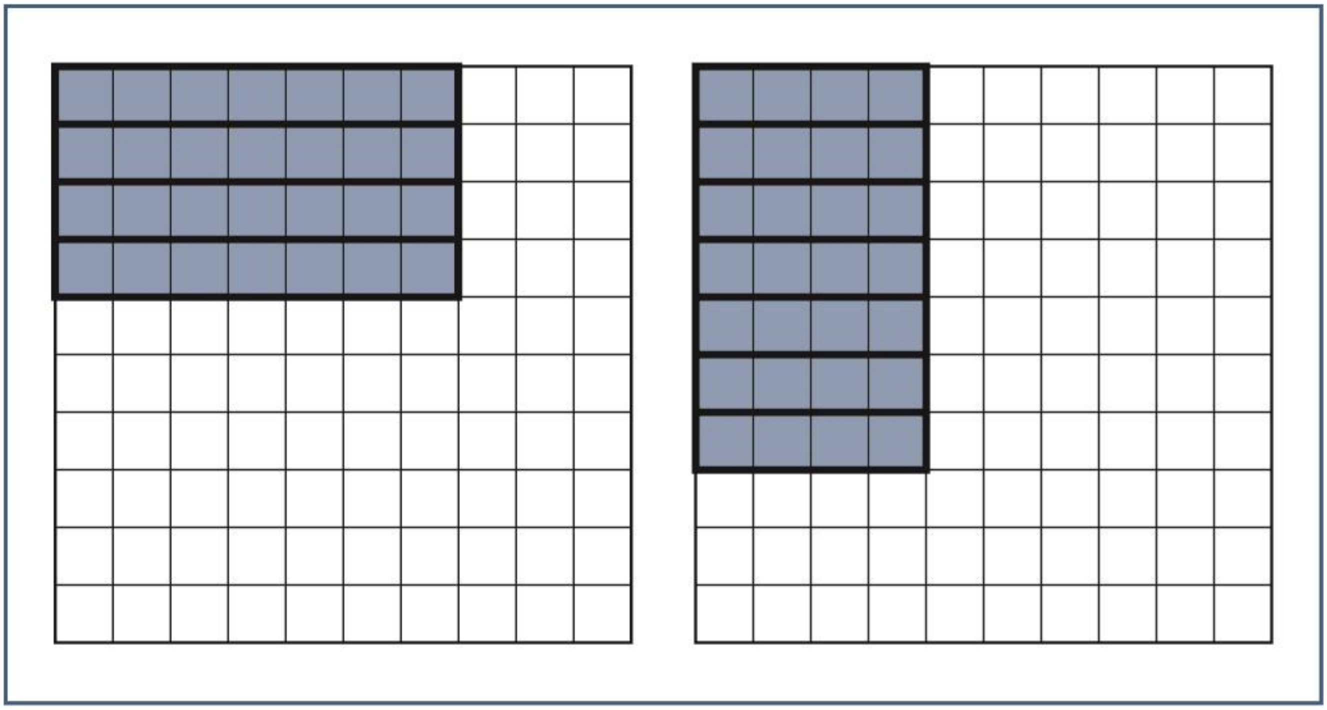
$$\begin{array}{r|rr} 5 & 45 & 60 \\ \hline 3 & 9 & 12 \\ \hline & 3 & 4 \end{array}$$

$$5 \bullet 3 = 15$$

- Ask students to explain why the greatest common factor of two distinct prime numbers must be 1. Students should understand that the only factors of a prime number are 1 and the prime number itself. Thus, the only common factor for two distinct prime numbers is 1. For example, the only factors of 13 are 1 and 13 and the only factors of 5 are 1 and 5, so the only common factor of 5 and 13 is 1.

- Ask students to find the greatest common factor of two numbers whose only common factor is 1. For example, consider the numbers 8 and 9. The only factors of 8 are 1, 2, 4, and 8. The only factors of 9 are 1, 3, and 9. Thus, the only common factor of 8 and 9 is 1. This means that the numbers 8 and 9 are relatively prime.

- Ask students to find the least common multiple of two numbers by modeling the multiples on a grid. For example, have students start with two 10 by 10 grids and give students the numbers 7 and 4. Have students shade in a group of 7 squares in the first grid and a group of 4 squares in the second grid. Then, ask students to continue to shade in squares by applying the following rules.
    - If there are more shaded squares in the first grid, shade in 4 squares in the second grid.
    - If there are more shaded squares in the second grid, shade in 7 squares in the first grid.
    - If the number of shaded squares in each grid is equal, stop.



► Identifying and Addressing Common Misconceptions

- Students may not understand that the greatest common factor of a set of numbers can be equal to one of the numbers in the set. Give students the numbers 12 and 24 and ask them to find the greatest common factor. Some students may not realize that 12 is a common factor and identify the greatest common factor as 6. Ask students to list all factors of each number without considering the other number and then find the greatest number that appears in both sets.
- Students may not understand that the least common multiple of two numbers may be one of the numbers. Give students the numbers 3 and 9 and ask them to find the least common multiple. Some students may not realize that 9 is a common multiple and identify 18 as the least common multiple. Ask students to create a table showing some of the multiples of the two numbers and identify the smallest multiple found in both lists.
- Students may factor out a common factor other than the greatest common factor and not recognize that it is possible to further factor the given numbers. Give students the expression 66 + 48 and ask them to factor out the greatest common factor of the two numbers. Some students may arrive at 2(33 + 24). Ask students to list all factors of the two numbers inside the parentheses and look for any additional common factors.

► Enrichment

- Ask students to find a real-world situation in which factoring out the greatest common factor of two numbers provides information that is not obvious in the sum of the two numbers. For example, give students this situation: “A store received two shipments of cans of soup. One shipment had 24 cans and another shipment had 42 cans.” By factoring out the greatest common factor, students can see that one shipment had 4 boxes of 6 cans and the other shipment had 7 boxes of 6 cans.
- Ask students to find the greatest common factor of a set of three or more numbers.
- Ask students to find the least common multiple of a set of three or more numbers.

► Key Terms

distributive property, greatest common factor, least common multiple, prime factorization

A-N.2.2.1 & A-N.2.2.2

► Reporting Category, Assessment Anchor, Anchor Descriptor

The Number System

Compute with multi-digit numbers and find common factors and multiples.

Apply number theory concepts (specifically, factors and multiples).

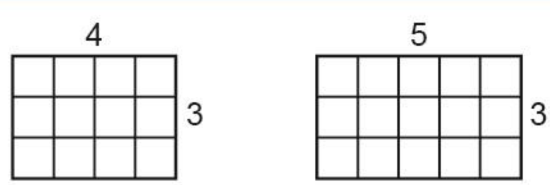
► Eligible Content

A-N.2.2.1 Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12.

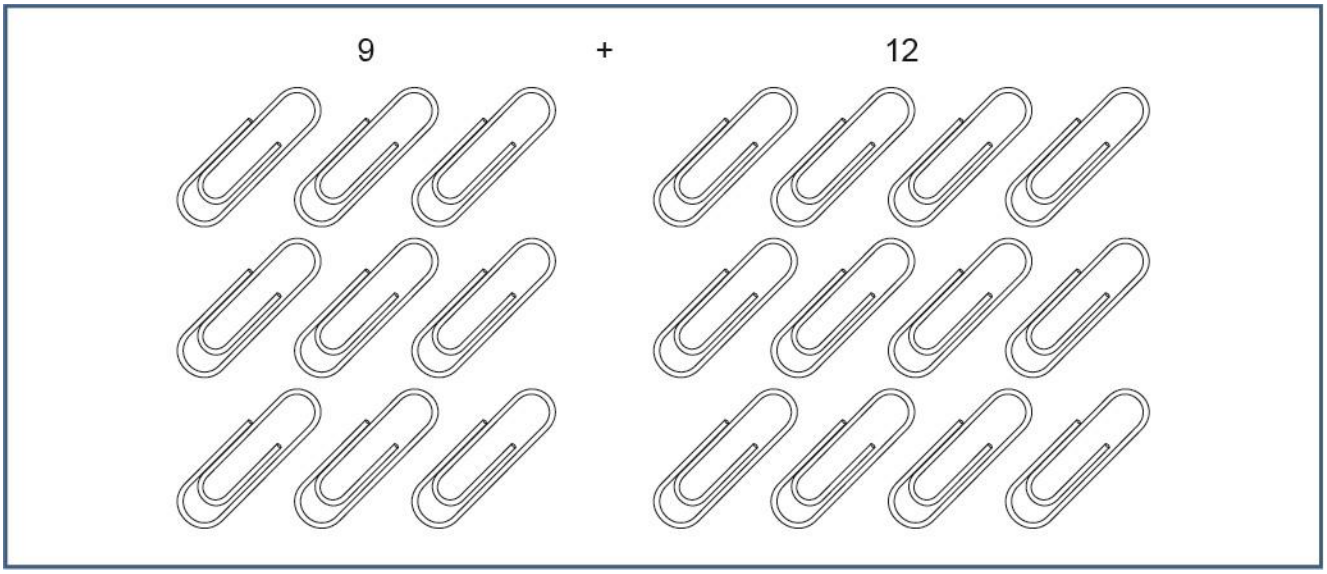
A-N.2.2.2 Apply the distributive property to express a sum of two whole numbers, 1 through 100, with a common factor as a multiple of a sum of two whole numbers with no common factor.

► Instructional Supports

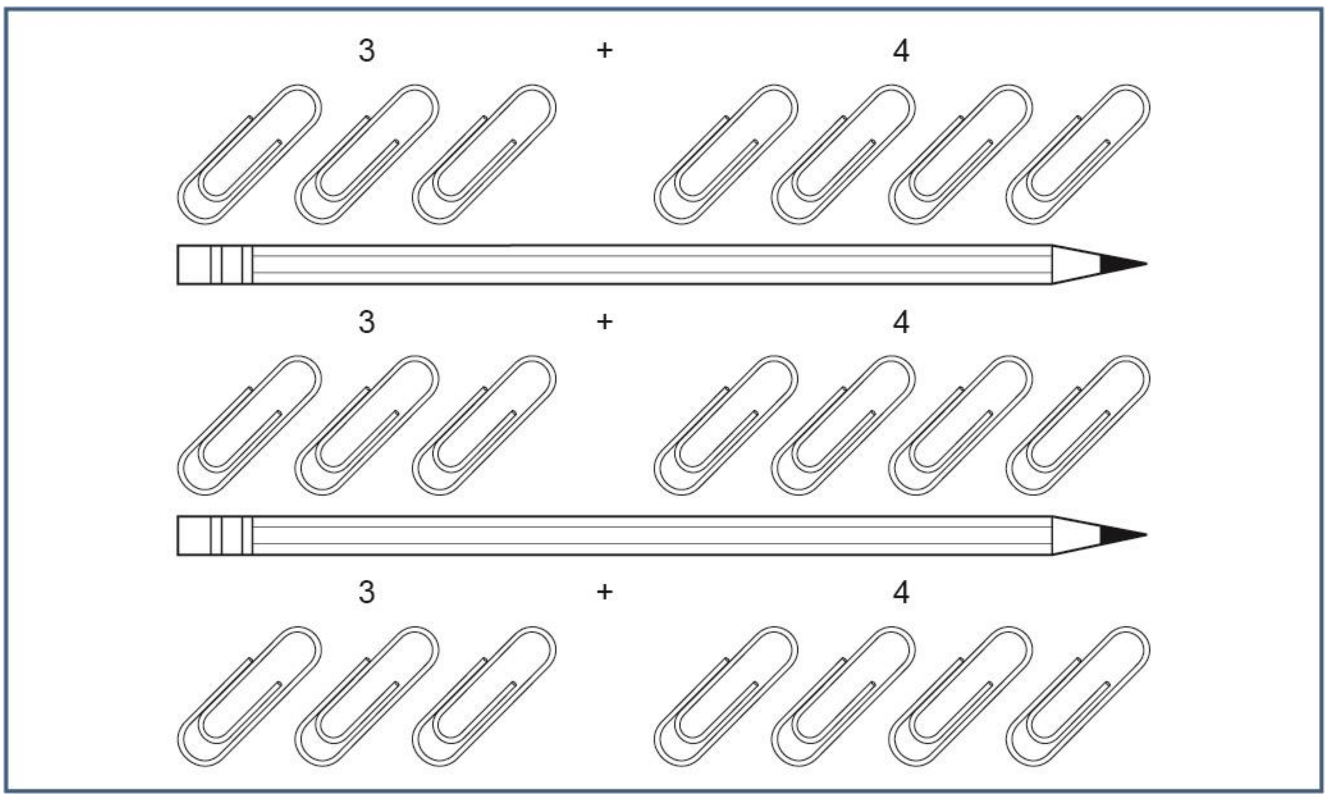
- Ask students to use area models to represent the relationship between two numbers and their common factors. For example, ask students to use models to find the common factors of 12 and 15. Have students create rectangular area models with the same height to represent each number. Ask students to discuss why this can only be done if the shared height is a common factor of the two numbers.



- Ask students to use objects to show how to rewrite a sum as a multiple of another sum. For example, give students the expression 9 + 12 and have them model the expression using paperclips. The paperclips used to represent each number can be arranged into rectangular arrays with a height equal to the greatest common factor of the two numbers, which in this case is 3.



Ask students to use objects to divide each array into equal groups.



The arrangement of the objects now shows that 9 + 12 is the same as 3 groups of 3 + 4, or 3(3 + 4).

- Ask students to use ladder diagrams to find the greatest common factor of two numbers. For example, give students the numbers 45 and 60. Students can create a ladder diagram to find the greatest common factor of the numbers by first dividing both numbers by a common factor of 5. The resulting quotients can then be divided by the common factor of 3. At this point, the resulting quotients, 3 and 4, have no common factors. The greatest common factor of 45 and 60 is found with the equation 5 • 3 = 15.

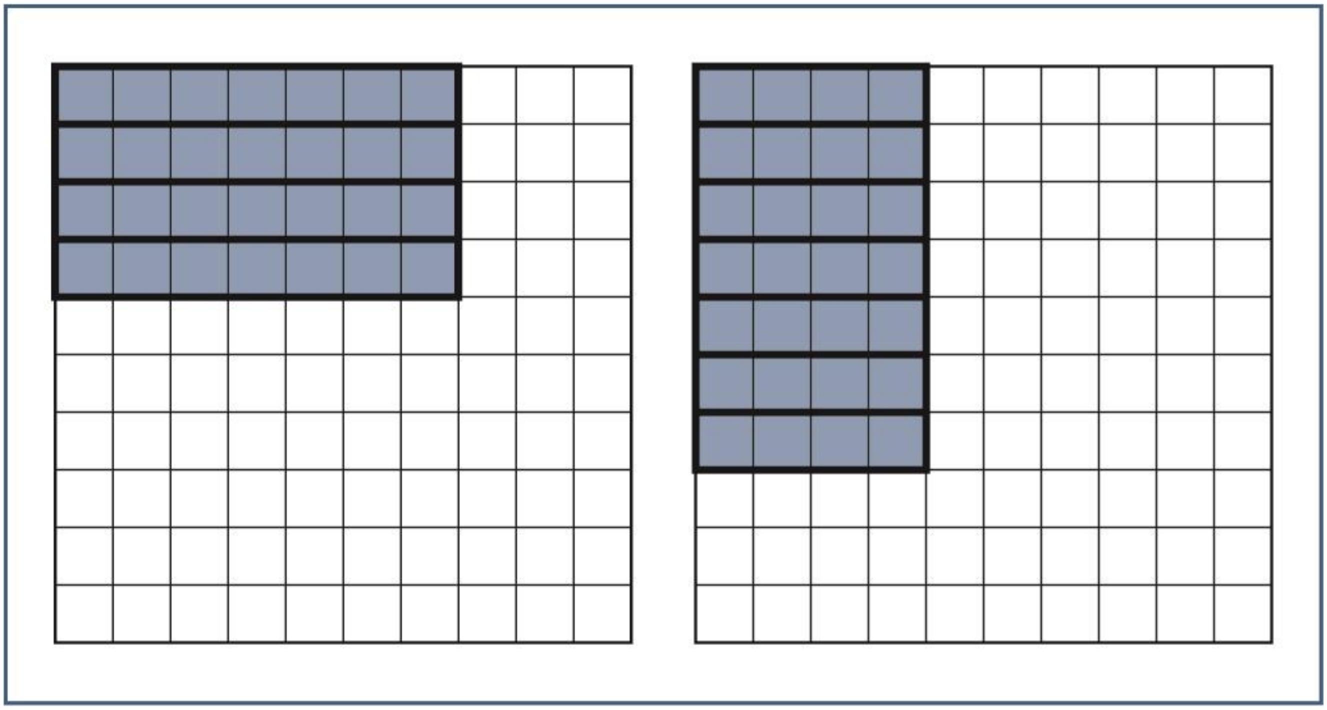
$$\begin{array}{r|rr} 5 & 45 & 60 \\ \hline 3 & 9 & 12 \\ \hline & 3 & 4 \end{array}$$

$$5 \cdot 3 = 15$$

- Ask students to explain why the greatest common factor of two distinct prime numbers must be 1. Students should understand that the only factors of a prime number are 1 and the prime number itself. Thus, the only common factor for two distinct prime numbers is 1. For example, the only factors of 13 are 1 and 13 and the only factors of 5 are 1 and 5, so the only common factor of 5 and 13 is 1.

- Ask students to find the greatest common factor of two numbers whose only common factor is 1. For example, consider the numbers 8 and 9. The only factors of 8 are 1, 2, 4, and 8. The only factors of 9 are 1, 3, and 9. Thus, the only common factor of 8 and 9 is 1. This means that the numbers 8 and 9 are relatively prime.

- Ask students to find the least common multiple of two numbers by modeling the multiples on a grid. For example, have students start with two 10 by 10 grids and give students the numbers 7 and 4. Have students shade in a group of 7 squares in the first grid and a group of 4 squares in the second grid. Then, ask students to continue to shade in squares by applying the following rules.
    - If there are more shaded squares in the first grid, shade in 4 squares in the second grid.
    - If there are more shaded squares in the second grid, shade in 7 squares in the first grid.
    - If the number of shaded squares in each grid is equal, stop.



► Identifying and Addressing Common Misconceptions

- Students may not understand that the greatest common factor of a set of numbers can be equal to one of the numbers in the set. Give students the numbers 12 and 24 and ask them to find the greatest common factor. Some students may not realize that 12 is a common factor and identify the greatest common factor as 6. Ask students to list all factors of each number without considering the other number and then find the greatest number that appears in both sets.
- Students may not understand that the least common multiple of two numbers may be one of the numbers. Give students the numbers 3 and 9 and ask them to find the least common multiple. Some students may not realize that 9 is a common multiple and identify 18 as the least common multiple. Ask students to create a table showing some of the multiples of the two numbers and identify the smallest multiple found in both lists.
- Students may factor out a common factor other than the greatest common factor and not recognize that it is possible to further factor the given numbers. Give students the expression 66 + 48 and ask them to factor out the greatest common factor of the two numbers. Some students may arrive at 2(33 + 24). Ask students to list all factors of the two numbers inside the parentheses and look for any additional common factors.

► Enrichment

- Ask students to find a real-world situation in which factoring out the greatest common factor of two numbers provides information that is not obvious in the sum of the two numbers. For example, give students this situation: “A store received two shipments of cans of soup. One shipment had 24 cans and another shipment had 42 cans.” By factoring out the greatest common factor, students can see that one shipment had 4 boxes of 6 cans and the other shipment had 7 boxes of 6 cans.
- Ask students to find the greatest common factor of a set of three or more numbers.
- Ask students to find the least common multiple of a set of three or more numbers.

► Key Terms

distributive property, greatest common factor, least common multiple, prime factorization

A-N.3.1.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

The Number System

Apply and extend previous understandings of numbers to the system of rational numbers.

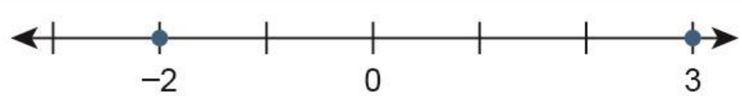
Understand that positive and negative numbers are used together to describe quantities having opposite directions or values and locations on the number line and coordinate plane.

► Eligible Content

A-N.3.1.1 Represent quantities in real-world contexts using positive and negative numbers, explaining the meaning of 0 in each situation (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge).

► Instructional Supports

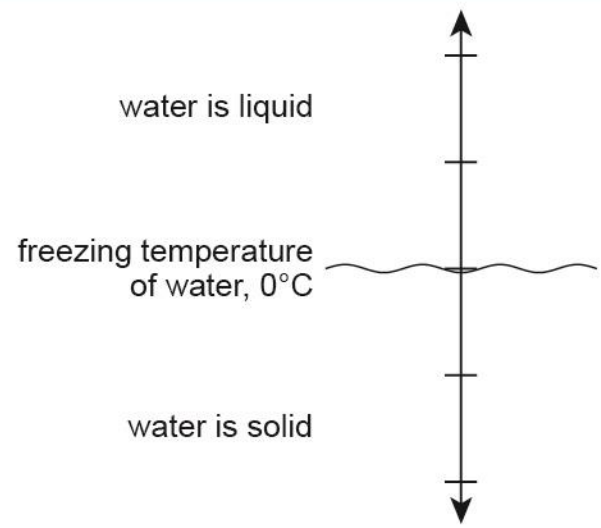
- Ask students to identify whether a given number is to the left or to the right of 0 on a number line, recognizing that numbers greater than 0 are represented with a (+) symbol or no symbol and are located to the right of 0, while numbers less than 0 are represented with a (–) symbol and located to the left of 0. For example, given a number line with 0 plotted in the middle, students should be able to locate –2 on the left of 0 and 3 on the right of 0.



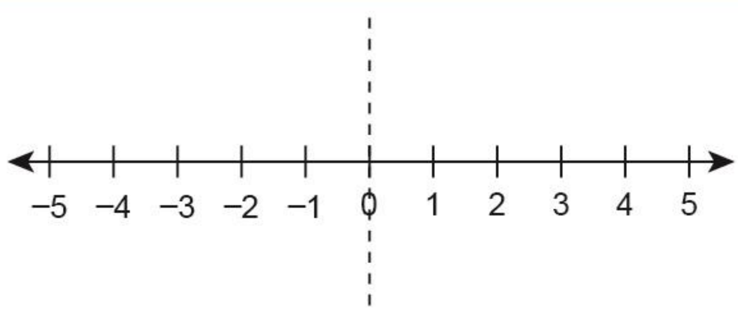
  Give students additional integers, and ask them which side of 0 the tick mark representing the integer is located on and to explain how they determined their answer.

- Ask students to represent situations involving positive and negative numbers. For example, a pair of frogs are sitting on the same rock. When one frog jumps to the right 5 feet, the other frog jumps to the left 3 feet. Ask students what numbers could be used to represent the locations of the two frogs.

- Ask students to create diagrams representing real-world situations with both positive and negative numbers. The diagrams should include a number line and show the location and the meaning of 0 in the real-world situation. For example, students may create a diagram showing significant temperatures above and below 0°C.



- Ask students to fold a piece of paper with a given number line on it so that the line of the fold is perpendicular to the number line and passes through the 0. Have students put a point on their number line on either side of 0 and then fold their paper and mark the image of their point on the other side of 0. Have students discuss patterns they notice with partners.



- Give students a pair of opposite numbers and ask them to identify which side of 0 each of the numbers is located on. Then, ask students to describe how the two numbers and their locations are related. For example, give students the numbers –6 and 6. Observe that –6 lies to the left of 0 and 6 lies to the right of 0. After more examples, conclude that a number and its opposite are always on opposite sides of 0.

- Ask students to explain the meaning of two opposite numbers in a given context. For example, students could explain that –30 feet is 30 feet below sea level and +30 feet is 30 feet above sea level.

▶ Identifying and Addressing Common Misconceptions

- Students may use positive numbers to represent both a value and its opposite in the same context. Ask students to represent elevations of 20 feet below sea level and 20 feet above sea level. Some students may use 20 for both situations. Give students statements representing a positive value and a negative value in a real-world context. Ask students to plot the points represented by the statements on a number line.
- Students may treat the initial value of a context as 0 when there is another location for 0 that is more useful within the context. Ask students to identify the meaning of 0 in this context: "A climber starts a climb at an elevation of 175 feet above sea level and climbs up a cliff that is 450 feet tall. What is the climber's final elevation above sea level?" Some students may identify the climber's starting point as 0 and therefore answer 450 feet as the climber's final elevation above sea level. Use a visual model with 0 indicating sea level and 175 feet as the starting point to demonstrate that the final elevation above sea level is 175 feet + 450 feet, which is 625 feet. Present students with a relatable context, such as a temperature change, and ask them to identify the 0 and its meaning. Then give students a context with a nonzero initial value and ask them to identify the 0 and its meaning.
- Students may associate certain keywords with negative numbers. Ask students to use a numeral to represent this statement: "The depth of the fish tank is fourteen inches." Students may give the answer of –14 because they associate depth with a downward measurement or because they think of elevation relative to sea level. Present students with a context where negative numbers are not commonly used. In the case of water in the fish tank, this could be finding the volume of the water. Observe that using a negative value to represent the depth would result in a negative value for the volume.

► Enrichment

- Ask students to identify contexts where both negative and positive numbers can be used. Ask them to define 0 in that context and give examples where positive numbers would be used and where negative numbers would be used. For example, students may describe a tree with 0 at ground level, where heights of branches are described using positive values and depths of roots are described using negative values.
- Ask students to use physical manipulatives to model positive and negative numbers. For example, use coins as counters where coins showing heads are positive numbers and coins showing tails are negative numbers.
- Ask students to identify situations in which a single quantity may be interpreted as either a positive value or as a negative value, depending on the perspective. For example, when a customer makes a purchase from a store, the customer might represent the cost as a negative number, while at the same time the store may represent the cost as a positive number. Similarly, if a hiker is on a hill and takes a trail that decreases the hiker's elevation, a negative number could be used to represent the height in relation to the starting height, but a positive number could be used to represent the height above sea level.

► Key Terms

negative number, opposites, positive number

A-N.3.1.2

► Reporting Category, Assessment Anchor, Anchor Descriptor

The Number System

Apply and extend previous understandings of numbers to the system of rational numbers.

Understand that positive and negative numbers are used together to describe quantities having opposite directions or values and locations on the number line and coordinate plane.

► Eligible Content

A-N.3.1.2 Determine the opposite of a number and recognize that the opposite of the opposite of a number is the number itself (e.g., –(–3) = 3, and that 0 is its own opposite).

► Instructional Supports

- Ask students to plot a point to represent a given number on a number line and indicate how far that point is from 0. Then, ask students to plot a point on the other side of 0 that is the same distance from 0 as the first point. These two numbers are called opposites. They are the same distance from 0 but on opposite sides of 0.
- Create an interactive number line on the floor and ask students to move to a location along the number line. For example, use masking tape to create a large number line on the floor and ask students to move to the location of the number 2. Then, ask students to move to the point representing the opposite of 2. Next, ask students to move to the location opposite of the current point on the number line. Use this exercise to emphasize the idea that the symbol "–" means opposite.
- Help students to further explore the concept of the opposite of an opposite using real-world situations. For example, use a light source, such as a flashlight or classroom lights, to demonstrate that the opposite of the opposite of a state is the original state. Ask students to find the opposite of "on." Then ask students to find the opposite of the opposite of "on."

▶ Identifying and Addressing Common Misconceptions

- Students may think that any number of negative signs indicates a negative number. Ask students to mark the location of $-(-7)$ on a number line. Some students may locate this value at -7 on the number line. Have students use a number line marked on the floor to find the location representing 7. Then have the students find the location representing the opposite of 7 and notice that the opposite of 7 is written as -7. Finally, have the students find the location representing the opposite of the opposite of 7. Note that the opposite of the opposite of 7 is written as $-(-7)$ and is a different location than -7.

- Students may incorrectly identify locations of negative mixed numbers and negative numbers with decimal fractions on number lines. They may correctly locate the integer part of the number and then move to the right to mark the fractional part, as they would with a positive number. Ask students to mark $-6\frac{1}{4}$ on a number line. Some students may correctly mark -6 but then incorrectly think that $\frac{1}{4}$ is positive and move $\frac{1}{4}$ to the right of -6 to place their final mark at $-5\frac{3}{4}$. Similarly, some students may locate -4.5 at -3.5 because they see the .5 as positive and therefore move to the right of -4. Remind students that a negative number is the opposite of its corresponding positive number and therefore is the same distance from zero. Ask students to mark positive $6\frac{1}{4}$ and 4.5 on a number line along with $-6\frac{1}{4}$ and -4.5. Then, ask students to fold their number line over a perpendicular line passing through 0 to emphasize that $-6\frac{1}{4}$ and -4.5 should be the same distance from 0 as their corresponding positive marks. Ask students to adjust the locations of their negative values as needed by emphasizing that the fractional and decimal fractional parts of the numbers are also negative.

- Students may not recognize that 0 is its own opposite or think that 0 doesn't have an opposite. To demonstrate that 0 is its own opposite, ask students to fold a piece of paper showing several number lines with different scales along a perpendicular line through the 0 on each number line (for ease of use, place each 0 in the middle of the page). Observe that all nonzero numbers line up with their opposites when the paper is folded. Ask students to identify which tick mark lines up with each 0 when the paper is folded.

► Enrichment

- Ask students to find a pattern in the values when the opposite of a number is taken any number of times. Specifically look for patterns when there are an even number of opposites and when there are an odd number of opposites.
- Ask students to identify real-world contexts in which knowing the opposite of a number is useful.

► Key Terms

number line, opposites, rational number

A-N.3.1.3

► Reporting Category, Assessment Anchor, Anchor Descriptor

The Number System

Apply and extend previous understandings of numbers to the system of rational numbers.

Understand that positive and negative numbers are used together to describe quantities having opposite directions or values and locations on the number line and coordinate plane.

► Eligible Content

A-N.3.1.3 Locate and plot integers and other rational numbers on a horizontal or vertical number line; locate and plot pairs of integers and other rational numbers on a coordinate plane.

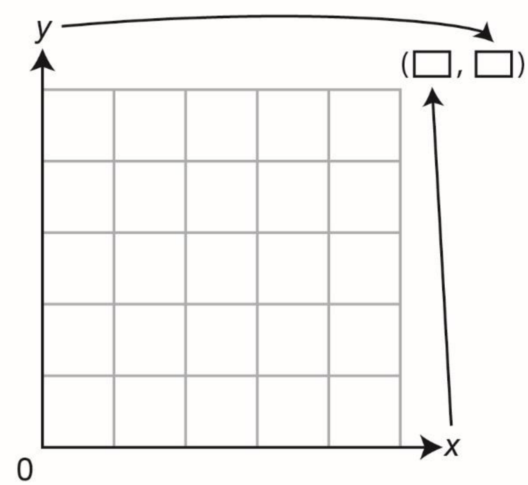
► Instructional Supports

- Ask students to plot a given rational number on both a horizontal number line and a vertical number line. Ask students to explain how changing the sign of the number affects the location of the number on each of the number lines.
- Ask students to describe the positions of rational numbers in relation to 0 on vertical and horizontal number lines. For example, given the numbers –1, $\frac{5}{6}$, –75, –0.38, 63, and $3\frac{2}{3}$, students should be able to identify that –1, –75, and –0.38 are below 0 on a vertical number line and to the left of 0 on a horizontal number line while $\frac{5}{6}$, 63, and $3\frac{2}{3}$ are above 0 on a vertical number line and to the right of 0 on a horizontal number line. Students should also identify –75 as being the point furthest below or to the left of 0 while 63 would be the point furthest above or to the right of 0 on the number lines.
- Ask students to plot a list of given ordered pairs on a coordinate plane. For example, ask students to plot the points (–2, –2), (4, $\frac{1}{2}$), (–1, 3.2), ($\frac{3}{2}$, 0), (–1, –3.2), and (2, –2) on a coordinate plane while explaining their process. Remind students that the first coordinate, also called the *x*-value, in an ordered pair indicates how far to move to the left (negative) or right (positive) from the origin, while the second coordinate, known as the *y*-value, indicates how far to move down

(negative) or up (positive) from the origin. For example, the ordered pair (–4, 6.5) indicates a location that is 4 units left of the origin and 6.5 units up.

▶ Identifying and Addressing Common Misconceptions

- Students may reverse the meaning of the *x*- and *y*-coordinates. For example, ask students to plot the point (3, 0). Students who plot a point at (0, 3) may have this misconception. Ask students to create a diagram like the following to show the relationship between the variable and the axis.



- Students may switch the positive and negative directions when working with number lines. For example, ask students to plot the point –2 on a number line. Students who plot the point at 2 may have this misconception. Ask students to draw a number line with values from 0 to 5. Then ask students to extend the number line to the left and label those numbers with negative signs. Then have the student plot the point representing –2.

- Students may incorrectly plot points with negative, non-integer coordinates in the wrong locations. They may correctly locate the integer parts of the coordinates and then move to the right or up to count the fractional part. For example, ask students to plot the point (–7.25, –4.75). Students who plot the point (–6.75, –3.25) may have this misconception. Ask students to count the distance of the plotted point from the axis and compare the count to the coordinates.

► Enrichment

- Give students a map drawn on a coordinate plane and a list of places on the map. Ask students to write directions telling how to get from the origin to each other place shown on the map. Have students create their own maps and trade with partners to write directions for locations on their partner's map.
- Ask students to use a number line to show real-world values, such as the highest and lowest points on each continent.
- Ask students to represent several points with the same ratio of x-value to y-value on a coordinate plane and describe what they observe about the points.

► Key Terms

axis, coordinate plane, horizontal, integer, negative, ordered pair, origin, positive, rational number, sign, vertical

A-N.3.2.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

The Number System

Apply and extend previous understandings of numbers to the system of rational numbers.

Understand ordering and absolute value of rational numbers.

► Eligible Content

A-N.3.2.1 Write, interpret, and explain statements of order for rational numbers in real-world contexts.

► Instructional Supports

- Ask students to create a real-world context for a given inequality. For example, give students the inequality $-8 < -4$. Students may come up with the context that a temperature of –8 degrees is less than a temperature of –4 degrees.
- Ask students to compare positive and negative values given in context. For example, give students a table showing the elevations of the lowest points in some states and ask them questions about the data in the table.

Lowest Points

Arizona	70 feet
California	–282 feet
Illinois	280 feet
Louisiana	–8 feet
New York	0 feet

- The lowest point of which state has the lowest elevation?
- The lowest point of which state has the highest elevation?
- Which state has a lowest point with an elevation lower than Arizona's lowest point and higher than Louisiana's lowest point?

Students can work together to answer these questions and explain that California has the lowest elevation for its lowest point, Illinois has the highest, and New York is between Arizona and Louisiana. Ask students to explain how they determined their answers.

► Identifying and Addressing Common Misconceptions

- Students may ignore negative signs and order positive and negative numbers by magnitude. For example, ask students to write an inequality comparing the temperatures of 1 degree Fahrenheit, –4 degrees Fahrenheit, and –7 degrees Fahrenheit. Some students may write an incorrect inequality such as $1 < -4 < -7$. Ask students to mark the numbers on a number line and then identify the order of the numbers from left to right. Then, have students use the order of the numbers on the number line to write a true inequality that correctly compares the numbers.
- Students may associate comparing two negative numbers with comparing the opposites of the numbers. For example, ask students to compare 6 feet and –9 feet. Then, have them compare –6 feet and –9 feet. Students who conclude that 6 is greater than –9 and –6 is less than –9 may have this misconception. Ask students to think of a real-world context for the numbers and then explain what it means for a number to be "greater" or "less" within the context.
- Students may not recognize when to assign negative numbers when comparing two values given in a real-world context. For example, ask students to write an inequality comparing a deposit of \$15 and a withdrawal of \$25 from an account. Some students may write $25 > 15$. Ask students to choose a starting value for the account and compare the final balance of the account in each case to demonstrate that the withdrawal is best represented with a negative number. Therefore, $-25 < 15$.

► Enrichment

- Ask students to create a real-world context involving rational numbers and then explain how the numbers compare. For example, a student could create a context about a diver who starts on a boat 1 meter above the surface of the water and then dives to a depth of –15 meters before returning to the surface of the water at 0 meters. The student could then explain that –15 meters is less than both 1 meter and 0 meters and that 0 meters is less than 1 meter.
- Ask students to assign given rational numbers to correctly complete a contextual situation. For example, give students this context: "A person enters a cave at an elevation of ____ feet. Immediately after entering the cave, the person descends to an elevation of ____ feet. The lowest point in the cave is at an elevation of ____ feet." Give students the numbers –60, 35, and 50, and ask students to place them in the blanks. Students should fill in the blanks in the order 50, 35, –60.
- Ask students to compare two numbers in a context and then compare the opposites of the numbers in the same context. Then, ask students to identify a pattern in these comparisons. For example, ask students to compare temperatures of 1 degree Celsius and 3 degrees Celsius and then compare −1 degree Celsius and –3 degrees Celsius. Students should reason that 1 degree is colder than 3 degrees, but −1 degree is warmer than –3 degrees. Help students recognize that when comparing the opposites of numbers, the comparison is reversed. Ask students to discuss why this is true.

► Key Terms

absolute value, greater than, inequality, less than, number line

A-N.3.2.2

▶ Reporting Category, Assessment Anchor, Anchor Descriptor

The Number System

Apply and extend previous understandings of numbers to the system of rational numbers.

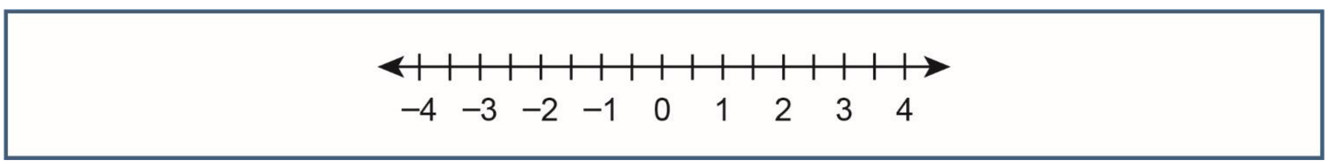
Understand ordering and absolute value of rational numbers.

▶ Eligible Content

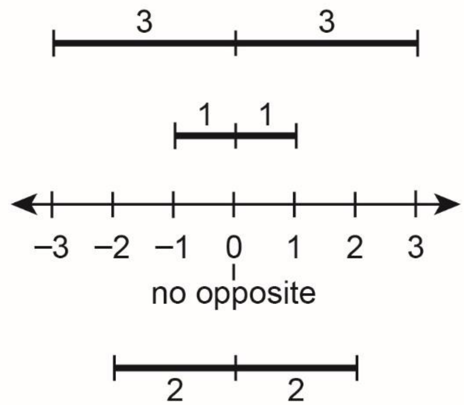
A-N.3.2.2 Interpret the absolute value of a rational number as its distance from 0 on the number line and as a magnitude for a positive or negative quantity in a real-world situation.

▶ Instructional Supports

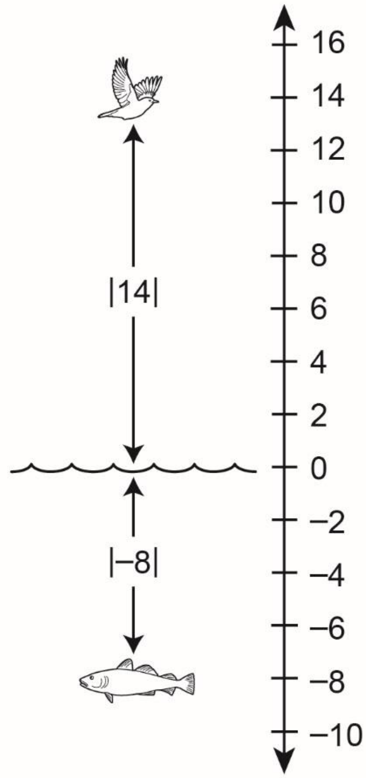
- Ask students to determine the absolute values of numbers by finding their distances from 0 on a number line. For example, give students the numbers –3, $1\frac{1}{2}$, and $-2\frac{1}{2}$, along with a number line similar to the following. Students can count units on the number line to help find the absolute values: 3, $1\frac{1}{2}$, and $2\frac{1}{2}$.



- Ask students to find all numbers with a given absolute value. For example, give students the absolute value of $\frac{7}{3}$. Students should be able to identify $\frac{7}{3}$ and $-\frac{7}{3}$ as having an absolute value of $\frac{7}{3}$. Ask students to discuss why it is possible for two different numbers to have the same absolute value.
- Ask students to find the opposite of 0 and explain what makes 0 different from other numbers when determining absolute value. Students should be able to recognize that 0 is its own opposite and is the only number 0 units away from 0. Students may create and use diagrams such as the following to aid in their explanations.



- Ask students to create two interpretations in the same context for a given absolute value. For example, give students the absolute value 12. Students could give the daily change in temperature. A change of 12 would indicate an increase of 12 degrees from one day to the next. A change of –12 would indicate a decrease of 12 degrees from one day to the next. An absolute value of 12 could indicate a change of 12 degrees in either direction.
- Give students a context and ask them to create an illustration for the context and explain the meaning of absolute value in that context. For example, give students this context: "At a lake, a bird flies at an elevation of 14 feet in relation to the surface of the water, and a fish swims at an elevation of –8 feet in relation to the surface of the water." Students should make an illustration representing the given context and explain the meaning of the absolute value of the elevations in context. Students should conclude that the absolute value 14 represents the distance, in feet, of the bird from the surface of the water and the absolute value 8 represents the distance, in feet, of the fish from the surface of the water.



► Identifying and Addressing Common Misconceptions

- Students may interpret finding absolute value as finding the opposite of a number. Ask students to find |8|. Students who answer –8 may have this misconception. Ask students to count the number of units between a number and 0 and explain that the number of units counted is the same as the absolute value of the number.

- Students may interpret an absolute value calculation as removing a negative sign even when the negative sign is outside of the absolute value symbols. For example, ask students to find –|14|. Students who find 14 as the result may have this misconception. Remind students of the correct order of operations with a reminder that the absolute value symbol is a type of grouping symbol and is therefore completed first, similar to working with parentheses first. Students should first determine that |14| = 14 and then apply the negative sign to get a final result of –14.

- Students may not understand that there can be two values with the same absolute value within a context. For example, ask students to identify the possible change in temperatures when the absolute value of the change in temperature is 26. Students who identify only 26 degrees or only −26 degrees may have this misconception. Ask students to find the distance from 0 on a number line using units. Encourage students to think about how absolute values of opposite numbers are related.

► Enrichment

- Ask students to plot points on a coordinate grid whose coordinates have given absolute values and make observations about the locations of the points on the grid. For example, ask students to plot all ordered pairs whose x-coordinates have an absolute value of 5 and whose y-coordinates have an absolute value of 3. Ask students to discuss patterns they notice between the locations of the points.
- Ask students to solve equations involving absolute values. For example, ask students to solve $|x| + 4 = 6$. Students can find that there are two possible values of x: –2 and 2. Ask students to explain how they determined their answers.
- Ask students to represent a given context using an inequality involving absolute value. For example, give students this context: "Some air and water samples are collected at different heights above and below the surface of a lake. All samples are collected between the heights of 10 feet below the surface and 10 feet above the surface." Ask students to write an inequality using absolute value to represent the heights at which samples are collected. Students can use the inequality $|x| < 10$ to represent the context. Have students explain how they determined their inequalities.

► Key Terms

absolute value, magnitude, number line, opposite

A-N.3.2.3

► Reporting Category, Assessment Anchor, Anchor Descriptor

The Number System

Apply and extend previous understandings of numbers to the system of rational numbers.

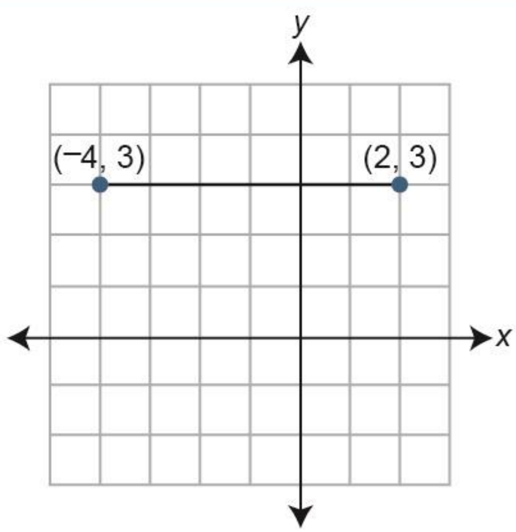
Understand ordering and absolute value of rational numbers.

► Eligible Content

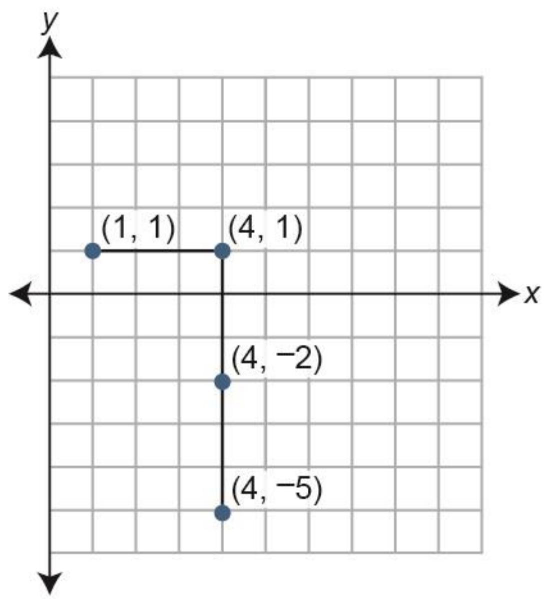
A-N.3.2.3 Solve real-world and mathematical problems by plotting points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

► Instructional Supports

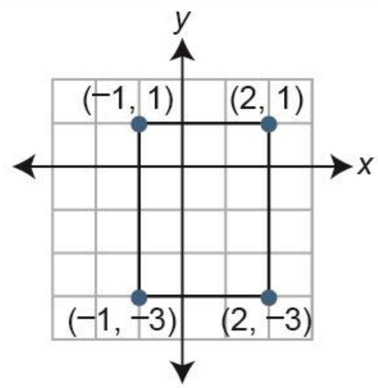
- Ask students to determine the length of a horizontal or vertical line segment on the coordinate plane. For example, give students a line segment with endpoints at (2, 3) and (–4, 3). Students should determine that the length of the segment is 6 units since the distance from (−4, 3) to the *y*-axis is |–4| = 4 units and the distance from (2, 3) to the *y*-axis is |2| = 2 units for a total of 6 units.



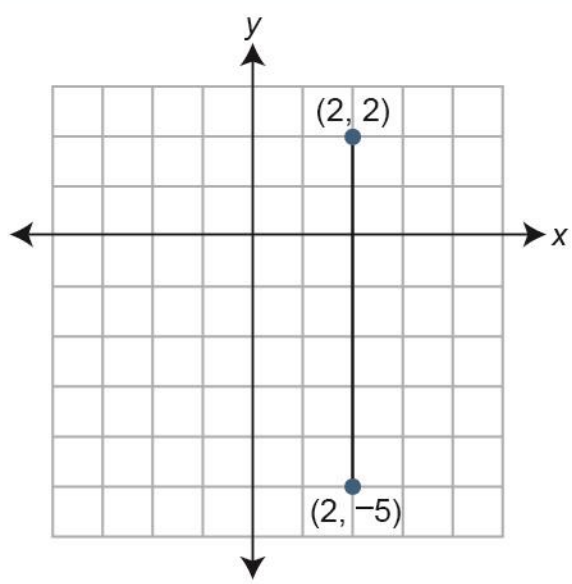
- Give students a horizontal or vertical line segment on the coordinate plane. Then, ask students to create a new horizontal or vertical line segment having the same length and sharing a common endpoint with the given line segment. For example, give students a line segment with endpoints at (4, 1) and (4, –2) on the coordinate plane. A possible horizontal line segment, from (4, 1) to (1, 1), and a possible vertical line segment, from (4, −2) to (4, −5), are shown.



- Give students four coordinate pairs representing the vertices of a rectangle and ask students to find the perimeter and area of the rectangle. For example, give students the coordinates (2, 1), (2, –3), (–1, –3), and (–1, 1). Ask students to plot the points on the coordinate plane and connect them to form a rectangle. Then, ask students to find the perimeter and area of the rectangle. Students should find that the perimeter is 14 units and the area is 12 square units.



- Give students one endpoint of a line segment, the length of the line segment, and the quadrant of the other endpoint and ask students to draw the line segment on the coordinate plane. For example, give students the endpoint of (2, –5) and ask them to draw a segment with a length of 7 units with the other endpoint in quadrant I. Students should draw the line segment by plotting the point (2, –5) and connecting it with a line to the other endpoint at (2, 2).



► Identifying and Addressing Common Misconceptions

- Students may confuse the quadrants or count clockwise from quadrant I when naming them. Give students a coordinate grid and ask them to plot a point in quadrant IV and name the coordinates of the point. Students who plot the point in any quadrant other than quadrant IV may have this misconception. Ask students to compare the quadrant of the point plotted to another grid with the quadrants labeled. Students should explore the signs of the coordinates of points in each quadrant.

- Students may find the sum instead of the absolute value of the difference when finding the distance between two points. Ask students to find the distance between the points (–4, –2) and (– 4, 3). Students who identify 1 as the distance may have this misconception. Ask students to plot the points and count the units between the two points on a coordinate grid.
- Students may use directions when finding the distance between points. Ask students to find the distance between (1, 6) and (–2, 6). Students who calculate a difference of –3 may have this misconception. Ask students to find the distance by counting the number of units between the points and compare the sign of the counted distance to the sign of the calculated distance. Also, ask students to identify the distance between two known locations. Then ask the students the distance between those same two locations in reverse order. Use this to remind students that distance is stated as a positive value no matter the direction you measure.

► Enrichment

- Ask students to find the vertical and horizontal distances between two points. For example, ask students to find the vertical distance and horizontal distance between (6, –4) and (–3, –3). Students should determine that the vertical distance is 9 and the horizontal distance is 1. Ask students to explain how they found their answers.
- Ask students to use the coordinates of the vertices to find the area of a triangle with one vertex that is not along the same vertical or horizontal line as another vertex. For example, give students the vertices (4, –1), (–2, –5), and (–2, 3). Have students plot the points on the coordinate plane and connect them to form a triangle. Students should determine that the area of this triangle is 24 square units. Ask students to use the diagram to explain how they determined their answer.
- Ask students to find a point given a horizontal and a vertical distance from a given point. For example, give students the point with coordinates (–4, 7) and ask them to find a new point 4 units below and 3 units to the right of the given point. Students should identify the new point as

having coordinates (–1, 3). Ask students to explain how to find the coordinates of the new point with and without the use of a coordinate plane.

► Key Terms

absolute value, coordinate plane, coordinates, magnitude, quadrant

A-R.1.1.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

Ratios and Proportional Relationships

Understand ratio concepts and use ratio reasoning to solve problems.

Represent and/or solve real-world and mathematical problems using rates, ratios, and/or percents.

► Eligible Content

A-R.1.1.1 Use ratio language and notation (such as 3 to 4, 3:4, 3/4) to describe a ratio relationship between two quantities.

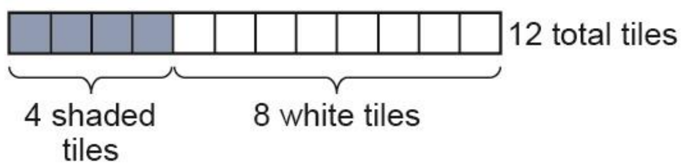
► Instructional Supports

Defining a ratio

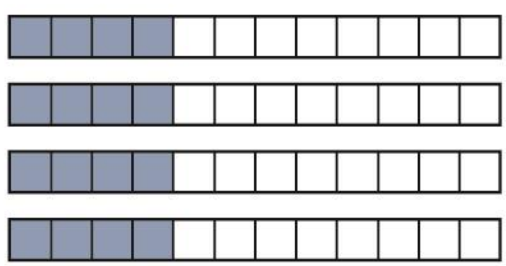
- A ratio is a quantitative comparison between two or more sets. Ratios can represent part-to-part and part-to-whole relationships. A ratio can also be interpreted as a composed unit relating a whole to a part-to-part relationship that is repeatable. For example, 3 parts red paint and 4 parts green paint can be combined to create a composed unit of a particular shade of yellow paint. For all quantities of paint in this particular shade of yellow, the ratio of red paint to green paint will be some multiple of the ratio 3 to 4. Composed units can also be partitioned into parts that maintain the original ratio. For example, a pitcher of juice is made by combining 4 parts concentrate with 5 parts water. A small portion of the juice would have reduced quantities of concentrate and water in the same ratio as the entire pitcher of juice.
- Ask students to practice using ratio language and notation. Give two whole numbers to students in context. Then, ask students to record part-to-part ratios using ratio language. For example, give students the information that 1 box of cake mix requires 3 eggs to make the cake. Students may explain the ratio in the following way.

For every box of cake mix, the recipe requires 3 eggs. This can be expressed as a ratio and represented as 1:3, $\frac{1}{3}$, or 1 to 3 because for every 1 box of cake mix, there are 3 eggs that are required.

- Ask students to represent equivalent ratios using a bar model. First give students a context and ask them to create a representation with a bar model. Then show how the ratio repeats with a continuation of the bar model. For example, students can create the following bar model showing 12 tiles used in a pattern to tile a kitchen floor, including 4 shaded tiles and 8 white tiles.



Now extend the context: "A carpenter uses a total of 48 tiles and maintains the same ratio of shaded tiles to white tiles to cover the kitchen floor." Have students create a new model showing the 48 tiles by iterating their first model four times. Students should see that there are 16 shaded tiles and 24 white tiles and that the ratio 4 to 8 is equivalent to the ratio 16 to 24.



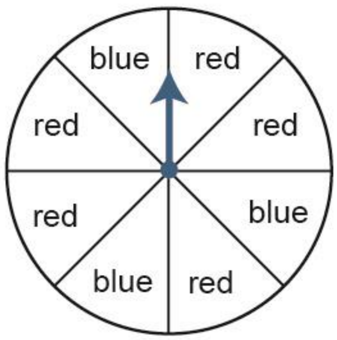
- Ask students to discuss how multiplication can be used to create equivalent ratios. For example, the label on an iced tea mix states that mixing iced tea requires 16 ounces of water and 2 tablespoons of the tea powder mix for one batch. How can the amount of powder mix be increased or decreased as needed while still maintaining the same ratio of water to powder mix? Ask students how much water and powder mix to use to make three batches of iced tea and have them explain their thinking. Students should explain that when the recipe is made three times, the 16 ounces of water and the 2 tablespoons of tea powder mix will each be tripled. In total, 48 ounces of water would be combined with 6 tablespoons of tea powder mix to make three batches. Help students see how this illustrates that the ratio 48:6 is equivalent to the ratio 16:2. Similarly, when 8 ounces of water is combined with 1 tablespoon of tea powder mix twice, there is a total of 16 ounces of water combined with 2 tablespoons of tea powder mix. Help students see how this illustrates that the ratio 8:1 is also equivalent to the ratio 16:2.

- Ask students to find real-world situations involving ratios and to describe them using ratio language. Have students explain why using ratios is appropriate for the situation. For example, the prices of different amounts of strawberries at a grocery store can be represented with a ratio. Students may create a description similar to the following.

  Mario bought 3 pounds of strawberries for a total of $6.00. This comparison can be appropriately described with a ratio because the relationship can be repeated or multiplied to maintain the same price per pound of strawberries. For example, at the same grocery store Nancy bought 6 pounds of strawberries for $12.00. Buying 6 pounds of strawberries is twice the original quantity of 3 pounds and results in a cost of $12.00, which is twice the original cost of $6.00. Also, we know that 1.5 pounds of strawberries will cost $3.00. The ratio between the number of pounds of strawberries to its cost in dollars can be expressed as 3:6, $\frac{3}{6}$, or 3 to 6, and is interpreted as "for every 3 pounds of strawberries, the cost is $6.00."

Representing ratios with visual models

- Ask students to use a geometric model to identify different ratios that can be created from the model. First ask students to create a geometric model of a fraction. Then have them identify and describe the ratio relationships that can be formed from the geometric model. For example, "The spinner shown is divided into 8 equal sections, each labeled 'red' or 'blue.'" Ask students to create ratios to represent the number of red sections to blue sections, the number of blue sections to red sections, the number of blue sections to total sections, and the number of red sections to total sections.



Red sections to blue sections is 5:3.
Blue sections to red sections is 3:5.
Red sections to total sections is 5:8.
Blue sections to total sections is 3:8.

Examples that are not ratios

- Ask students to discuss non-examples of real-world ratios. Then have students explain why their non-examples do not represent ratio relationships. For example, give students the following situation: "Karissa has 5 postcards and Sophie has 2 postcards. Karissa and Sophie each get 5 more postcards." Students should explain why this situation does not represent a ratio relationship. A sample student response is shown.

  This situation is not appropriate for using ratios because the initial comparison for the number of postcards of Karissa to Sophie is 5:2. After each girl adds 5 more postcards to her collection, the comparison for the number of Karissa's postcards to Sophie's postcards is 10:7. Since the original situation was represented with the ratio 5:2, any equivalent situation should have a ratio that is a multiple, such as 10:4, 15:6, or 20:8, which are multiples of two, three, and four, respectively. The addition of the

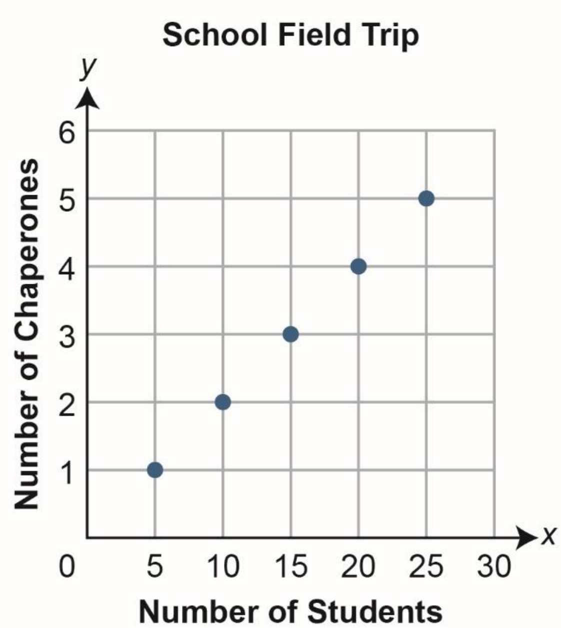
5 postcards for each girl results in a ratio comparison of 10:7, which is not equivalent to 10:4.

► Identifying and Addressing Common Misconceptions

- Some students may misinterpret addition relationships as ratio relationships. Ask students to explain which of the following two statements represents a ratio relationship between 3 blue marbles and 2 red marbles: "I have 1 more blue marble than the number of red marbles I have" or "I have 1.5 times as many blue marbles as the number of red marbles I have." The first statement is an additive relationship because "1 more" indicates that 1 marble needs to be added for the quantities to be equal. The second statement is a multiplicative comparison because "1.5 times" indicates that the number of marbles should be multiplied by 1.5 in order to make an equivalent ratio. Remind students that equivalent ratios stem from multiplicative comparisons.
- Some students assign units incorrectly when creating ratios. Ask students to write a ratio involving the number of sixth graders and the number of seventh graders on a field trip where there are 4 sixth graders for every 3 seventh graders. The two ratios of 4 sixth graders:3 seventh graders or 3 seventh graders:4 sixth graders are both correct, but 4 seventh graders:3 sixth graders, for example, is not correct. Remind students that the order of the units is just as important as the order of the numbers. Similarly, the statement "the ratio of sixth graders to seventh graders is 4:3" is a correct statement because the first group, sixth graders, matches the first quantity, 4. The statement "the ratio of sixth graders to seventh graders is 3:4" is incorrect because the first group, sixth graders, does not match the first quantity, 3.
- Some students use differences to determine ratio equivalence. Ask students to determine whether the ratio 10:7 is equivalent to 8:5. Some students may think they are equivalent since $10 - 7 = 3$ and $8 - 5 = 3$ or that $10 - 2 = 8$ and $7 - 2 = 5$. Remind students that ratios can be illustrated using multiplication. For example, multiplying a ratio of 10 cats to 7 dogs four times results in an equivalent ratio of 40 cats to 28 dogs. However, multiplying a ratio of 8 cats to 5 dogs five times results in an equivalent ratio of 40 cats to 25 dogs, which is not equivalent to 40 cats to 28 dogs.

► Enrichment

- Ask students to determine the equivalence of ratios in which the repetition of one ratio does not yield the other ratio. For example, ask students to determine whether the ratios of 6:9 and 8:12 are equivalent and why. Since 6 and 8 are not multiples of each other and because 9 and 12 are not multiples of each other, one of the ratios cannot be repeated to create the other ratio. However, guide students to see that by repeating both ratios an appropriate number of times, both ratios can be shown to be equivalent to 24:36.
- Provide students a real-world graph on a coordinate plane containing several plotted points. Ask students to determine whether a ratio relationship is appropriate to describe the situation and, if so, to determine the ratio and its meaning. For example, the graph shown represents information teachers must follow when planning a school field trip.



The ratio for this situation is 5:1, and for every 5 students who attend the school field trip, 1 chaperone attends.

▶ Key Terms

part-to-part, part-to-whole, quantity, ratio, relationship

A-R.1.1.2

▶ Reporting Category, Assessment Anchor, Anchor Descriptor

Ratios and Proportional Relationships

Understand ratio concepts and use ratio reasoning to solve problems.

Represent and/or solve real-world and mathematical problems using rates, ratios, and/or percents.

▶ Eligible Content

A-R.1.1.2 Find the unit rate a/b associated with a ratio a:b (with b ≠ 0), and use rate language in the context of a ratio relationship.

▶ Instructional Supports

- Ask students to identify and explain unit rates that are used in everyday real-world situations. Emphasize that a unit rate is a rate involving 1 unit for the second measure. A unit rate can be expressed as ratio a:b (or $\frac{a}{b}$) with $b = 1$. Some example real-world unit rates and possible student explanations are shown.
  - miles per gallon

    This is a unit rate that could describe the distance, in miles, a vehicle can travel with 1 gallon of gasoline. Mr. Peterson records that his vehicle can travel 26 miles per gallon, which means for each 1 gallon of gasoline, the vehicle can travel 26 miles.

- dollars per hour

  This is a unit rate that could describe the amount, in dollars, an employee earns for each hour of work. An employee that earns $12 per hour means that for each 1 hour of work, the employee will earn $12.

- inches per foot

  This is the unit rate that compares the number of inches equivalent to 1 foot. For example, there are 12 inches in 1 foot, which means for each 1 foot measured, there are 12 inches measured.

- Given a written description of a unit rate, ask students to express the rate as both a unit rate fraction with 1 in the denominator and as a single compound unit. Some examples are shown.
  - James earns $18 for each hour he works at his job.

    The unit rate is $\frac{18 \text{ dollars}}{1 \text{ hour}}$ *or $18 per hour.*

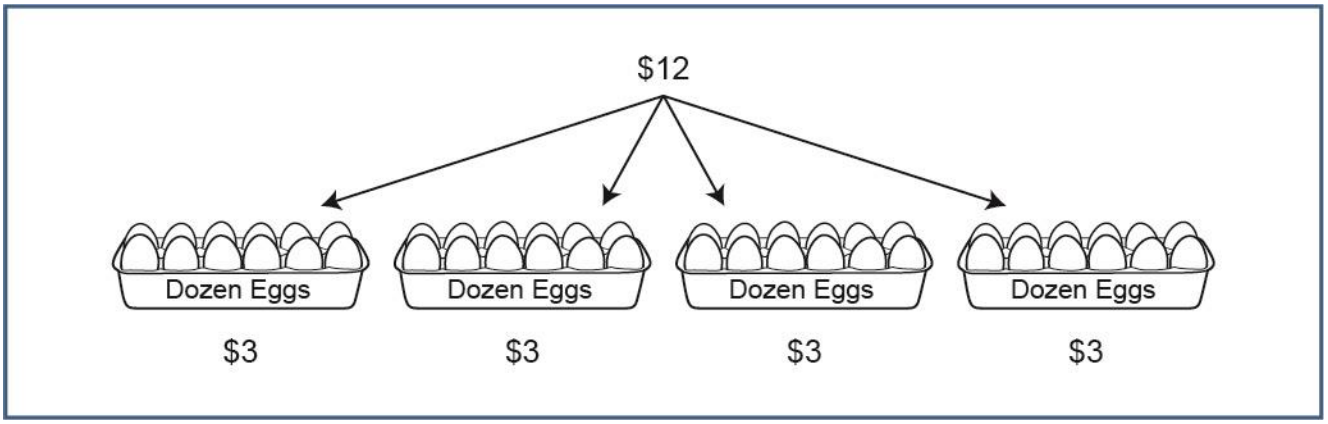
  - Keisha's truck used 1 gallon of gasoline to travel 16 miles.

    The unit rate is $\frac{16 \text{ miles}}{1 \text{ gallon}}$ *or 16 miles per gallon.*

  - It took 1 minute for Sabrina's printer to print 36 pages.

    The unit rate is $\frac{36 \text{ pages}}{1 \text{ minute}}$ *or 36 pages per minute.*

- Ask students to create a model to represent a given situation and use the model to determine the unit rate for the situation. For example, the diagram shows a model and method to determine the cost of a carton of one dozen eggs when a restaurant owner buys 4 cartons of a dozen eggs for a total cost of $12.



Students should confirm that the unit rate is $3 per carton of a dozen eggs by observing that buying 1 carton of a dozen eggs for $3 four times results in the same situation as buying 4 cartons of a dozen eggs for $12. This situation can be taken even further to determine the unit rate for the cost per egg within a carton of a dozen eggs.

- Ask students to create a table that can be used to find a unit rate. For example, give students the following situation: "An apple farm has 5 rows of trees with the same number of apple trees in each row. There are 25 apple trees in total on the farm. What is the unit rate, in apple trees per row?"

Number of Rows	1	2	3	4	5
Number of Apple Trees	5	10	15	20	25

The table shows that repeating 1 row with 5 apple trees five times results in 5 rows containing 25 apple trees. Therefore, the unit rate is 5 apple trees per row.

- Ask students to show work to determine the unit rate for a given situation. For example, give students the following situation: "It took Santos 24 minutes to run 3 miles. What is Santos's running rate in minutes per mile?" Ask students to determine the unit rate and show their work or explain their thinking. A possible student response is shown.

$$\frac{24 \text{ minutes}}{3 \text{ miles}} = \frac{8 \text{ minutes}}{1 \text{ mile}} = 8 \text{ minutes per mile}$$

- Ask students to compare costs of items by determining and comparing the unit rates. For example, ask students which has a lower price per ounce: a can containing 12 ounces that costs

$3.00 or a can containing 18 ounces that costs $3.60. Ask students to determine the unit rates for two different cans while showing or explaining their work, as in the following example.

$$\text{12-ounce can: } \frac{\$3.00}{12 \text{ ounces}} = \frac{\$0.25}{1 \text{ ounce}}$$

$$\text{18-ounce can: } \frac{\$3.60}{18 \text{ ounces}} = \frac{\$0.20}{1 \text{ ounce}}$$

Students should find that the unit rate for the 12-ounce can is $0.25 per ounce, while the unit rate for the 18-ounce can is $0.20 per ounce. The 18-ounce can has a lower price per ounce because it costs $0.05 less per ounce.

► Identifying and Addressing Common Misconceptions

- Some students think that the numbers in a ratio must be factors or multiples of one another in order to calculate a unit rate. Ask students to find the unit rate in the problem "Lacy ran 5 miles in 38 minutes." Some students may think that a unit rate cannot be calculated because 5 is not a factor of 38. Show students that 7.6 minutes per mile is a correct unit rate by creating a table or by showing that repeating addition of 7.6 five times results in a total of 38.
- When comparing prices, some students may assume that the larger item or larger quantity of items has the best value. Conversely, some students may assume that the item or quantity of items with the lesser cost has the best value. Ask students to determine whether buying 15 gallons of gas for $35 or buying 26 gallons of gas for $47.50 is a better value and to explain why. If a student says buying 26 gallons is a better value simply because it is larger, ask the student if this still holds true if the cost was $60, $70, or more. Students should realize that the larger size or quantity is not the only factor. Similarly, if a student says buying 15 gallons is a better value simply because it costs less, ask the student if this still holds true if the number of gallons is decreased to 10, 5, or less for the same cost. Students should realize that the lower price is not the only factor.
- Some students may automatically associate a larger unit rate with things like "faster," "more expensive," or "bigger"; students may also automatically associate a smaller unit rate with things like "slower," "less expensive," or "smaller" without considering the units. Ask students to explain whether a speed of 8 minutes per mile or a speed of 10 minutes per mile is faster. Some

students may choose 10 minutes per mile because 10 is a greater number. Ask students to explain the meaning of each unit rate and help them to recognize that running 1 mile in 8 minutes is faster than running 1 mile in 10 minutes.

- Some students will think fractions with numerators of 1 are unit fractions. Ask students to consider the situation of purchasing 3 oranges for $1.50 and determine a unit fraction for this situation. Some students will say 1 orange for $0.50 or $\frac{1}{0.50}$ is the unit rate for the oranges because the oranges were listed first in the situation and they could easily determine 1 orange would cost $0.50. To guide students through this, ask them what their units would be. Most students will say $0.50 per orange. Help them to recognize the second term in their unit rate is the denominator of their unit.

► Enrichment

- Ask students to rewrite unit rates using the reciprocal of the given unit rate. For example, if the unit rate is 45 miles per 1 hour, what would be the unit rate in hours per mile? Help students to recognize that there are always two unit rates that can be created from a ratio comparison and to describe the difference in their meanings. For example, the pace of a car that travels 24 miles in 30 minutes can be described in miles per minute or in minutes per mile.
- Help students to connect vocabulary with other terms they have learned. Discuss the idea of a unit as "one" in the contexts of unit rates, unit squares, and unit cubes and how the meaning of the terms are similar and different in each of those situations.

► Key Terms

part-to-part, part-to-whole, quantity, rate, ratio, relationship, unit rate

A-R.1.1.3

► Reporting Category, Assessment Anchor, Anchor Descriptor

Ratios and Proportional Relationships

Understand ratio concepts and use ratio reasoning to solve problems.

Represent and/or solve real-world and mathematical problems using rates, ratios, and/or percents.

► Eligible Content

A-R.1.1.3 Construct tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and/or plot the pairs of values on the coordinate plane. Use tables to compare ratios.

► Instructional Supports

- Ask students to create a table to represent equivalent ratios by extending a ratio in a given situation. For example, give students the following situation: "It costs $3 for 5 tickets to ride each of the attractions at the amusement park. Each ride ticket costs the same amount." A possible table of equivalent ratios is shown.

Number of Ride Tickets	Total Cost (dollars)
5	3
10	6
15	9

- Ask students to determine and use a pattern in a ratio table to solve a real-world ratio problem. For example, give students the following situation and table: "Ivan made 1 quart of fruit punch by mixing 8 tablespoons of fruit punch mix with 4 cups of water. How many tablespoons of fruit punch mix should Ivan use if he only wants to make 1 cup of fruit punch while maintaining the same ratio of fruit punch mix to water?"

Fruit Punch Mix (tablespoons)	8	12	16	20
Water (cups)	4	6	8	10

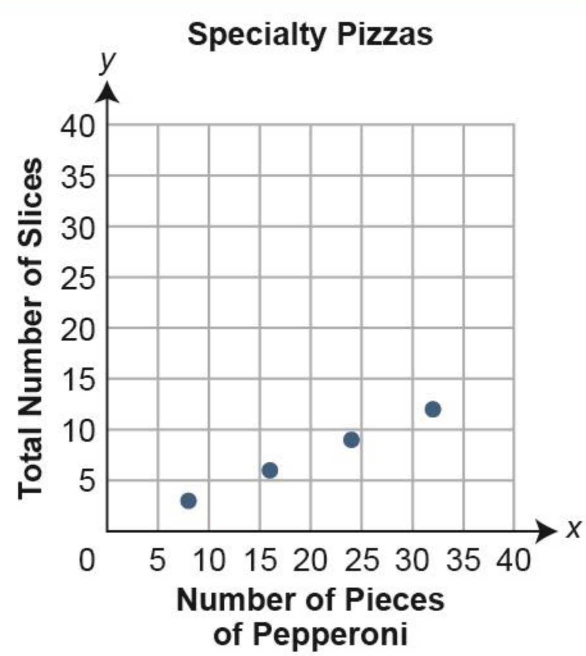
Help students to recognize that the number of tablespoons of fruit punch mix is double the number of cups of water. Therefore, 2 tablespoons of fruit punch mix are needed for 1 cup of water because doubling 1 makes 2.

- Ask students to represent equivalent ratios by graphing given points on a coordinate plane and to make observations about the plotted points. For example, give students the following table showing the number of pieces of pepperoni on a specialty pizza compared to the total number of slices of pizza.

Specialty Pizzas

Number of Pieces of Pepperoni	8	16	24	32
Total Number of Slices of Pizza	3	6	9	12

A possible graph and student response are shown.



Once all the points from the table have been plotted on the coordinate plane, it appears that a straight line can be drawn passing through the *origin and all the plotted points. For every 8 units of increase along the x-axis on the coordinate plane, there are 3 units of increase along the y-axis. Therefore, this specialty pizza appears to contain 8 pepperoni slices for every 3 slices of the pizza.*

- Ask students to determine and analyze patterns in a given ratio table and to use those patterns to help find a missing value in the table. For example, give students a table such as the following.

x	y
2	16
3	24
5	40
8	?

Use the values in the table to determine that the ratio between the values is 1 to 8, or that y is 8 times the value of x, and that the missing value associated with 8 is 64.

► Identifying and Addressing Common Misconceptions

- Some students assume that constant differences imply constant ratios. Ask students to find patterns in the table below. For example, there is a pattern of a constant increase of 9 minutes for each increase in 1 mile. However, the ratio of Number of Minutes to Number of Miles is not constant. There is an additive pattern with the sequence of the Number of Minutes, but that does not imply a multiplicative pattern (ratio) relating the Number of Miles to the Number of Minutes.

Number of Minutes	Number of Miles
8	1
17	2
26	3
35	4

- When determining patterns in a table, some students focus on the patterns of each variable based on the previous value in the table rather than by determining the relationship between the two variables. Ask students to determine the missing values in the following table that shows numbers of square pyramids and the numbers of triangular faces on the square pyramids.

Number of Square Pyramids	Number of Triangular Faces
1	4
2	8
3	12
5	?
?	28

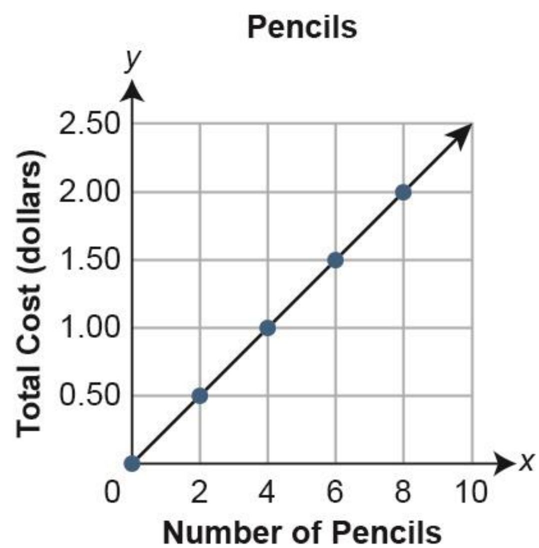
When asked to find the missing value in the first column, some students may look at the numbers in that column only and determine that the value is 8 because the two values above it, 5 and 3, add to 8, just as 2 and 3 add to 5 and 1 and 2 add to 3. Rather than determining the next value based on the previous values, students should focus on the relationship between the number of square pyramids and the number of triangular faces. Ask students to determine the ratio of Number of Triangular Faces to Number of Square Pyramids by dividing to get a constant ratio of 4. In terms of geometry, one square pyramid has 4 triangular faces, so the number of triangular faces will always be a multiple of the number of pyramids. If there are 28 faces, there must be 28 divided by 4, or 7, pyramids. When asked to find the missing value in the Number of Triangular Faces column, students should determine the value based on the relationship with the number of triangles and determine the value to be 5 times 4 or 20 and not 16 by incorrectly adding 4 to the previous value of 12.

► Enrichment

- Ask students to solve a ratio or rate problem involving positive rational numbers. For example, give students the following situation: “Marlene buys 2 pencils for a total of $0.50. Each pencil costs the same amount. What is the cost of 9 pencils?” Ask students to represent the situation by creating a ratio table and then graphing the ordered pairs. Then, have them determine the cost of 9 pencils. Students may create tables and graphs similar to the following examples.

Pencils

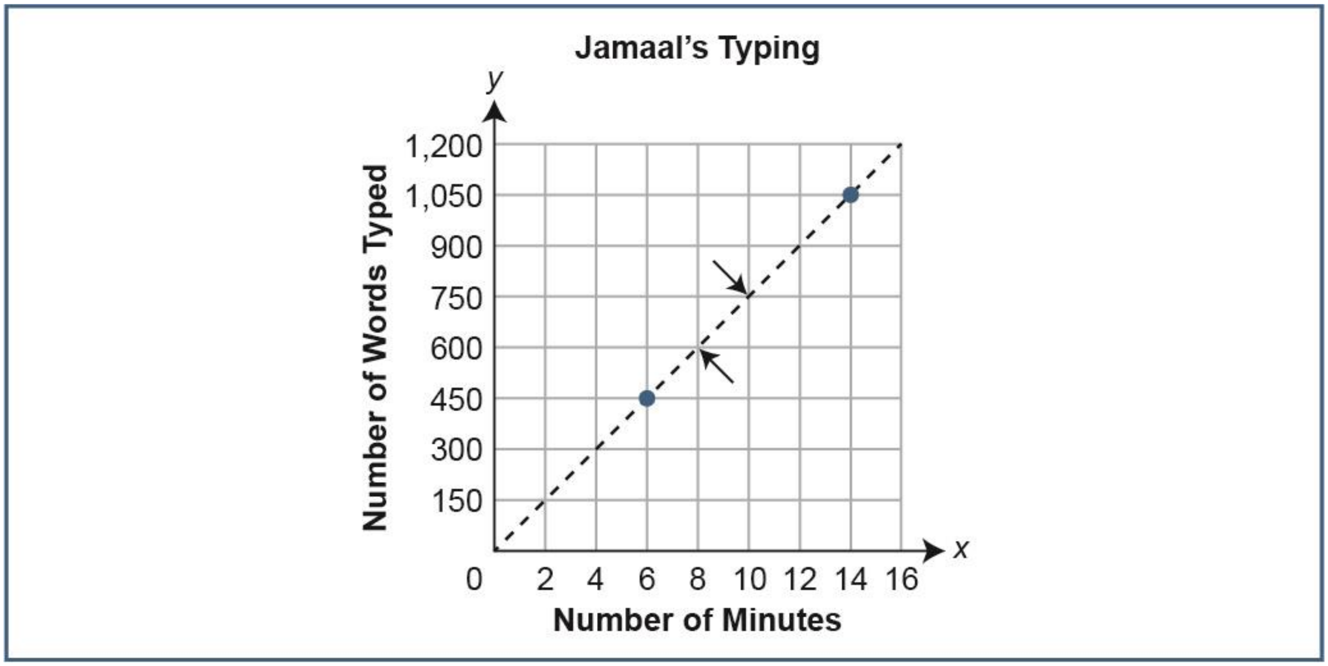
Number of Pencils	0	2	4	6	8
Total Cost (dollars)	0	0.50	1.00	1.50	2.00



Using the information from the table and graph, students can determine that the cost for 1 pencil is $0.25, so the cost of 9 pencils is $2.25.

- Ask students to create and use a graph to find the missing values in a table. For example, Jamaal types at a constant rate, as shown in the table.

Number of Minutes	Number of Words Typed
6	450
8	?
?	750
14	1050



Guide students to see that the points with the given x- and y-coordinates that fall on the same line as the graphed points are (8, 600) and (10, 750). Therefore, the values that complete the table are 600 and 10.

► Key Terms

coordinate plane, equation, equivalent ratio, rate, ratio, table, tape diagram

A-R.1.1.4

▶ Reporting Category, Assessment Anchor, Anchor Descriptor

Ratios and Proportional Relationships

Understand ratio concepts and use ratio reasoning to solve problems.

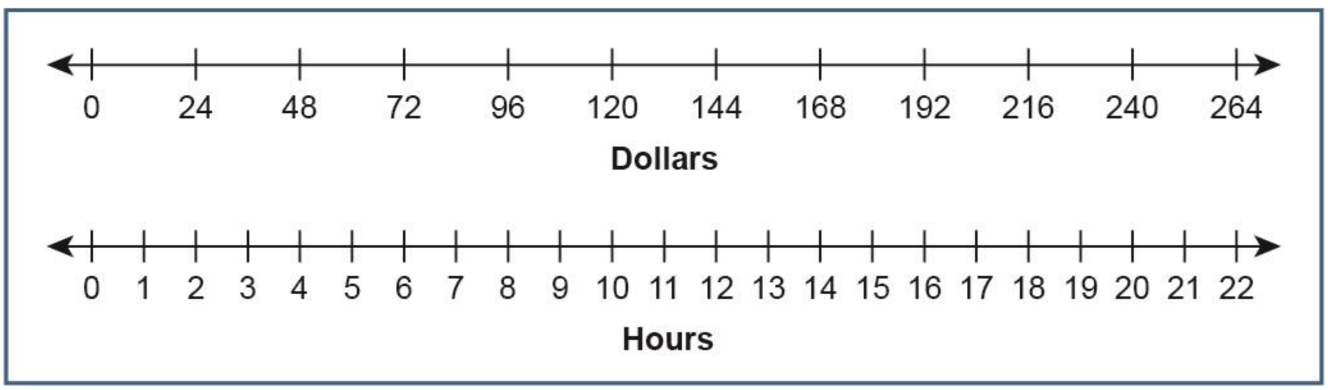
Represent and/or solve real-world and mathematical problems using rates, ratios, and/or percents.

▶ Eligible Content

A-R.1.1.4 Solve unit rate problems including those involving unit pricing and constant speed.

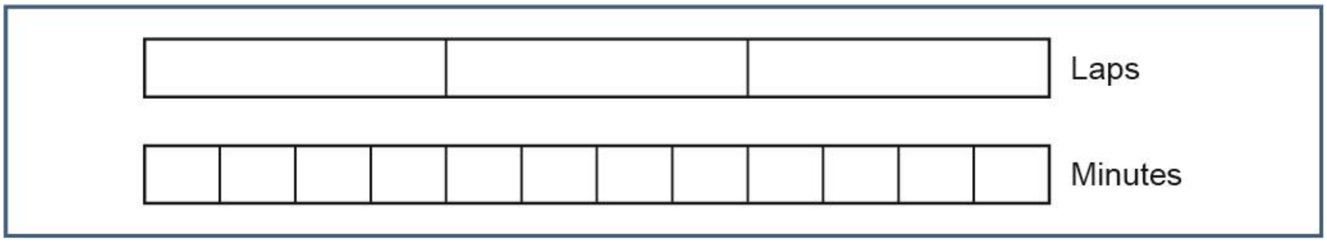
▶ Instructional Supports

- Ask students to use double number line diagrams to help determine the unit rate when solving real-world problems. For example, give students the double number line diagram shown, as well as the following problem: “Rudy works at least 20 hours each week. Last week, he earned $264 for 22 hours of work. At this rate, how much does Rudy earn working 26 hours for the week?”



  Students can use the diagram to determine that the unit rate for the given situation is $12 per hour. They should also calculate that Rudy earns $312 for working 26 hours for the week by multiplying 12 by 26.

- Ask students to use bar models to help determine the unit rate when solving real-world problems. For example, give students the bar model shown to represent the following situation: “It took Christina 12 minutes to run 3 laps around the school track. At this rate, how many laps around the school track does she run in 56 minutes?”



Students can determine that it takes Christina 4 minutes to complete 1 lap around the school track. They should also calculate that Christina completes 14 laps around the school track in 56 minutes at this rate by multiplying 14 times 4.

- Ask students to determine a unit rate and to use it to solve a given real-world problem. For example, give students the following problem: "Albert took a two-part road trip during his vacation. During the first part of his road trip, he traveled 256 miles in 4 hours at a constant speed. During the second part of his road trip, Albert drove 6 hours at the same constant speed. What was Albert's constant speed, in miles per hour, and how many miles did he travel during the second part of his road trip?" Students should determine that Albert traveled at a speed of 64 miles per hour during the first part of the road trip. Students should use the unit rate to also determine that Albert traveled 384 miles during the 6 hours of the second part of his road trip.
- Ask students to set up a proportion to represent and solve a real-world problem. For example, give students the following situation: "A store sells 8 hot dogs for \$2. At this rate, what is the cost of 12 hot dogs?" Students should set up a proportion in which the unit values are correctly aligned on each side of the proportion, as shown.

$$\frac{8 \text{ hot dogs}}{2 \text{ dollars}} = \frac{12 \text{ hot dogs}}{x \text{ dollars}}$$

Students should then cross multiply to begin solving the proportion, as shown.

$$8x = 24$$

Students should solve for x to determine that the cost of 12 hot dogs is \$3.

► Identifying and Addressing Common Misconceptions

- Some students may misinterpret or incorrectly calculate unit rates when solving given problems. Ask students to use unit rates to solve the following problem: "A bag of 12 marbles costs \$3. At this rate, what would a bag of 18 marbles cost?" Some students may determine a unit rate of 4 marbles per dollar and then multiply 4 by 18 to get an answer of $72. Use a double number line diagram to demonstrate that if 12 marbles cost $3, then each marble must cost less than $1. An answer of $72 leads to a situation where each marble costs more than $1. Help students to recognize that 18 marbles should be multiplied by the unit rate of 0.25 dollars per marble to determine a cost of $4.50 for a bag of 18 marbles.

- Some students may not assign units correctly when setting up a proportion. Ask students to set up a proportion for the following problem: "Denice drove 150 miles in 2 hours while traveling at a constant speed. At this rate, how many miles does Denice drive in 5 hours?" Some students may set up the following proportion.

$$\frac{150}{2} = \frac{5}{x}$$

Help students understand that when setting up the proportion it is helpful to include the units to avoid confusion.

$$\frac{150 \text{ miles}}{2 \text{ hours}} = \frac{x \text{ miles}}{5 \text{ hours}}$$

If the first fraction is set up with miles in the numerator and hours in the denominator, the other fraction should be set up the same way. If the first fraction has hours in the numerator and miles in the denominator, then the other fraction should match.

► Enrichment

- Ask students to solve problems involving multiple unit rates. For example, give students the following problem: "Roy and James each started collecting trading cards at the same time. Each week, Roy adds 15 trading cards to his collection. Each week, James adds 7 trading cards to his collection. How many trading cards will James have in his collection when Roy has 120 trading cards in his collection?" Students should use Roy's unit rate to determine that it takes Roy 8 weeks to add 120 trading cards to his collection. Then students should use James's unit rate to determine that James will have 56 trading cards in his collection after 8 weeks.
- Show students how unit analysis can be used to determine the final unit on a unit rate problem involving multiple rates. For example, give students the following problem: "There are

2.54 centimeters in 1 inch, 12 inches in 1 foot, and 60 minutes in 1 hour. If an earthworm can move at the rate of 30 centimeters per minute, how many feet can it travel in 1 hour?" The following equation can be set up to solve this problem.

$$\frac{30\ \cancel{\text{cm}}}{1\ \cancel{\text{min}}} \bullet \frac{1\ \cancel{\text{in}}}{2.54\ \cancel{\text{cm}}} \bullet \frac{1\ \text{ft}}{12\ \cancel{\text{in}}} \bullet \frac{60\ \cancel{\text{min}}}{1\ \text{hr}} = \frac{59.06\ \text{ft}}{1\ \text{hr}}$$

► Key Terms

constant speed, double number line diagram, equation, equivalent ratio table, unit pricing, unit rate, unit analysis

A-R.1.1.5

▶ Reporting Category, Assessment Anchor, Anchor Descriptor

Ratios and Proportional Relationships

Understand ratio concepts and use ratio reasoning to solve problems.

Represent and/or solve real-world and mathematical problems using rates, ratios, and/or percents.

▶ Eligible Content

A-R.1.1.5 Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent.

▶ Instructional Supports

- Ask students to express fractions and decimals as percentages.
  - Ask students to express $\frac{2}{5}$ as a percentage. Students convert a fraction to a percentage by renaming the fraction with a denominator of 100. Percent means "per 100," so $\frac{2}{5} \bullet \frac{20}{20} = \frac{40}{100}$. Therefore, the fraction $\frac{2}{5}$ is equivalent to 40%.
  - Ask students to express 0.35 as a percentage by first representing 0.35 as a fraction, $\frac{35}{100}$, and then by recognizing that $\frac{35}{100}$ can be interpreted as "35 per 100." Therefore, 0.35 is 35%. By the same reasoning, students can also express a decimal value as a percentage by multiplying by 100 because $0.35 = \frac{35}{100}$ and $\frac{35}{100} \bullet 100 = 35$.
- Ask students to express percentages as fractions and decimals.
  - Ask students to express 11% as a fraction. Students convert a percentage to a fraction by writing it with a denominator of 100. Here, 11% is equivalent to $\frac{11}{100}$.
  - Ask students to express 64% as a decimal. Students convert a percentage to a decimal by dividing by 100 because 64% means $\frac{64}{100}$. Evaluate $64 \div 100$ to get 0.64.

- Ask students to describe a part as a percentage of a whole.
  - Give students the situation that 8 out of 20 marbles are blue. Ask students to write the ratio as a fraction, such as $\frac{8}{20}$. Since the denominator is a factor of 100, students can multiply the fraction by $\frac{5}{5}$ so that the denominator is equal to 100. The fraction is then rewritten as $\frac{40}{100}$. Therefore, 40% of the marbles are blue.
  - Give students the situation that 8 out of 32 students will be eating a lunch brought from home. Ask students to write the ratio as a fraction, such as $\frac{8}{32}$. Since the denominator is not a factor of 100, students can divide 8 ÷ 32 to change their fraction into the decimal 0.25. Multiplying the decimal by 100 shows that 25% of the students will be eating a lunch brought from home.

- Ask students to use proportions to solve percentage problems. Help students recognize that the setup for a proportion involving percentages is always the same equation based on the equality of two ratios.

$$\frac{\text{part}}{\text{whole}} = \frac{\text{percent}}{100}$$

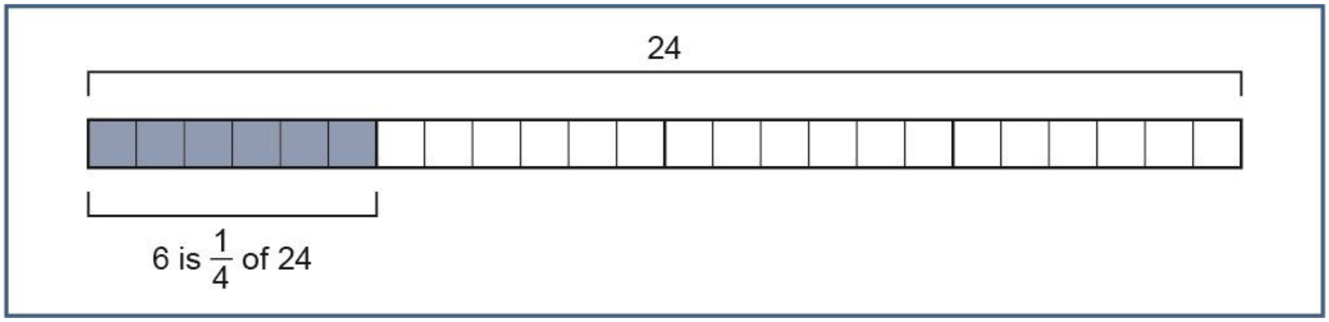
The "part" represents the portion of the whole and is generally the number associated with the word "is." The "whole" represents the total size of the group and is generally the number associated with the word "of." The "percent" represents the number with the percent sign (%). Give students the following examples.

  - "What number is 20% of 80?" Ask students to first determine the "part" and the "whole" and then to create a proportion. Students determine that 80 is the whole, and the part, which in this case is unknown, can be represented with x. Set up the proportion $\frac{x}{80} = \frac{20}{100}$. Students then multiply both sides of the equation by 80. The resulting equation is $x = \frac{1{,}600}{100}$. Students divide both sides of the equation by 100 and conclude that 20% of 80 is 16.

- "The number 6 is what percent of 24?" Ask student to first determine the "part" and the "whole" and then to create a proportion. Students determine that 24 is the whole and 6 is the part. In this case, the percent, or the part out of 100, is unknown and can be represented with x. Students set up the proportion $\frac{6}{24} = \frac{x}{100}$. Students then multiply both sides by 100. The resulting equation is $\frac{600}{24} = x$. Students divide by 24 to conclude that 6 is 25% of 24.

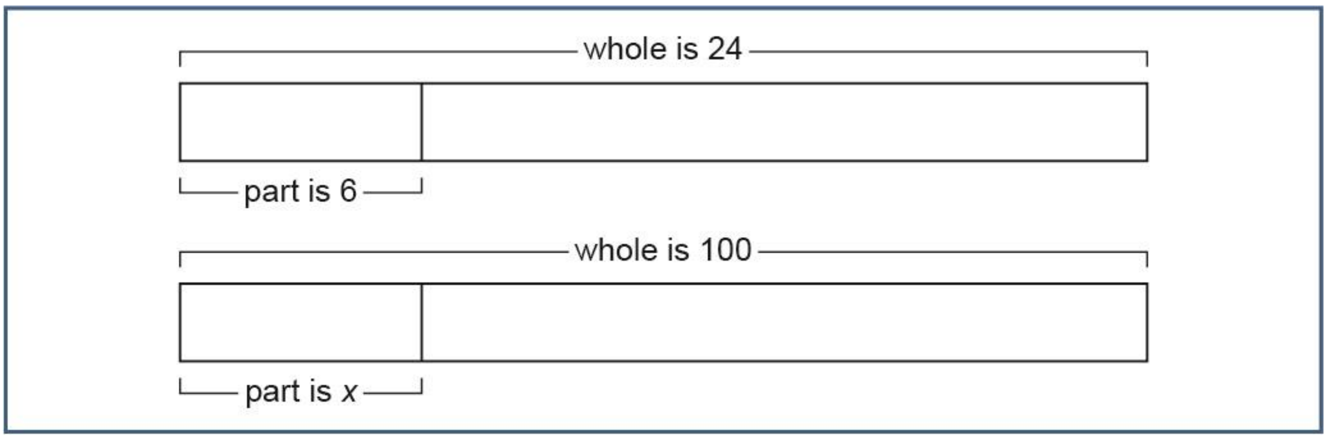
- "The number 9 is 75% of what number?" Ask student to first determine the "part" and the "whole" and then to create a proportion. Students determine that 9 is the part and that the whole, which in this case is unknown, can be represented with x. Students set up the proportion $\frac{9}{x} = \frac{75}{100}$. Students then cross multiply 100 with 9 and x with 75. The resulting equation is $75x = 900$. Students divide both sides of the equation by 75 to conclude that 9 is 75% of 12.

- Ask students to find a whole when given a part and a percentage using ratios. For example, give students the following problem: "At a stable, 32% of the horses are black. There are 8 black horses at the stable. How many horses are in the stable?" The percentage of horses that are black (32%) can be expressed as the ratio $\frac{\text{32 black horses}}{\text{100 horses}}$. The total number of black horses can be determined by creating an equivalent ratio with a numerator of 8 black horses. Because 32 is a multiple of 8, students can divide both the numerator and the denominator by 4 to create the equivalent ratio $\frac{\text{8 black horses}}{\text{25 horses}}$. Therefore, the stable has 8 black horses out of a total of 25 horses.
- Ask students to create a visual model to represent and solve percentage problems. For example, provide students with the following problem: "What percent of 24 is 6?" A possible student model is shown.



There are 4 groups of 6 squares in a set of 24 squares. Because 6 is $\frac{1}{4}$ of 24 and 25 is $\frac{1}{4}$ of 100, it follows that 6 is 25% of 24.

Another visual model for the same problem is to compare a bar model of 6 parts from a whole of 24 with a bar model of x parts from a whole of 100.



From the bar models, students determine that the ratios of 6 to 24 and x to 100 are equal, which leads to the proportion $\frac{6}{24} = \frac{x}{100}$.

▶ Identifying and Addressing Common Misconceptions

- Some students may incorrectly convert single-digit percentages into decimals. Ask students to write 8% as a decimal. Some students may give an answer of 0.8. Help students to understand that percentages represent a part per one hundred and to think of 8 percent as 8 per hundred or 8 hundredths. Ask students to write 8 hundredths as a decimal.
- Some students may incorrectly convert decimals involving tenths into percentages. Ask students to write 0.5 as a percentage. Some students may give an answer of 5%. Ask students to identify

the place value of the 5 as tenths and to rewrite 0.5 as $\frac{5}{10}$. Remind students that percentages represent the number of parts per hundred; therefore, it is helpful to represent $\frac{5}{10}$ with its equivalent fraction of $\frac{50}{100}$ to show that 0.5 is the same as 50%.

- Some students may think that greater percentages always mean greater amounts without considering the whole. Ask students to explain whether “50% of 20” or “10% of 200” is greater. Some students may think that 50% is always greater than 10%. Remind students of the importance of considering the whole when comparing fractions. Then ask them to calculate and compare both percentages.
- Some students may confuse the part and the whole when interpreting statements about percentages. Ask students to determine what number is 10% of 30 and also to determine what number 30 is 10% of. Some students may not recognize the difference between the two questions. Help students recognize that the wording “of 30” indicates that 30 is the whole and is represented by the denominator in a fraction. The wording “30 is 10% of” indicates that 30 is the part and is represented by the numerator in a fraction.

► Enrichment

- Help students understand that the word percent has two parts: “per” means each and “cent” means hundred. Ask students to discuss other situations in which they encounter both of these words. Students may be familiar with cost per person and miles per hour, which indicate a rate. Students may also be familiar with cents and centuries, both of which indicate the number 100.
- Ask students to identify and write general strategies that can be used to solve various percentage problems. Some examples are shown.
    - To find 60% of a value, first find 10% and then multiply by 6 because 10% • 6 = 60%.
    - To find 75% of a value, first find 50%. Then find half of the 50% value, which is 25%. Then add the 50% value to the 25% value because 50% + 25% = 75%.
    - To find 20% of a value, first multiply the value by 2. Then find 10% of the doubled value because 2 • 10% = 20%.

- Ask students to discuss what it means to have a percentage that is greater than 100%. Help students understand a percentage greater than 100% can be written as a decimal or as a fraction in the same manner as other percentages. A percentage greater than 100% indicates a quantity that is greater than one whole in the same way that a mixed number indicates a fraction that is greater than one whole.
- Ask students to work with percentages that are smaller than 1%.

► Key Terms

part, per 100, percentage, quantity, whole

B-E.1.1.1

▶ Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Apply and extend previous understandings of arithmetic to numerical and algebraic expressions.

Identify, write, and evaluate numerical and algebraic expressions.

▶ Eligible Content

B-E.1.1.1 Write and evaluate numerical expressions involving whole-number exponents.

▶ Instructional Supports

Structure of exponential expressions

- Discuss the structure of exponential notation in the form a^b where *a* and *b* are whole numbers. A value written in this form contains two parts: the base, *a*, and the exponent, *b*. This notation can be thought of as repeated multiplication where *a* is multiplied by itself *b* times. For example, 2^3 means that the number 2 is being multiplied by itself 3 times, where 2 is the base and 3 is the exponent (2 • 2 • 2).
- Ask students to rewrite a repeated multiplication expression using exponential notation. For example, in the expression 7 • 7 • 7 • 7, the repeated factor is 7, so 7 is the base. The number 7 appears as a factor 4 times, so 4 is the exponent. Therefore, the expression 7 • 7 • 7 • 7 can be rewritten as 7^4.
- Ask students to use repeated multiplication to rewrite an expression given in exponential form. For example, the expression 5^3 is equivalent to an expression in which the factor of 5 is multiplied 3 times, which can be rewritten as 5 • 5 • 5.
- Ask students to investigate the value of an exponential expression with a power of 0 by investigating patterns of powers. For example, ask students to look for patterns in powers of 2 and then extend that pattern to determine the value of 2^0. Give students a table to complete in order to help them find and discuss patterns.

Powers of 4	Value
2^4	16
2^3	8
2^2	4
2^1	2
2^0	?

Students should determine that as the power of 2 increases by 1, the value of the expression is multiplied by 2, and as the power of 2 decreases by 1, the value of the expression is divided by 2. For example, 16 ÷ 2 is 8, 8 ÷ 2 is 4, and 4 ÷ 2 is 2. Since 2 ÷ 2 = 1, students should conclude that $2^0 = 1$. Help students extend this to conclude that $a^0 = 1$ for any nonzero number a.

Comparing exponential expressions

- Ask students to compare the values of expressions written in exponential notation which have like bases or like exponents. For example, students can compare 3^5 and 2^5. Because the exponents are the same, a comparison of the bases can be used to determine which expression has the greater value. In this case, the base 3 is greater than the base 2, so $3^5 > 2^5$. Additionally, students can compare 4^5 and 4^7. Because the bases are the same, a comparison of the exponents can be used to determine which expression has the greater value. The exponent 5 is less than the exponent 7, so $4^5 < 4^7$.

Evaluating exponential expressions

- Ask students to evaluate expressions involving whole-number exponents by applying the order of operations. For example, give students the expression shown.

$$5^2 + 3(4 + 3^2) - 2(8 + 4)$$

Following the order of operations, students should start by evaluating the parts of the expression inside the parentheses. The part of the expression inside the first set of parentheses is equivalent to 13 (because 4 + 9 = 13), while the part of the expression inside the second set of parentheses is equivalent to 12. The expression can now be rewritten as shown.

$$5^2 + 3(13) - 2(12)$$

The value 5^2 can be simplified to 25, while 3(13) = 39 and 2(12) = 24. Therefore, the expression is equivalent to 25 + 39 – 24, which is equivalent to 40.

► Identifying and Addressing Common Misconceptions

- Some students treat exponential notation as standard multiplication, viewing a^b as $a \cdot b$ or "*a* added to itself *b* times." Ask students to evaluate 3^5 and explain their thinking. Some students might interpret 3^5 as $3 \cdot 5$ or as 3 + 3 + 3 + 3 + 3. Remind students that exponentiation represents repeated multiplication just as multiplication represents repeated addition. The notations are different because the operations are different. Additionally, when the exponents are small enough, encourage students to always rewrite expressions presented in exponential notation as expressions using repeated multiplication. The process of writing out this step may further prevent students from accidentally multiplying a base by an exponent.

- Some students perform operations out of order, applying other operations to the base before applying the exponent. Ask students to evaluate $1 + 2^4$ using the order of operations. Some students may interpret the expression as equivalent to 3^4, performing the operations from left to right and adding 1 and 2 as the first step. Remind students that applying exponents is a step in the order of operations that comes before multiplication and after grouping symbols. Further, explain to students that an exponent only applies to its base, and unless there are grouping symbols to show otherwise, the base is always the **single** number that immediately precedes the exponent. Therefore, in the first example above, 2^4 needs to be evaluated first, and then that result is added to 1.

- Some students reverse the roles of the base and the exponent and misinterpret a^b to mean "*b* multiplied by itself *a* times." Some students may think the base and the exponent are interchangeable, believing a^b and b^a to be equivalent expressions. Ask students to determine whether 3^2 is greater than, less than, or equal to 2^3 and explain their thinking. Some students might claim that 2^3 is greater or that the two expressions are equal. Ask students to consider specific examples of a^b and b^a. For example, ask students to evaluate 2^5 and 5^2 by rewriting each value as repeated multiplication. Observe that 2^5 is the same as $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ or 32, and it is not equal to 5^2, which is the same as $5 \cdot 5$ or 25.

► Enrichment

- Ask students to explore comparisons between a^b and b^a for whole number values of a and b. If $b > a$, then which is greater: a^b or b^a? Ask students to discuss whether this is always true.
- Ask students to evaluate exponential expressions with fractional bases less than 1, such as $\left(\frac{2}{3}\right)^3$, and explain their thinking and share their observations.
- Ask students to evaluate an expression involving whole-number exponents. Then ask students to evaluate similar expressions created by adding, removing, or adjusting the location of grouping symbols. For example, given the expression $5 \cdot 4^2 + 7 \cdot 3^3$, have students place a set of parentheses around different numbers and operations and record the resulting values. Challenge students to find as many unique values as possible for the expression by adding and manipulating the grouping symbols. Further, challenge students to find the highest possible value or the lowest possible value just by adding and manipulating the grouping symbols and justify how they know it is the highest or lowest possible value. Ask students to discuss their findings in small groups.
- Ask students to evaluate expressions involving whole-number exponents that are applied to quantities within a grouping symbol. For example, have students evaluate the expression $4(10 - 2^3 + 5)^2 \cdot 3$ and explain their thinking.
- Ask students to investigate and explain patterns in the ones digits of powers of a number. For example, the powers of three are 3, 9, 27, 81, 243, 729, 2,187, ... The ones digits of powers of three follow the pattern 3, 9, 7, 1, 3, 9, 7, 1, ... Ask students to discuss the patterns with a partner and as a class.

► Key Terms

base, exponent, exponential notation, factor, order of operations

B-E.1.1.2

► Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Apply and extend previous understandings of arithmetic to numerical and algebraic expressions.

Identify, write, and evaluate numerical and algebraic expressions.

► Eligible Content

B-E.1.1.2 Write algebraic expressions from verbal descriptions.

► Instructional Supports

Define variables

- Ask students to discuss the role of variables in mathematical expressions. Variables may be used for a number of reasons, and they may be represented using a variety of symbols. Variables are most commonly used to represent unknown or changing quantities, and they are expressed as letters. It is important for students to understand that regardless of how variables are used or written, they are placeholders for numbers, and even though variables may not look like numbers, they are treated as numbers.
- Help students see parallels between numeric and algebraic expressions. For example, ask students to write a numeric expression to represent the statement “3 less than 7.” Then ask students to write an algebraic expression to represent the statement “x less than 7.” Discuss the parallels in structure between the answers $7 - 3$ and $7 - x$ and note that the only thing changing in both the statement and the numeric expression is the replacement of 3 by x.

Representing situations using variables

- Ask students to use algebraic expressions to model mathematical and real-world problems. Discuss the use of contextual clues and keywords to help identify relevant quantities and to determine which operations to use on which quantities. For students needing extra support, suggest replacing the variable with a number and writing a numeric expression first. Some sample descriptions and corresponding expressions are shown in the following table.

Verbal Description	Mathematical Expression	Meaningful Characteristics of the Variable
Divide x by 6.	$x \div 6$ or $\frac{x}{6}$	x is any number
Multiply 13 and y.	$13 \bullet y$ or $13y$	y is any number
There are m students signed up for the Math Club. Then 4 more students join. How many students are in the Math Club?	$m + 4$	m represents a number of students m is positive and whole
Ms. Patel received 6 items in yesterday's mail. She received x fewer items in today's mail than in yesterday's mail. How many items did Ms. Patel receive in today's mail?	$6 - x$	x represents a number of pieces of mail x is positive and whole
A chef slices a 9-inch-long stalk of celery into p equal pieces. How many inches long is each piece?	$\frac{9}{p}$	p represents a length, in inches p is positive and whole p is nonzero
Tom had \$20. Then he bought 3 cookies for c dollars each. How much money does Tom have?	$20 - 3c$	c represents a cost, in dollars c is positive (could be whole or a decimal extended to the hundredths place) c is nonnegative

- Ask students to discuss possible constraints on values represented by variables based on real-world contexts. For example, if a variable in a situation is being used to represent a length or a distance, then that variable would most likely never be equal to a negative number. The previous table includes notes about the possible values for each example.

► Identifying and Addressing Common Misconceptions

- Some students believe that a variable is a letter that can only represent one specific number. Explain to students that a variable can also represent a set of numbers. For each of the following situations, ask students to determine if the variable n can represent only one number or many numbers.

- 5 more than n is 8
- A baker knows how many pastries he expects to sell each day based on experience. The baker does not want to run out of pastries, so he bakes five dozen extra pastries each day. The expression $n + 5$ represents the dozens of pastries he bakes on a day when he expects to sell n pastries.

Further, ask students to determine what value n represents in the expression "5 more than n is 8." Students should realize that in this expression, n represents 3. Then, for the bakery example, ask students to determine several possible values n could represent in the expression "$n + 5$." Examples include 10, 11, 2, 8, etc., because n represents the number of pastries the baker expects to sell each day, and this amount can change from day to day. Additionally, use common formulas, such as the area of a rectangle, to further illustrate situations where variables can take on multiple values. Students should understand that multiplying the length of a rectangle by the width of the rectangle gives the area of the rectangle, and they should understand this to be true for many different values for the length and width.

- Some students think variables take on properties of operations that are different from those of numbers. Ask students to consider whether $2 + x$ is equivalent to $x + 2$. Some students may think the expressions are not equivalent because they do not look the same or that they cannot be compared without knowing the value of x. Remind students that while variables are presented as letters, they are used to represent numbers. Therefore, the properties of operations (the commutative, associative, identity, inverse, and distributive properties), as well as the order of operations, apply to variables in the same way as they apply to numbers. For instance, students learn in previous grades that subtraction is not commutative (i.e., $2 - 5 \neq 5 - 2$). Ask students to discuss how the same rule applies to subtraction involving variables (i.e., $2 - x \neq x - 2$).
- Some students do not attend to precision in defining what a variable represents in a real-world context and may misinterpret expressions and parts of expressions involving variables. Present students with the following sample context:

  There are m students signed up for the Math Club. Then 4 more students join. How many students are in the Math Club?

Ask students to define what m represents in this context. Some students may say that "m represents the number of students in the Math Club." Ask these students to then define what $m + 4$ represents in this context. Some students may say that "$m + 4$ represents the number of students in the Math Club." Guide students to realize that the definition of m should not be the same as the definition of $m + 4$, because this would imply that $m = m + 4$. Help students specify more precise definitions for m and $m + 4$. Ask these students to clarify their original definitions, guiding them to see that "m represents the number of students currently signed up for the Math Club before any more students join," and "$m + 4$ represents the number of students signed up for the Math Club after 4 more students join." Discuss with students the importance of being very clear and specific in their definitions of variables and help them realize that once a variable is defined in context, it should not take on a different meaning. Emphasize that while the exact quantity a variable represents may be unknown or fluctuating, the description of the quantity it represents—or the definition of the variable—is unchanging.

► Enrichment

Ask students to rewrite verbal descriptions as algebraic expressions involving two or more operations. Discuss the fact that grouping symbols should be used if the verbal description indicates an order that does not follow the order of operations. For example, ask students to rewrite the following verbal description as an algebraic expression.

Add 13 and y, then multiply by 2.

Students may write "$13 + y \cdot 2$," however the order of operations calls for 2 to be multiplied by y before adding 13. Since the verbal description indicates the addition should be done first, a correct expression would be $2(13 + y)$.

- Ask students to create a real-world problem that can be represented by a given algebraic expression. For example, give students the expression "$3g + 5$" and ask them to create a corresponding real-world problem. One example response is given: "Sam ate g grapes. Lin ate 5 more than 3 times as many grapes as Sam. How many grapes did Lin eat?"
- Ask students to write an algebraic expression to describe the rule for extending a numerical pattern. For example, present students with the following number pattern.

3, 5, 7, 9, 11, . . .

Students may recognize that the numbers are increasing by 2 and use this pattern to find the next several terms. However, challenge students to identify a term farther out, such as the 80th term of the pattern. To do so, students will need to investigate how the term number relates to the term itself. It may be helpful for students to create a table similar to the following.

Term	Number
1	3
2	5
3	7
4	9
5	11

Ask students to share patterns they notice. Guide students to realize that each number in the pattern is 1 more than double the term number. For instance, the 5th term is 11 because $2(5) + 1 = 11$; therefore, the 80th term is found using the equation $2(80) + 1 = 161$. Then ask students to write an algebraic expression to represent the *n*th term in the pattern.

► Key Terms

algebraic expression, variable

B-E.1.1.3

► Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Apply and extend previous understandings of arithmetic to numerical and algebraic expressions.

Identify, write, and evaluate numerical and algebraic expressions.

► Eligible Content

B-E.1.1.3 Identify parts of an expression using mathematical terms (e.g., sum, term, product, factor, quotient, coefficient, quantity).

► Instructional Supports

- Review and/or define mathematical terminology that can be used to describe an algebraic expression or parts of an algebraic expression. Ensure students are familiar with the following mathematical terms:
    - Term: an addend of an algebraic expression, typically in the form of a single number or variable or the product of numbers and/or variables
    - Sum: the result of adding two or more values
    - Product: the result of multiplying two or more values
    - Factor: a value that is being multiplied by another value
    - Quotient: the result of dividing two or more values
    - Coefficient: a constant value that is being multiplied by a variable
- Ask students to identify parts of an algebraic expression using mathematical terminology. For example, give students the following sample expression and ask them to identify at least one instance of each vocabulary word.

$$3(5x - 6) + \frac{4y}{7}$$

The chart shows key features the students should be looking for.

Term	$\boxed{3(5x - 6)} + \boxed{\frac{4y}{7}}$ The expression has two individual pieces that are being added together: $3(5x - 6)$ and $\frac{4y}{7}$.
Sum	$3(5x - 6) \boxed{+} \frac{4y}{7}$ The addition symbol indicates a sum of the term before the symbol and the term after the symbol.
Product	$\boxed{3(5x - 6)} + \frac{\boxed{4y}}{7}$ The first boxed expression shows the product of 3 and the difference of $5x$ and 6. The second boxed expression shows the product of 4 and the variable y. Note: Students may also interpret $\frac{4y}{7}$ as the product of $\frac{4}{7}$ and the variable y.
Factor	$\boxed{3(5x - 6)} + \frac{\boxed{4y}}{7}$ The first boxed expression shows the product of two factors: 3 and $(5x - 6)$. The second boxed expression shows the product of two factors: 4 and y. Note: Students may also interpret $\frac{4y}{7}$ as the product of two factors: $\frac{4}{7}$ and y.
Quotient	$3(5x - 6) + \boxed{\frac{4y}{7}}$ The fraction bar indicates a quotient of $4y$ and 7.
Coefficient	$3(\boxed{5}x - 6) + \frac{\boxed{4}y}{7}$ The expression has two variables, and therefore two coefficients; 5 is the coefficient in $5x$, and 4 is the coefficient in $4y$.

	Note: Students may also interpret $\frac{4y}{7}$ as the product of $\frac{4}{7}$ and the variable y. In this interpretation, the coefficient of y is $\frac{4}{7}$.

- Ask students to verbally describe a whole expression and parts of the expression using mathematical terminology. For example, given the expression $(x \div 5) + (3 - x)$, a valid description is "the sum of the quotient of x and 5 and the difference of 3 and x." Students might also describe the first term as a quotient with a dividend of x and a divisor of 5.

- Ask students to use multiple mathematical words to describe a single quantity in an expression. For example, given the expression $x \div (y + 1)$, students describe $(y + 1)$ as both a sum and a divisor. If the expression were changed to $x \div 2(y + 1)$, students would describe $(y + 1)$ as both a sum and a factor.

- Ask students to identify and describe the different terms in a given expression, including constant terms. For example, given the expression $14p - 9q + 10$, identify the three terms of the expression as $14p$, $9q$, and 10. Remind students that terms are the parts that are being added or subtracted. Note that some students may interpret the subtraction as addition of a negative, or $14p + (-9q)$, in which case the second term would be classified as $-9q$. Further, ask students to identify terms that are products and list the factors. The term $14p$ is a product of the factors 14 and p, and the term $9q$ is a product of the factors 9 and q. Discuss that terms with a fixed value (or terms without a variable) are called "constant terms." In the preceding example, identify the final term of 10 as a constant term.

► Identifying and Addressing Common Misconceptions

- Some students confuse the meanings of the terms "sum," "difference," "product," and "quotient." Present students with the following expressions and ask them to label each as a sum, difference, product, or quotient.

$$8 - x \qquad x \div 8 \qquad 8(x) \qquad x + 8$$

Some students may make the common mistake of thinking $x \div 8$ represents a difference or that $8(x)$ represents a sum. For these students, practice and exposure is key. Focus on modeling the use of these terms yourself during discussions. Additionally, have students create a graphic organizer for each of the four basic operations that includes all related terminology and common representations.

- Some students do not recognize the various ways to express products and quotients in an algebraic expression. Most students will recognize the symbols × and ÷ as indications to multiply and divide, respectively; however, some students will not recognize alternative notations, such as the use of parentheses to indicate a product or a fraction bar to indicate a quotient. Present students with the following expressions and ask them to indicate whether each expression shows a product, a quotient, or neither.

$$3a \qquad a \bullet 3 \qquad a/3 \qquad a - 3$$

$$\frac{a}{3} \qquad a(3) \qquad 3 * a \qquad (3)(a)$$

  Some students think, for instance, that $3a$ represents the sum $(3 + a)$, or that $3a$ is not indicative of any operation at all. Some students may not identify bullet points or asterisks as multiplication symbols or forward slashes or fraction bars as division symbols. For these students, practice and exposure is key. Regularly "quiz" students with sample expression lists such as the one above until they correctly identify all indicated operations. Additionally, have students create a graphic organizer for each of the four basic operations that includes all related terminology and common representations.

- Some students confuse the meanings of the words "coefficient" and "constant term." Give students the expression $-6x + 15 + 3y$ and ask them to name the coefficients and constant terms. Some students will say that -6 and 3 are constant terms and that 15 is a coefficient. Some students may say that the entire terms of $-6x$ and $3y$ are coefficients. Some students may be confused by the idea that a single variable has an invisible coefficient of 1 and instead think that 15 is a coefficient of an invisible variable. Encourage students to consider the everyday meaning of the word "constant." Just as the meaning suggests, a constant term is a term that has a constant, or fixed and unchanging, value. A term that includes a variable cannot be said to have a fixed or unchanging value; however, a term that is a single number or a product of numbers does have a constant value. Note that it is also important to distinguish the meanings of "constant value" and "constant term." A constant value is simply a number, and so a coefficient is in fact a constant value that is being multiplied by a variable; however, the term including the coefficient is not a constant term.

- Some students think that a term comprised of a single variable has no coefficient or a coefficient of zero. Ask students to list the coefficients in the expression $7x + y + 4z$. Some students will say the only coefficients are 7 and 4. Some students may say the coefficients are 7, 0, and 4. Remind students that a coefficient is defined as the number (or constant value) that is being multiplied by a variable. Therefore, when considering the term y of the example expression, ask students what number is being multiplied by the y. Guide students to realize that any single value can be rewritten as a product of two factors by multiplying by 1, and so the term y can be rewritten as the term $1y$, showing that the coefficient is indeed 1.
- Some students think that multiplication or division symbols can separate terms in an expression. Ask students to identify the terms in the expression $12y \div 4 + 5x$. Some students may think there are three terms: $12y$, 4, and $5x$. Remind students that the individual terms of an expression can only be separated by addition or subtraction symbols, and since a division symbol is separating the $12y$ and 4, this entire quotient is just one term. Therefore, the expression has only two terms: $(12y \div 4)$ and $5x$. It may also help to ask students to rewrite the division using a fraction bar. Viewing the expression as $\frac{12y}{4} + 5x$ may make it easier to see that the expression has only two terms.

► Enrichment

- Ask students to write algebraic expressions that match given mathematical terminology. For example, given the statement, "a product of three factors, two of which are variables and one of which is a difference of two numbers," students could write the algebraic expression $(7 – 3)xy$. Likewise, given the statement, "a quotient of a sum and a product, where the sum has three addends, one of which is a variable, and one factor of the product is a variable," students could write the algebraic expression $(x + 14 + 15) \div (8y)$.

- Given a real-world context and a representative algebraic expression, ask students to identify parts of the expression using mathematical terminology and define what each part means in context. For example, given that a sunflower is 12 centimeters tall and grows at an approximate rate of 5 centimeters per week, along with the expression $12 + 5w$, identify the following:
  - The expression shows a sum of two terms. The whole expression represents the total height, in centimeters, of the sunflower after w weeks.
  - The first term is 12. It is a constant term. It represents the starting height of the sunflower.
  - The second term is $5w$. It is the product of the factors 5 and w. It represents the additional growth of the sunflower after w weeks.
  - In the term $5w$, the number 5 is a coefficient. It represents the approximate number of centimeters the sunflower grows in one week.
- Ask students to identify the factors and the coefficient of a term made up of multiple variables with exponents. For example, given the expression $3x^2y(5z)$, identify the following:
  - The expression is a single term written as a product of four factors: 3, x^2, y, and $(5z)$. Note that x^2 is itself a product of the factors x and x. Also note that the factor $(5z)$ is itself a product of factors: 5 and z.
  - The coefficient of the term is 15. As written, the coefficient of x^2 is 3, the coefficient of y is 1, and the coefficient of z is 5. However, each of these products are factors of the larger product that makes up the term. Since multiplication is commutative and associative, the expression could be rewritten as $3 \bullet x^2 \bullet y \bullet 5 \bullet z = (3 \bullet 5) \bullet x^2 \bullet y \bullet z = 15x^2yz$, showing that 15 is the numeric factor of a product containing variables, and therefore 15 is the coefficient of the term.

► Key Terms

algebraic expression, coefficient, constant term, expression, factor, product, quotient, sum, term, variable

B-E.1.1.4

▶ Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Apply and extend previous understandings of arithmetic to numerical and algebraic expressions.

Identify, write, and evaluate numerical and algebraic expressions.

▶ Eligible Content

B-E.1.1.4 Evaluate expressions at specific values of their variables, including expressions that arise from formulas used in real-world problems.

▶ Instructional Supports

- Ask students to examine real-world formulas and determine what each variable represents. For example, give students the formula for the area of a rectangle, $A = l \cdot w$. Students should determine that
    - A represents the area of a rectangle,
    - l represents the length of the rectangle, and
    - w represents the width of the rectangle.

  Likewise, given the formula for the volume of a rectangular prism, $V = l \cdot w \cdot h$, students should determine that V represents the volume of a rectangular prism, l represents the length of the base of the prism, w represents the width of the base of the prism, and h represents the height of the prism.

- Ask students to evaluate algebraic expressions with one variable at specific values of the variable. For example, ask students to evaluate $5 + n \cdot 3$ when $n = 7$. Students should first rewrite the algebraic expression as a numeric expression by substituting 7 for n: $5 + 7 \cdot 3$. Then, students should follow the order of operations to find the value of the resulting numeric expression: $5 + 7 \cdot 3 = 5 + 21 = 26$.

- Ask students to evaluate algebraic expressions in which the same variable appears in multiple terms. For example, give students the expression $8x + 5(3 - 2x)$. Ask students to evaluate the expression when $x = 5$. Students start by substituting 5 for x in the expression and then follow the order of operations as shown.

$$8(5) + 5(3 - 2(5))$$

$$8(5) + 5(3 - 10)$$

$$8(5) + 5(-7)$$

$$40 + -35$$

$$5$$

- Ask students to evaluate algebraic expressions with two variables at specific values of their variables. For example, ask students to evaluate $4 \cdot p + pq - 19$ when $p = 6$ and $q = 11$. Students should first rewrite the algebraic expression as a numeric expression by substituting 6 for p and 11 for q; then, students should follow the order of operations to find the value of the resulting numeric expression: $4 \cdot 6 + 6 \cdot 11 - 19 = 24 + 66 - 19 = 90 - 19 = 71$.

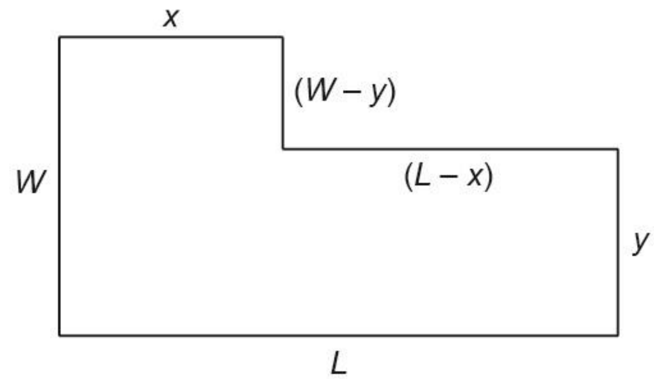
► Identifying and Addressing Common Misconceptions

- When substituting in a value for a variable that is being multiplied by a coefficient, some students may see it as a two-digit number instead of as a representation of multiplication. Ask students to evaluate the expression $4w$ when $w = 7$. Some students may write 7 in place of w and say the expression has a value of 47. Remind students that a number written directly in front of a variable implies multiplication between the number and the variable, so $4w$ represents the product of 4 and w. Ask students to rewrite $4w$ using a multiplication symbol or parentheses. Students may write $4 \cdot w$. Ask students to evaluate the rewritten expression when $w = 7$. Most students will find that the expression has a value of 28. Point out that since $4w$ and $4 \cdot w$ are equivalent expressions, they should have an equivalent value when $w = 7$.

- Some students may incorrectly apply the order of operations when substituting in values for variables, especially when the expression includes a variable with a coefficient other than 1. Ask students to evaluate the expression $8 \div 2x$ when $x = 4$. Some students will multiply 2 and 4 first to get $8 \div 8 = 1$. Ask students to review the basic order of operations. Ask students to describe how to handle various instances of multiplication/division or addition/subtraction in an expression and help them recall that all multiplication and division operations are performed left to right, followed by all addition and subtraction operations performed left to right. Then ask students to carefully reconsider the expression $8 \div 2x$. Guide them to see that the expression contains a quotient and a product. It may be helpful to some students to rewrite $2x$ as $2 \bullet x$. Since all multiplication and division must be performed from left to right, the quotient in this expression must come before the product, resulting in 16.

► Enrichment

- Give students an algebraic expression with one variable, along with the value of the expression, and ask them to find the value of the variable that yields the given value of the expression. For example, give students the expression $3x + 7$ and ask them to find the values for x that give the expression values of 25, 40, and 52.
- Give students an algebraic expression with two variables, along with the value of the expression, and ask them to find pairs of values of the variables that yield the given value of the expression. For example, give students the expression $3xy$ and ask them to find as many possible pairs of values for x and y that give the expression a value of 48. For further enrichment, ask students to list the pairs of values as ordered pairs and graph the results on a coordinate plane.
- Give students two algebraic expressions, each having one variable, and ask them to substitute a specific value for the variable into the first expression and then substitute the resulting value of the first expression for the variable into the second expression. For example, give students the expressions $2x + 1$ and $5y - 8$. Ask students to evaluate $2x + 1$ when $x = 7$. Then ask them to evaluate $5y - 8$ when y is the resulting value of $2(7) + 1$.
- Ask students to create more than one formula for expressing a quantity and evaluate the formulas for specific values of their variables. For example, present students with the following composite rectilinear figure.



Challenge students to write multiple different formulas for expressing the area of the figure, each corresponding to a different calculation method. Then, ask students to confirm that all four formulas are equivalent by providing specific values for the variables and observing that each formula yields the same total area. Possible student formulas are shown.

- Method 1: Find the area of the single rectangle that envelopes the entire figure and subtract the area of the missing rectangular section: $(L \cdot W) - [(L - x) \cdot (W - y)]$.
- Method 2: Find the area of the tall rectangle on the left and add it to the area of the shorter rectangle on the right: $(W \cdot x) + [(y \cdot (L - x)]$.
- Method 3: Find the area of the long rectangle across the bottom and add it to the area of the shorter rectangle on the top: $(L \cdot y) + [x \cdot (W - y)]$.
- Method 4: Find the area of the tall rectangle on the left, add it to the area of the long rectangle across the bottom, and subtract the area of the rectangular overlap: $[(W \cdot x) + (L \cdot y)] - (x \cdot y)$.

Given $L = 15$, $W = 8$, $x = 6$, and $y = 5$, the area of the figure is shown to be 93 square units regardless of the chosen method.

► Key Terms

exponent, expression, formula, order of operations, parentheses, variable

B-E.1.1.5

► Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Apply and extend previous understandings of arithmetic to numerical and algebraic expressions.

Identify, write, and evaluate numerical and algebraic expressions.

► Eligible Content

B-E.1.1.5 Apply the properties of operations to generate equivalent expressions.

► Instructional Supports

- Ask students to use different representations of multiplication to write equivalent expressions. For example, give students the expression $21z$. Some possible responses include $21(z)$, $21 \bullet z$, $(3 \bullet 7)z$, and $3(7z)$.
- Ask students to use different representations of division to write equivalent expressions. For example, given the expression $w \div 12$, students write $\frac{w}{12}$, $w \bullet \frac{1}{12}$, and $\frac{1}{12}w$, among others.
- Ask students to use the distributive property to combine like terms in an expression or decompose an expression into like terms.
    - Give students the expression $12z$. The 12 can be decomposed into $10 + 2$, so $12z$ can be rewritten as $12z = (10 + 2)z = 10z + 2z$ using the distributive property.
    - Give students the expression $13q + 3q + 9q$. The distributive property can be used to rewrite it as $(13 + 3 + 9)q$ or $25q$.
- Ask students to factor expressions containing sums of terms, some of which have variables, by identifying and factoring out a common factor of the coefficients. For example, given the expression $8x + 40$, students decompose it as $2(4x + 20)$, $4(2x + 10)$, or $8(x + 5)$. Students should write two equivalent expressions for a given expression and then compare their expressions to expressions written by other students.

- Ask students to create expressions equivalent to given expressions when the given expressions include a variable with an unwritten coefficient of one. For example, give students the expression $6x + x$. Students rewrite it as the equivalent expression $6x + 1x$ and then add the coefficients of the two like terms to create the equivalent expression $7x$. Students should understand that rewriting x as $1x$ is not necessary but may clarify the process of combining like terms.
- Ask students to represent multiplication or division in different ways in order to facilitate the process of creating equivalent expressions.
    - Give students the expression $(z \div 8) + \left(\frac{5z}{8}\right)$. Students can rewrite it as $\frac{z}{8} + \frac{5z}{8}$ in order to facilitate the process of adding the fractions, which gives a sum of $\frac{6z}{8}$.
    - Give students the expression $\frac{5p}{9} + 10$. Students can rewrite it as $5 \bullet \frac{p}{9} + 10$ in order to facilitate the process of factoring out the common factor of 5 to create the equivalent expression $5\left(\frac{p}{9} + 2\right)$.
- Give students an expression and ask them to write multiple equivalent expressions. For example, give students the expression $8t + 4s$. Some equivalent expressions are shown.
    - $t + 7t + 4s$
    - $4(2t + s)$
    - $4t + 4(t + 1s)$
    - $3t + 3t + 2t + 3s + s$

► Identifying and Addressing Common Misconceptions

- Some students may reverse the numerator and denominator when rewriting division expressions. Ask students to rewrite $\frac{3w}{7}$ in a variety of ways. Some students may respond with expressions such as $7 \bullet \frac{w}{3}$. For expressions such as those, ask students how to write 7 as a fraction (i.e., $\frac{7}{1}$), and then multiply $\frac{7}{1} \bullet \frac{w}{3}$ to determine whether it is equivalent to the original expression.

- Some students may omit or forget about the unwritten coefficient of 1 when adding like terms. Ask students to solve $6x + x$. Some students may respond with an answer of $6x$. Ask students to rewrite the $6x$ of the original expression using only addition. Students should rewrite the entire first expression as $x + x + x + x + x + x + x$. Then, have students rewrite the addition expression as a single multiplication expression. Students should recognize that the addition expression can be rewritten as $7x$, and it is equivalent to $6x + x$.

- Some students may completely remove a term from an expression when factoring out a constant that is equal to the term or the coefficient of a variable. For example, ask students to factor (or apply the distributive property to) the expression $8 + 24x$. Some students may factor an 8 from both terms, think that factoring an 8 from 8 leaves 0 rather than 1, and respond with the expression $8(3x)$. Have students determine whether or not their factored expressions are equivalent to the original by applying the distributive property again. Students should recognize that $8(3x) = 24x$, not $8 + 24x$. Ask students how they determined that the coefficient of x should be 3 after factoring out the 8. Once students realize they used the fact that $24 \div 8 = 3$, have students use the same idea to factor out the 8 from the constant term 8. Students find the other term to be 1 because $8 \div 8 = 1$.

- Some students may try to add unlike terms. Ask students to write an expression that is equivalent to $3x + 5$. Some students may respond $8x$. Ask students to write both their expression and their answer as sums of 1's and x's. Help them recognize that the only way for the expressions to be equivalent is if the x's and the 1's are equivalent, which is generally not the case.

► Enrichment

- Ask students to determine missing values in equivalent expressions. For example, give students the two equivalent expressions $8(3 + \_\_\_)$ and $4(\_\_\_ + 2y)$ and ask them to determine the missing values.

- Ask students to determine how many different whole numbers (other than 1) can be factored out of a given expression based on the values of the constants and coefficients. For example, give students the expression $20 + 24x$. Students should identify that 20 and 24 have the common factors of 2 and 4 and the expression can be rewritten as $2(10 + 12x)$ or $4(5 + 6x)$. The number of possible ways that an expression can be factored is determined by the number of common factors the constant and coefficients have.

- Ask students to apply properties to rewrite known formulas. For example, given that a rectangular prism has sides with areas X, Y, and Z, where the values of all three are different, have students write two expressions to represent the surface area. In this case, two possible expressions are $2X + 2Y + 2Z$ and $2(X + Y + Z)$. Students can repeat this activity given actual dimensions (length, width, and height) of rectangular prisms.
- Ask students to use equivalent expressions to explain "number magic tricks." For example, give students the following instructions:
    - Think of a number.
    - Double the number.
    - Then add 10.
    - Next divide by 2.
    - Subtract your original number.

  The result is always 5. Help students use each step to create an expression to represent the given instructions.

    - x
    - $2x$
    - $2x + 10$
    - $\frac{1}{2}(2x + 10)$
    - $\frac{1}{2}(2x + 10) - x$

  Ask them to use equivalent expressions to show how the trick works.

► Key Terms

equivalent, expression, like terms, properties of operations, variable

B-E.2.1.1

▶ Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Interpret and solve one-variable equations and inequalities.

Create, solve, and interpret one-variable equations or inequalities in real-world and mathematical problems.

▶ Eligible Content

B-E.2.1.1 Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

▶ Instructional Supports

- Ask students to explain what it means to find a solution (or the solutions) to an equation or inequality. A value is a solution to an equation or inequality if that value can be substituted into the equation or inequality to create a true statement. Finding the solutions to an equation or inequality entails finding all such values.
- Give students an equation or inequality and ask them to determine whether a given value is a solution of the equation or inequality by substituting the value into the equation or inequality and determining whether the resulting mathematical statement is true. Some examples are shown.
    - Give students the equation $(2z - 3)(z + 1) = 25$ and the value $z = 4$. Students show that when 4 is substituted for z, the equation becomes $(2(4) - 3)(4 + 1) = 25$. Further simplification yields $25 = 25$, which is a true statement. Therefore, the value $z = 4$ is a solution to the equation.
    - Give students the equation $3(4m + 1) = 5m + 10$ and the value $m = 2$. Students show that when 2 is substituted for m, the equation becomes $3(4(2) + 1) = 5(2) + 10$. Further simplification yields $27 = 15$, which is not a true statement. Therefore, the value $m = 2$ is not a solution to the equation.

- Give students the inequality $w + 2 < 10$ and ask them to determine whether values fitting the inequality $w < 5$ are solutions to $w + 2 < 10$. Students can select some values that satisfy $w < 5$, such as $w = 0$ or $w = 4$, and show that they are all solutions to $w + 2 < 10$. Students should be aware that checking any finite number of values does not guarantee that all values satisfying $w < 5$ are solutions, and it does not guarantee that $w < 5$ encompasses all solutions.

- Ask students to determine the solution to an equation or the solution set to an inequality by reasoning about which values can be substituted to make the equation or inequality true. For example, give students the inequality $y - 7 > 2$. Students can reason that the value of y that makes the two sides of the inequality equal is 9 because $9 - 7 = 2$. Increasing the value of y increases the value $y - 7$, so any value of y that is greater than 9 results in $y - 7$ being greater than 2. Therefore, the solutions can be represented by the inequality $y > 9$.

► Identifying and Addressing Common Misconceptions

- Some students may think that inequalities can have only one solution. Ask students to find all solutions to $x + 3 > 8$. Some students may respond with a single value. Ask students to substitute other values that are also solutions to the inequality. Students should recognize that these values are also solutions to the inequality, and that there are many more. In this case, there are infinitely more.
- Some students may think that a value that makes both sides of an inequality equal is a solution. Ask students whether $z = 4$ is a solution to the inequality $3z > 12$. Some students may respond that it is because they are used to solving equations and forget about the inequality symbol or because the first thing they do when determining the solution set for an inequality is to set the two sides of the inequality equal to one another. Ask students to substitute the value $z = 4$ into the inequality and read the resulting statement aloud. Students should read the statement as “12 is greater than 12.” Ask students whether that statement is true. Students should recognize that, while 4 determines the boundary of the solution set, it is not actually part of the solution set.
- Some students may simply put digits together rather than multiplying when substituting a value into an equation or inequality. Ask students to determine whether $z = 2$ is a solution to the

equation $4z = 42$. Some students may respond that it is because $4z$ has a value of 42 when $z = 2$. Ask students to rewrite the equation by using an operator to represent multiplication, such as $4 \cdot z = 42$. Make sure students understand that $4z = 42$ and $4 \cdot z = 42$ are the same equation and ask students if $z = 2$ is a solution to $4 \cdot z = 42$.

► Enrichment

- Ask students to explore the idea of a system of equations or inequalities. For example, have students work in pairs and have each student write a simple, one-operation inequality using the same variable. For example, one pair might write $z + 4 > 8$ and $3z < 20$. Then, have students write some possible solutions to their own inequalities. The students then exchange their possible solutions and see if any of the solutions to one inequality are also solutions to the other inequality. Have students work together to generate a list of values that are solutions to both inequalities or explain why there cannot be any common solutions.
- Ask students to write equations or inequalities with a given solution.
  - Give students the solution set $z > 3$. Students could write $z + 6 > 9$, $5z > 15$, or more complicated inequalities like $2(z + 3) - z - 12 > -3$.
  - Give students the solution $a = 4$. Over time, students may discover that a fast way to create an equation is to first write an expression, then substitute a given value for the variable in the expression, and finally set that expression equal to the resulting value. For example, start with $a^2 - 4 + \frac{a+2}{3}$, then substitute $a = 4$ to get $4^2 - 4 + \frac{4+2}{3} = 14$. Therefore, students can create the equation $a^2 - 4 + \frac{a+2}{3} = 14$.
- Ask students to check solutions for equations and inequalities involving more than one variable. For example, ask students to determine whether $x = 5$ and $y = 2$ is a solution for the equation $2x + y = 10$.

► Key Terms

equation, inequality, solution, substitution

B-E.2.1.2

► Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Interpret and solve one-variable equations and inequalities.

Create, solve, and interpret one-variable equations or inequalities in real-world and mathematical problems.

► Eligible Content

B-E.2.1.2 Write algebraic expressions to represent real-world or mathematical problems.

► Instructional Supports

- Ask students to write an expression for a real-world problem using variables to represent unknown numbers. For example, give students this situation: "A tree is 7 feet tall when it is planted. Each year, the tree grows 2 feet." Students determine that the increase in the height of the tree can be represented as $2y$ where y is the number of years. The total height of the tree can then be found by adding the starting height and the increase each year, resulting in the expression $7 + 2y$.
- Ask students to write and evaluate an expression to solve a real-world problem. For example, give students this situation: "A person drove 184 miles before lunch and will continue to drive at 54 miles per hour after lunch. How far will the person have driven 4.5 hours after lunch?" The expression $184 + 54h$ can represent the total distance driven h hours after lunch. To determine the number of miles driven 4.5 hours after lunch, the h can be replaced with 4.5. The number of miles can be found by first multiplying 54 by 4.5 to find a product of 243. This can then be added to 184 to arrive at a final value of 427 miles.
- Ask students to describe the set of values for a variable that are meaningful in the context of the expression. Two examples are shown.
    - Give students this situation: "A student has read 77 pages of a book. The student continues to read at a rate of 1 page every 2 minutes." Give students the expression

$77 + 0.5m$ to represent this situation. Students determine that meaningful values for m, or the time spent reading, cannot be negative.

- Give students this context: "The cost for a ticket to a concert is \$47. The expression $47t$ can be used to represent the total cost of purchasing t tickets." Students can determine that only whole number values for t make sense in the context.

▶ Identifying and Addressing Common Misconceptions

- Some students may assign an incorrect meaning to a variable. Give students this context: "The expression $6x + 8$ is used to represent the number of birds that are counted at a birdfeeder based on the amount of time, in minutes, the birdfeeder is observed." Ask students what the meaning of the variable is in this context. Students who identify the variable as the number of birds seen at the birdfeeder may have this misconception. Ask students to identify the significance of the expression $6x + 8$. Help them recognize that the cue "represent the number of birds" refers to the expression as a whole. Then ask them to identify the quantities in the expression that determine the number of birds. The three quantities in the expression are 6, x, and 8. Help students interpret the phrase "based on the amount of time." Because the expression is based on 6, x, and 8, and because it is also based on the amount of time, the amount of time must be one of these three quantities. And because the amount of time is a variable, it is represented by x.
- Some students may not recognize a variable as having multiple possible values. Ask students to identify the number of possible values for x that give the expression $0.75x – 7.5$ a value of at least 0. Students who identify only one value of x may have this misconception. Ask students to evaluate the expression at several values of x that meet the stated criteria (e.g., 10 and 100).
- Some students may not recognize appropriate limitations on values for a variable. Give students this context: "The expression $4g$ can be used to determine the number of baseballs needed per game for a city baseball team." Ask students what values could be used for g in this situation. Students who respond with any positive number may have this misconception. Ask students what the meaning of $4g$ is in this context. Then ask them to evaluate the expression for $g = 3.2$. Help students understand that 12.8 does not make sense in this context as the team cannot use 12.8 baseballs during a game.

► Enrichment

- Ask students to write an expression in which each term has a different variable. For example, give students this context: "A student spends 4 hours reading articles at a constant rate for a research project. Another student spends 4.5 hours reading articles at a different constant rate for the research project." Ask students to write an expression representing the total number of articles the two students read. The expression $4x + 4.5y$ could be used to represent this situation.
- Ask students to add and subtract expressions involving variables. For example, ask students to add the expressions $–5x – 9$ and $4x + 4$. Students can start by writing $(–5x – 9) + (4x + 4)$. Students can then apply the commutative and associative properties to rewrite the expression as $(–5x + 4x) + (–9 + 4)$, which can be rewritten as $–x – 5$.
- Ask students to write an expression in which two variables are multiplied by each other. For example, ask students to write an expression representing this context: "A number of painters, x, are assigned to a project, and each painter can paint y square feet per minute." Students can write the expression xy to represent the total number of square feet that can be painted per minute.

► Key Terms

equation, expression, solution set, variable

B-E.2.1.3

► Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Interpret and solve one-variable equations and inequalities.

Create, solve, and interpret one-variable equations or inequalities in real-world and mathematical problems.

► Eligible Content

B-E.2.1.3 Solve real-world and mathematical problems by writing and solving equations of the form x + p = q and px = q for cases in which p, q, and x are all non-negative rational numbers.

► Instructional Supports

- Ask students to solve equations using inverse operations to isolate a variable. Some examples are shown.
    - Give students the equation $x + \frac{1}{2} = \frac{7}{8}$. The variable can be isolated by subtracting $\frac{1}{2}$ from each side of the equation. The subtraction on the right side of the equation can be performed by converting $\frac{1}{2}$ to the equivalent fraction $\frac{4}{8}$.

$$x + \frac{1}{2} = \frac{7}{8}$$

$$x + \frac{1}{2} - \frac{1}{2} = \frac{7}{8} - \frac{1}{2}$$

$$x + 0 = \frac{7}{8} - \frac{4}{8}$$

$$x = \frac{3}{8}$$

- Give students the equation $\frac{3}{7}x = \frac{6}{7}$. The variable can be isolated by dividing both sides by $\frac{3}{7}$, which is equivalent to multiplying both sides by $\frac{7}{3}$.

$$\frac{3}{7}x = \frac{6}{7}$$

$$\frac{7}{3} \bullet \frac{3}{7}x = \frac{6}{7} \bullet \frac{7}{3}$$

$$x = \frac{42}{21}$$

$$x = 2$$

- Ask students to identify and correct errors made in the process of solving an equation. For example, give students this scenario: "A student solves the equation $x + 2\frac{2}{3} = 7\frac{1}{3}$ by adding $2\frac{2}{3}$ to each side of the equation to arrive at the solution $x = 10$." Students identify that the error is that $2\frac{2}{3}$ should be subtracted from each side, not added, in order to isolate the variable. Students then determine that the correct solution is $x = 4\frac{2}{3}$.
- Ask students to write and solve an equation representing a given context. For example, give students this context: "A customer makes a purchase at a store that adds 5% sales tax to each purchase. The amount of tax added to the customer's purchase is \$1.95. What was the amount of the customer's purchase to which the sales tax was added?" The amount to which the tax was added can be represented as p, so the amount of sales tax can be represented by the expression $0.05p$. Therefore, the equation $0.05p = 1.95$ can be written to represent the situation. The equation can then be solved to find $p = \$39$.

$$0.05p = 1.95$$

$$0.05p \div 0.05 = 1.95 \div 0.05$$

$$p = 39$$

- Ask students to identify and correct errors made when writing an equation to represent a situation. For example, give students this situation: "A piece measuring $1\frac{3}{4}$ feet is cut from a board that is $11\frac{1}{2}$ feet long. A student determines the new length of the board (x) by solving the equation $x - 1\frac{3}{4} = 11\frac{1}{2}$ and concludes that the board is now $13\frac{1}{4}$ feet long." Students should determine that the given equation uses the incorrect operation. The shortened length of the board plus the length of the piece cut from the board is equal to the length of the board before the cut. The correct equation is thus $x + 1\frac{3}{4} = 11\frac{1}{2}$, and the correct solution is $x = 9\frac{3}{4}$ feet.

▶ Identifying and Addressing Common Misconceptions

- Some students may not use the inverse operations when isolating a variable. Ask students to solve the equation $\frac{2}{3}x = \frac{1}{15}$ for x. Students who find a solution of $x = \frac{2}{45}$ may have this misconception. Ask students to substitute the value found back into the given equation and determine whether the resulting equation is true. Then ask them to write the resulting equation after their initial operation of multiplying by $\frac{2}{3}$. The resulting equation is shown.

$$\frac{2}{3} \bullet \frac{2}{3}x = \frac{2}{3} \bullet \frac{1}{15}$$

$$\frac{4}{9}x = \frac{2}{45}$$

  Help students recognize that when the inverse operation is not used, the resulting equation does not isolate the variable as needed.

- Some students may use different operations on each side of an equation. Ask students to solve the equation $\frac{3}{4} = x - \frac{5}{8}$ for x. Students who add $\frac{5}{8}$ to one side and subtract $\frac{5}{8}$ from the other side to find a solution of $x = \frac{1}{8}$ may have this misconception. Remind students that the two sides of the original equation are equal. Then ask students whether applying different operations to each side of the equation maintains the equality. Help students recognize that while it is possible for different operations to maintain the equality, such as adding 0 to one side and subtracting 0 from the other side, it is only guaranteed to always be true if the same operation is applied on both sides of the equation.

► Enrichment

- Ask students to investigate the effect of multiplying an equation by a constant. For example, give students the equation $\frac{3}{8}x = \frac{3}{5}$. Ask students to solve for x. Then ask students to multiply both sides of the equation by 6 and solve the resulting equation for x. Students find that in both cases the solution is $x = \frac{8}{5}$.
- Ask students to interpret each value within an equation in the context of a real-world situation. For example, give students the equation $0.25x = 3$ and this situation: “A baker uses 0.25 cups of sugar in each batch of cookies. The baker uses a total of 3 cups of sugar to make cookies.” Students can interpret 0.25 as the rate at which sugar is used per batch, x as the number of batches of cookies the baker makes, and 3 as the total number of cups of sugar the baker uses to make cookies.
- Ask students to investigate cases in which multiple values can be combined to arrive at an equation of the form $x + p = q$ or $px = q$. For example, give students this situation: “An employee at a store receives a paycheck of $384.60. From this paycheck, the employee makes purchases for $62.14 and $11.07. The employee deposits the rest of the paycheck in a bank account.” Ask students to write an equation representing this situation. Students can write the equation $62.14 + 11.07 + x = 384.60$ or $73.21 + x = 384.60$.

► Key Terms

equation, inverse operation, isolate, unknown, variable

B-E.2.1.4

► Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Interpret and solve one-variable equations and inequalities.

Create, solve, and interpret one-variable equations or inequalities in real-world and mathematical problems.

► Eligible Content

B-E.2.1.4 Write an inequality of the form x > c or x < c to represent a constraint or condition in a real-world or mathematical problem and/or represent solutions of such inequalities on number lines.

► Instructional Supports

Inequalities in context

- Ask students to write an inequality to represent a given situation. For example, give students this situation: "The temperature for the next several days will be hotter than 90 degrees Fahrenheit." To represent the temperature, students use a variable, t. Since the temperature is going to be over 90 degrees, the inequality $t > 90$ can be used to describe the set of possible temperatures, in degrees Fahrenheit, for the next several days.

- Ask students to describe the solution set to a given inequality using words. For example, give students the inequality $c < 12$. Students determine that the solution set is the set of numbers (including integers, fractions, and decimals) that are less than 12 because the inequality states that c must be less than 12. Students should observe that this includes all negative numbers as well as the positive numbers that are less than 12.

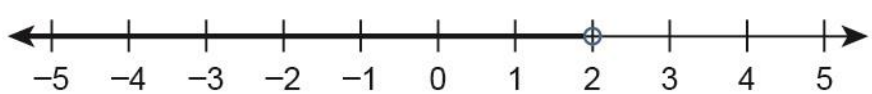
- Ask students to make a list of possible words or phrases that can help identify which inequality to use in different situations. For example, students may note that phrases like "taller than," "warmer than," and "heavier than" generally indicate the use of the greater than symbol, >. Similarly, students may note that phrases like "fewer than," "cheaper than," and "younger than" generally indicate the use of the less than symbol, <.

- Ask students to identify restrictions to the solution set of an inequality when interpreted within a given context. For example, give students this situation: "A sign in an elevator states that the capacity of the elevator is less than 13 people." The inequality $p < 13$, where p is the number of people in the elevator, can be used to represent this situation. Students explain that the solution set to the inequality is the infinite set of numbers, including negative numbers, that are less than 13. However, since the variable represents the number of people in the elevator, only the whole, non-negative values in the solution set of the inequality are reasonable in the context of the situation. Therefore, only the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 are solutions to the inequality in the context of this situation.

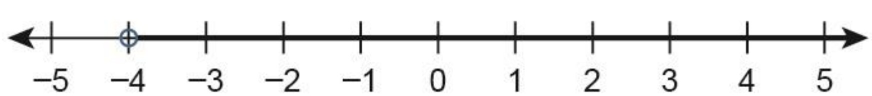
- Give students an inequality and ask them to write a real-world situation that can be represented by the given inequality. For example, give students the inequality $x > -5$. The solution set represents all the numbers that are greater than –5. Students can write this situation: "The low temperature for tomorrow will be warmer than –5°F."

Graphs of inequalities

- Ask students to graphically show the solution set to an inequality using a number line. For example, give students the inequality $x < 2$. Ask students to start by constructing a number line containing the point 2 along with some values greater than 2 and some values less than 2. Then ask students to plot a point at 2 using an open circle. Students should understand that the endpoint is an open circle because that value is not included in the solution set. Lastly, ask students to draw an arrow to the left of the endpoint, as the inequality symbol indicates that the solution set is all values less than 2 on a number line.



- Give students a graph on a number line and ask them to write an inequality that represents the given graph. For example, give students the number line shown.



Students determine that the inequality $m > -4$ represents the solution set shown on the number line. Any variable can be used to represent the set of numbers that are greater than −4.

▶ Identifying and Addressing Common Misconceptions

- Some students may confuse the less than (<) and greater than (>) symbols. Ask students to interpret the inequality $x > 8$. Some students may give an answer of "the value of x is less than 8." Help students recall the meaning of each symbol and develop mnemonics for remembering their meanings (e.g., the notion that the open end ("mouth") of the inequality symbol opens to the greater value or the pointed end ("point") points to the lesser value).
- Some students may incorrectly use integer thinking when interpreting the solution set of an inequality. Ask students to describe the solution set of the inequality $n > 5$. Some students may interpret the solution set as all whole numbers 6 and greater. Ask students whether 5.5 is a solution to the inequality. Help students understand and explain that while 5 is not a solution to the inequality, there are numbers between 5 and 6, all of which make the inequality true.
- Some students may not consider the context when interpreting solutions to inequalities. Give students this situation: "Brittany and Lori each won game tickets. Brittany won 28 game tickets, and Lori won fewer game tickets than Brittany. How many tickets could Lori have won?" Ask students to write an inequality and state possible solutions. Some students may state that possible solutions for the number of game tickets Lori could have won include any number less than 28, which includes values such as 12.25, 19, 6.5, and –7. Ask students if it makes sense to win 12.25 tickets. Help students understand that even though those values are true for the inequality, some of those values do not make sense in the context of the situation. The number 19 is part of the solution set and reasonable in the context of the situation, but the other values are not reasonable.

▶ Enrichment

- Ask students to interpret the solution set to a given inequality when the inequality symbol is changed but the relationship is maintained by rearranging the variables and constants. For example, give students the inequality $z > 8$. Students understand that the solution set is the set of numbers that are greater than 8. Then ask students to understand and explain why $8 < z$ represents the same solution set as $z > 8$.

- Ask students to solve one-step inequalities of the form of $ax > c$ or $ax < c$ where a is positive.
- Ask students to solve one-step inequalities of the form of $x + b > c$ or $x + b < c$ where b is less than c.
- Ask students to explain why an open circle is needed when graphing inequalities, even when that value is not part of the solution set. For example, ask students to explain why an open circle is needed to graph the solution set of $r < 7$ rather than just shading all values less than 7. Because decimal and fraction values get infinitely close to 7, the shading of the number line is so close to 7 that the values less than 7 are indistinguishable from the value exactly at 7. Therefore, some sort of symbol is needed to show that 7 is not included but everything leading up to 7 is included.

► Key Terms

equal, equation, feasible, greater than, inequality, infinitely many, less than, number line, open circle, solution set

B-E.3.1.1 & B-E.3.1.2

► Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Represent and analyze quantitative relationships between dependent and independent variables.

Use variables to represent two quantities in a real-world problem that change in relationship to one another.

► Eligible Content

B-E.3.1.1 Write an equation to express the relationship between the dependent and independent variables.

B-E.3.1.2 Analyze the relationship between the dependent and independent variables using graphs and tables, and/or relate these to an equation.

► Instructional Supports

- Ask students to identify the independent and dependent variables in a real-world scenario presented as a problem, graph, or table. For example, ask students to identify the independent and dependent variables in each of the scenarios shown.
    - Olivia does 24 math problems each night for homework. How many math problems (p) does Olivia do after n nights?

      The number of nights determines the total number of problems. Therefore, the independent variable is n, and the dependent variable is p.

- The table shown represents the total number of words (w) Paige can type in m minutes.

Paige's Typing

Time (minutes)	Words Typed
2	82
5	205
8	328
11	451
14	574

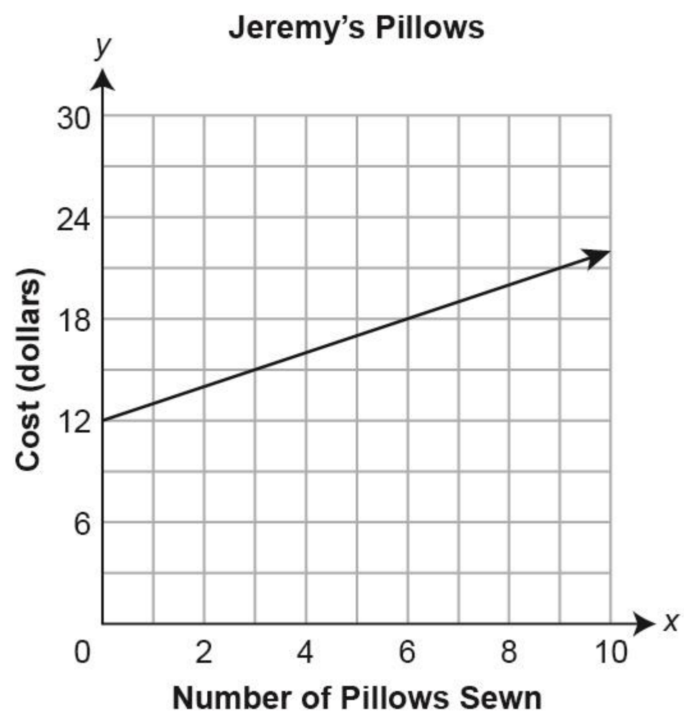
The number of minutes determines the total number of words typed.

Therefore, the independent variable is m, and the dependent variable is w.

- Ask students to determine how a dependent variable changes based on the independent variable and interpret values of the dependent variable when the independent variable is zero. For example, give students the following scenarios.
  - Uri uses the equation $c = \frac{2}{3}d$ to determine the number of people signed up for the chess club (c) when there are d people signed up for the drama club.

    The relationship between the two variables is $\frac{2}{3}$. This means every time the number of people signed up for the drama club increases by 3, the number of people signed up for the chess club increases by 2. When there are no people signed up for the drama club, there are no people signed up for the chess club.

- The following graph shows the amount of money (*d*), in dollars, it costs Jeremy to sew *p* pillows.



The slope of the line is 1, and when the number of pillows sewn is 0, the initial cost is $12. This means Jeremy has to spend $12 before he sews any pillows, and every additional pillow costs him $1 to sew.

- The following table represents the relationship between the amount of sugar (*s*), in grams, in different sized juice drinks (*j*), in ounces.

Juice Drinks

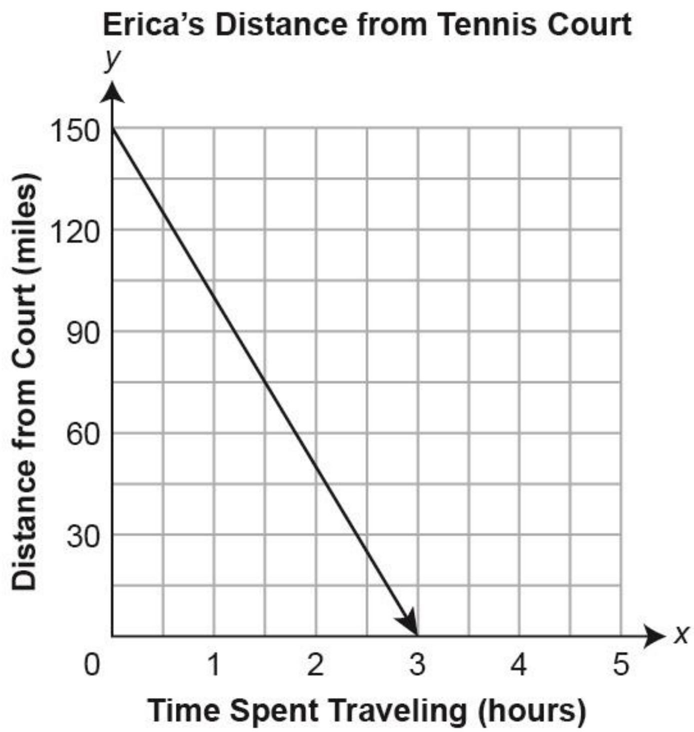
Amount of Juice (fluid ounces)	Sugar (grams)
8	24
16	48
24	72
32	96

The relationship between the variables is that the amount of sugar in grams is 3 times as great as the amount of juice in fluid ounces. This means that every ounce of juice drink has 3 grams of sugar, and if the value of the amount of juice is zero, the grams of sugar is also zero.

- Ask students to write an equation with two unknowns to represent the independent variable and the dependent variable in a real-world scenario presented as a problem, graph, or table. Some examples are shown.
    - Lonnie fills a swimming pool with water. The hose he uses pours 45 gallons of water every 6 minutes. Write an equation Lonnie could use to show how much water (w), in gallons, the hose has poured into the swimming pool after m minutes.

      The relationship between the variables is that the number of gallons increases by $\frac{45}{6}$ (which is equivalent to $\frac{15}{2}$) every minute, and the initial value is 0. Therefore, Lonnie could use the equation $w = \frac{15}{2}m$ to show how much water the hose has poured into the swimming pool.

    - Erica is traveling across the state to play in a tennis match. The graph shows Erica's distance from the court (d), in miles, after she has traveled for h hours. Write an equation Erica could use to show how far she is from the court.



The slope of the line is −50, and the initial distance is 150 miles. That means Erica is initially 150 miles from the court. It also means that distance is decreasing by 50 miles every hour. Therefore, Erica can use the equation $d = 150 - 50h$ to show how far she is from the court.

- Keegan is raking leaves in his neighborhood. The table represents the amount of money (m), in dollars, Keegan earns for raking r yards in his neighborhood. Write an equation Keegan could use to show how much money he earns raking yards.

Keegan's Yard Raking

Yards Raked	Earnings ($)
3	45
5	75
8	120

Keegan earns \$15 for each yard he rakes. Therefore, Keegan can use the equation $m = 15r$ to show how much money he earns raking yards.

► Identifying and Addressing Common Misconceptions

- Some students may reverse the independent variable and the dependent variable. Ask students to identify the independent and dependent variables in this scenario: “Erik wants to bring his family of 5 people to an amusement park that charges \$7.25 per person. How much money does he need?” Some students may identify the amount of money as the independent variable and the number of people as the dependent variable. Ask students to identify whether the amount of money is known and used to determine the number of people who can go to the park, or whether the number of people is known and used to determine the amount of money needed.

► Enrichment

- Ask students to work with scenarios in which the independent and/or dependent variables can be negative values.
- Ask students to discuss scenarios where either variable could be independent or dependent. For example, consider the relationship between the total number of students (x) in a school and the number of classrooms (y) in a school. If there is a fixed number of students that can be in a classroom, then the total number of students that can be in a school will depend on the number of classrooms available. However, if a new school is being built, then the number of classrooms needed will depend on the number of students expected to be enrolled in the school.
- Ask students to work with scenarios that are not linear.
- Ask students to investigate how a scenario changes when the independent and dependent variables are reversed.

► Key Terms

dependent variable, equation, graph, independent variable, table, variable

B-E.3.1.1 & B-E.3.1.2

► Reporting Category, Assessment Anchor, Anchor Descriptor

Expressions and Equations

Represent and analyze quantitative relationships between dependent and independent variables.

Use variables to represent two quantities in a real-world problem that change in relationship to one another.

► Eligible Content

B-E.3.1.1 Write an equation to express the relationship between the dependent and independent variables.

B-E.3.1.2 Analyze the relationship between the dependent and independent variables using graphs and tables, and/or relate these to an equation.

► Instructional Supports

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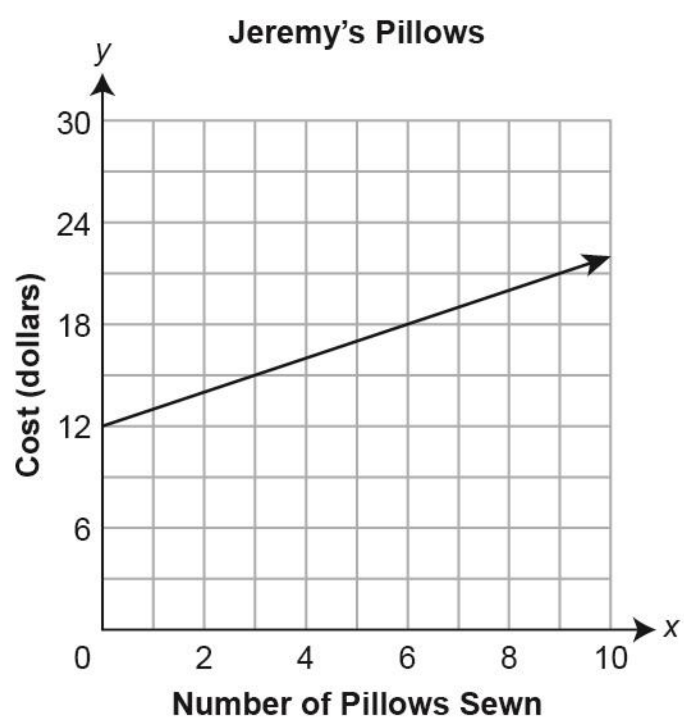

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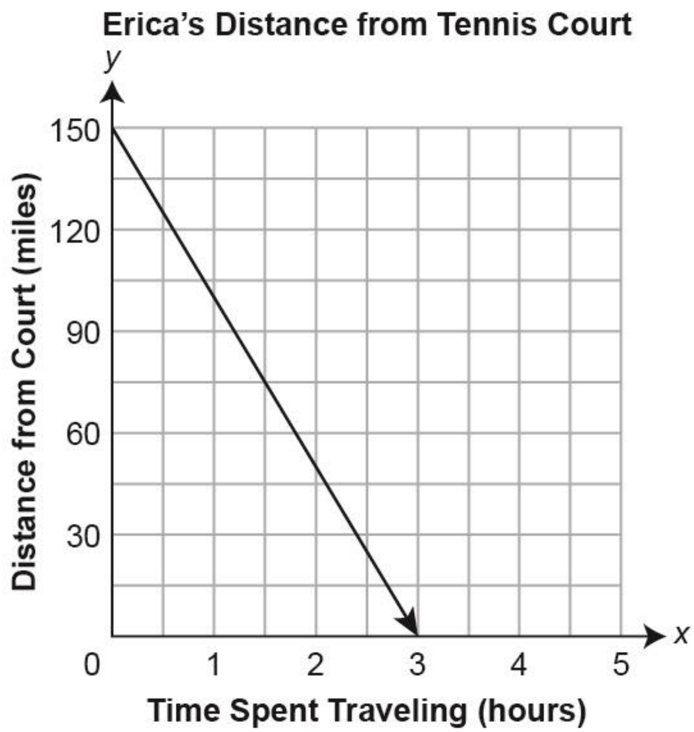

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- Ask students to work with scenarios that are not linear.
- Ask students to investigate how a scenario changes when the independent and dependent variables are reversed.

► Key Terms

dependent variable, equation, graph, independent variable, table, variable

C-G.1.1.1 & C-G.1.1.2

► Reporting Category, Assessment Anchor, Anchor Descriptor

Geometry

Solve real-world and mathematical problems involving area, surface area, and volume.

Find area, surface area, and volume by applying formulas and using various strategies.

► Eligible Content

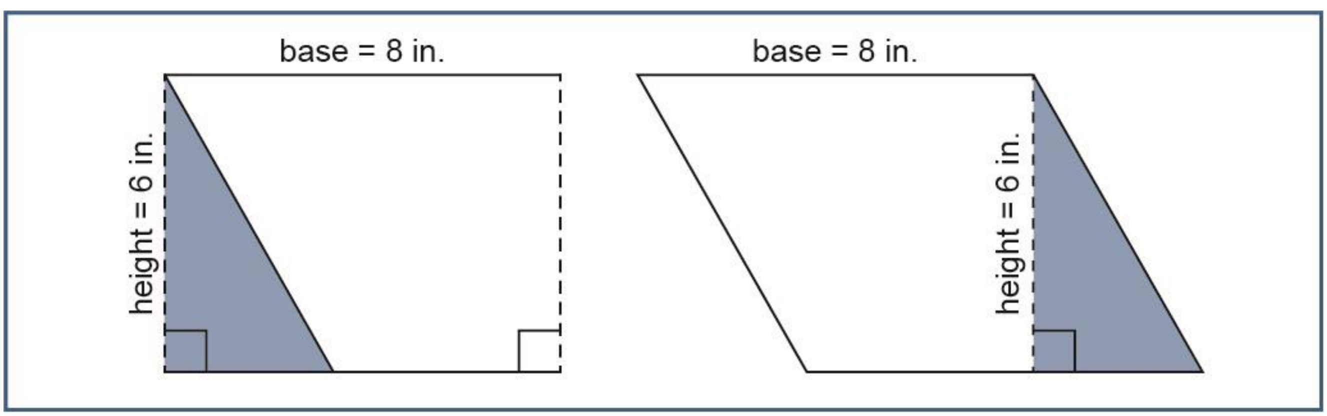
C-G.1.1.1 Determine the area of triangles and special quadrilaterals (i.e., square, rectangle, parallelogram, rhombus, and trapezoid). Formulas will be provided.

C-G.1.1.2 Determine the area of irregular or compound polygons.

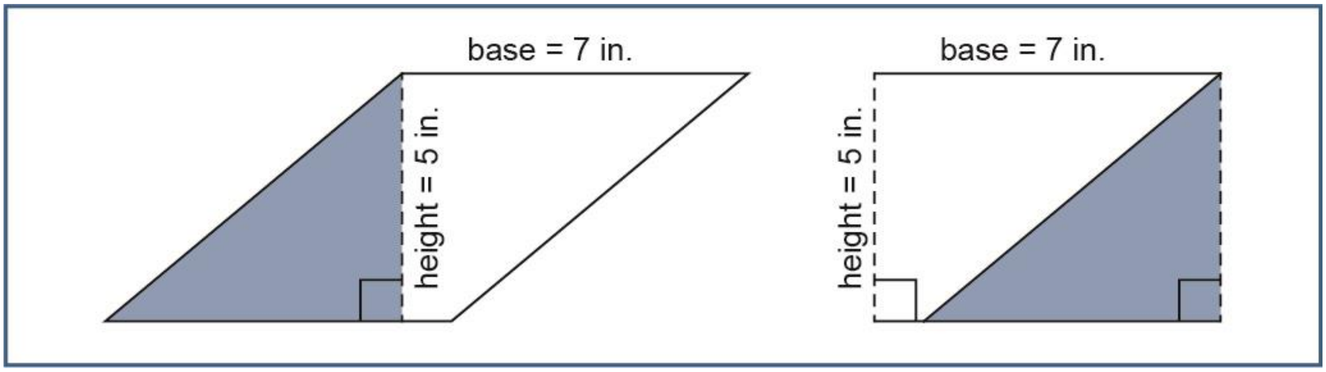
► Instructional Supports

Parallelograms

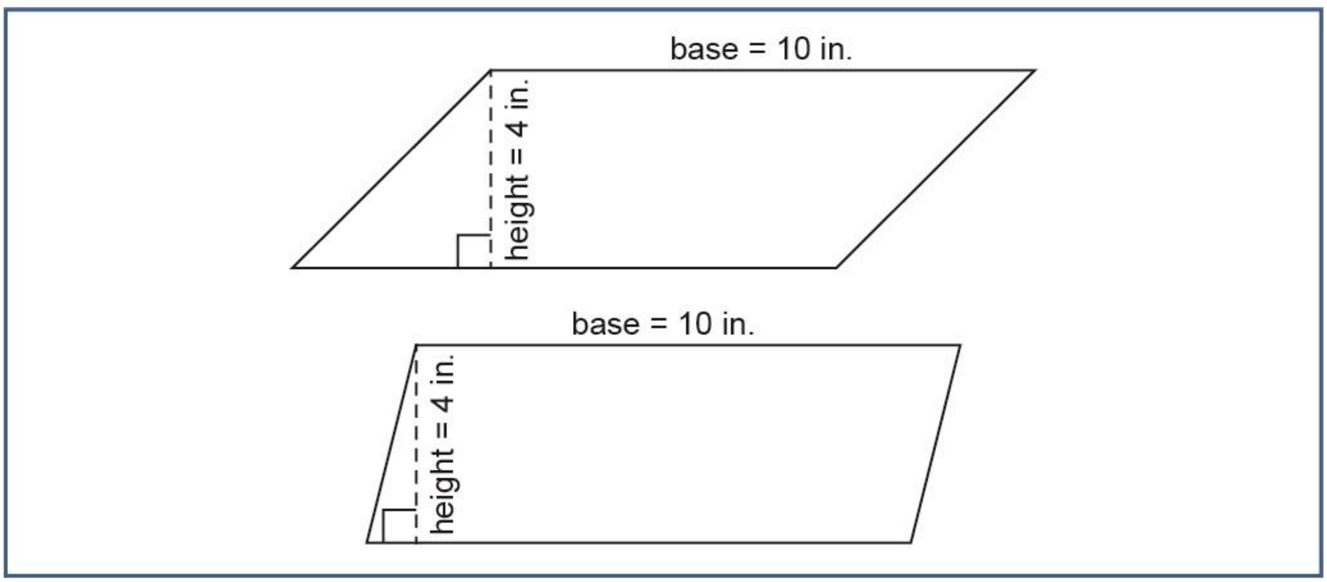
- Ask students to manipulate a model of a rectangle into a parallelogram with the same area by removing a triangular section from one end and placing it at the other end. For example, provide a rectangle with a base of 8 inches and a height of 6 inches. Ask students to draw a line to create a right triangle and move the shaded triangle to place it at the other end, creating a parallelogram with the same area of 48 square inches.



- Ask students to manipulate a model of a parallelogram into a rectangle in order to find the area of the parallelogram. For example, provide a parallelogram with a base of 7 inches and a height of 5 inches. Allow students to manipulate the triangular section of the figure formed by the dashed line to reposition the triangular piece to form a rectangle. The area of the rectangle and, consequently, the area of the parallelogram can be calculated as 35 square inches.

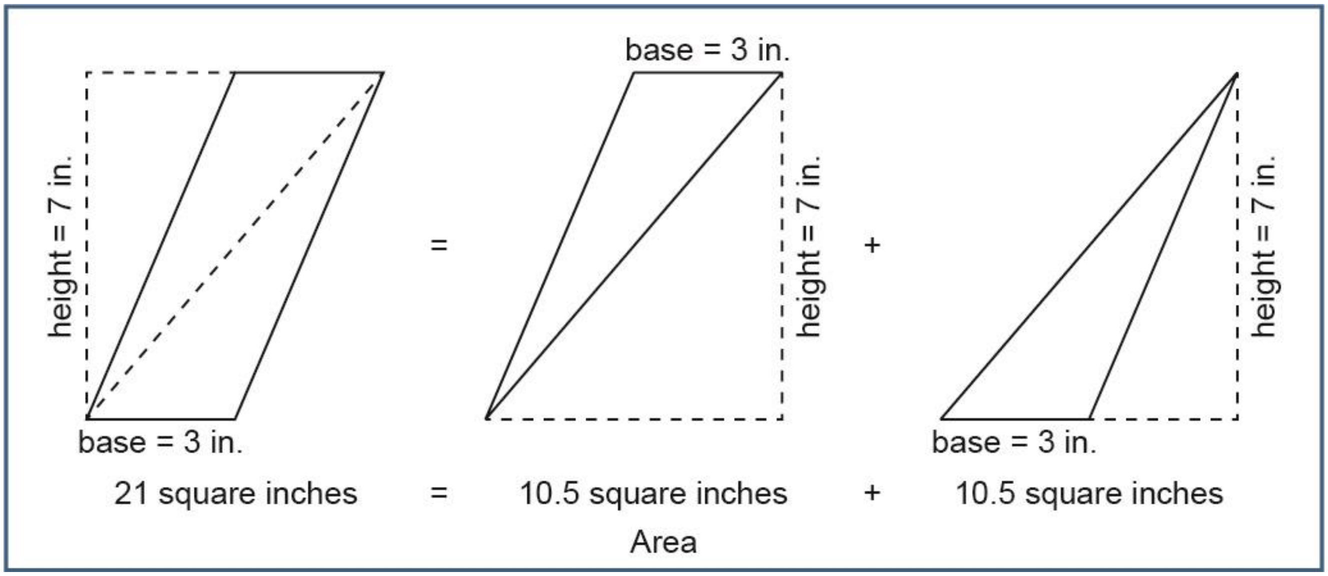


- Ask students to compare parallelograms and observe that even when parallelograms have sides with different lengths, if the bases and heights are equal, the areas are also equal. For example, provide two parallelograms with heights of 4 inches and bases of 10 inches. Discuss with students that, despite differing slant heights, the parallelograms have the same area of 40 square inches.

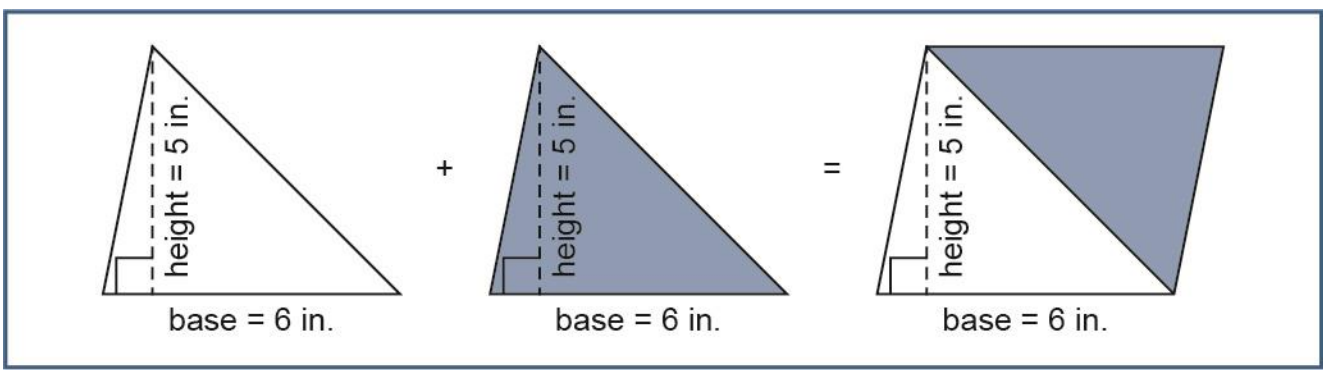


Triangles

- Ask students to decompose a model of a parallelogram into two triangles of equal size so that the area of each triangle measures half the area of the parallelogram. For example, provide a parallelogram with a base of 3 inches and a height of 7 inches that may be bisected to create two triangles, each with an area of 10.5 square inches.

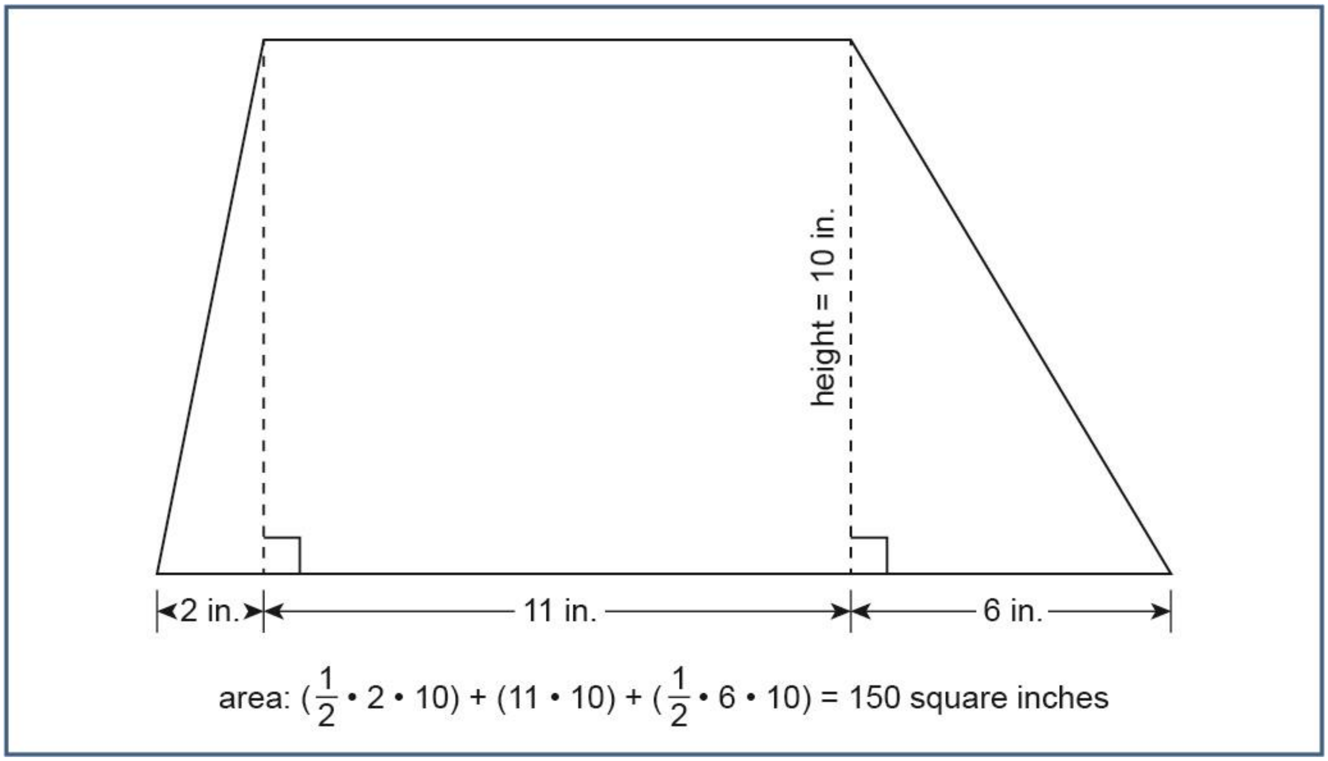


- Give students a triangle and ask them to join copies of the triangle together to form a parallelogram that has twice the area of each individual triangle. For example, provide two triangles, each with a base of 6 inches and a height of 5 inches. Allow students to align the triangles along corresponding sides in order to determine that since the constructed parallelogram has an area of 60 square inches, each triangle must have an area of 30 square inches.

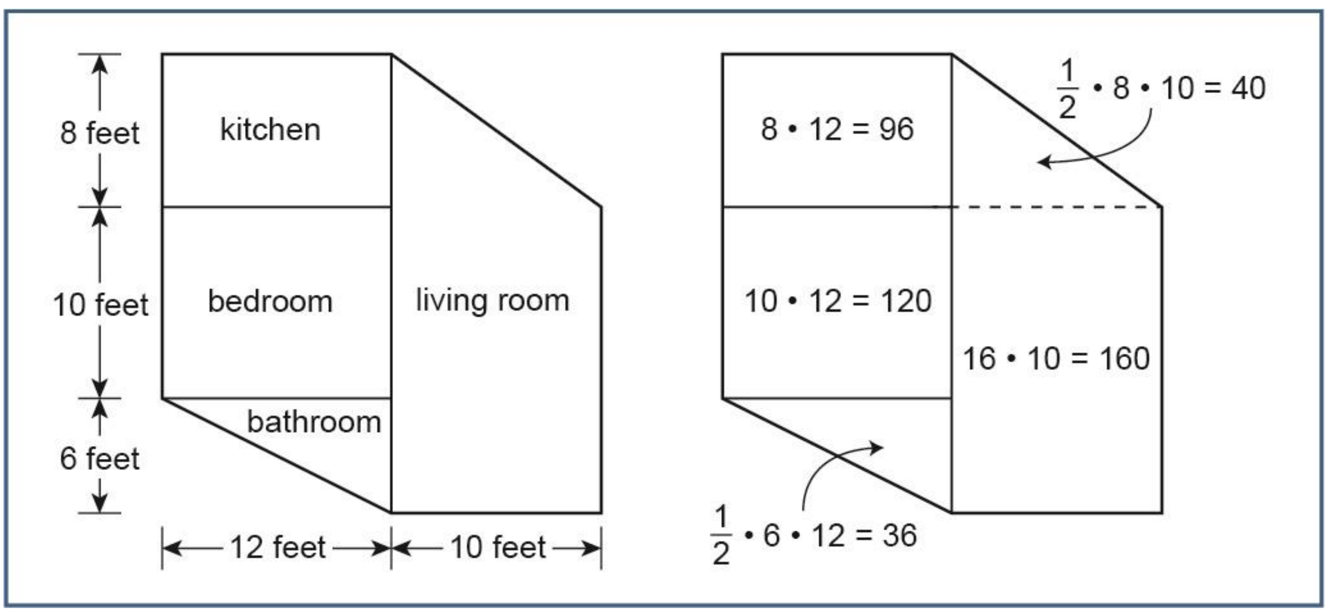


Composite Figures

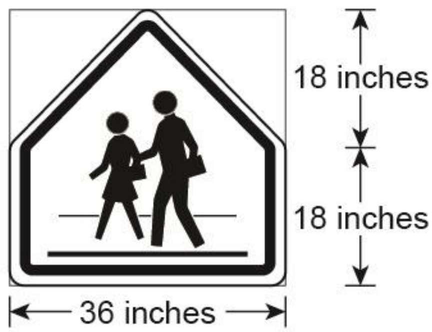
- Ask students to find the area of a trapezoid or other special quadrilateral by decomposing the shape into triangles and/or rectangles. For example, give students the trapezoid shown and ask them to decompose the shape and use either known formulas or other strategies to find the area.



- Ask students to determine the area of a composite figure by decomposing it into smaller figures and calculating the sum of the areas of the smaller figures. For example, give students the following apartment floor plan and ask them to determine the square footage by decomposing it into rectangles and triangles. The total area is equal to the sum of the three rectangles and two triangles: 96 + 120 + 36 + 40 + 160 = 452 square feet.

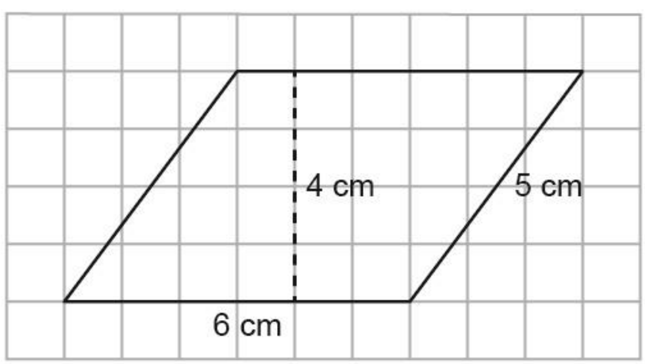


- Ask students to calculate the area of a shape that is "missing" pieces from the whole. For example, provide a diagram of a school crosswalk sign. The area can be calculated by subtracting the areas of two right triangles on the corners from the encompassing square. The image shows a square of 36 inches • 36 inches = 1,296 square inches, and each corner triangle is $\frac{1}{2} \cdot 18 \cdot 18 = 162$ square inches. Therefore, the area of the school crosswalk sign is 1,296 – 162 • 2 = 972 square inches.



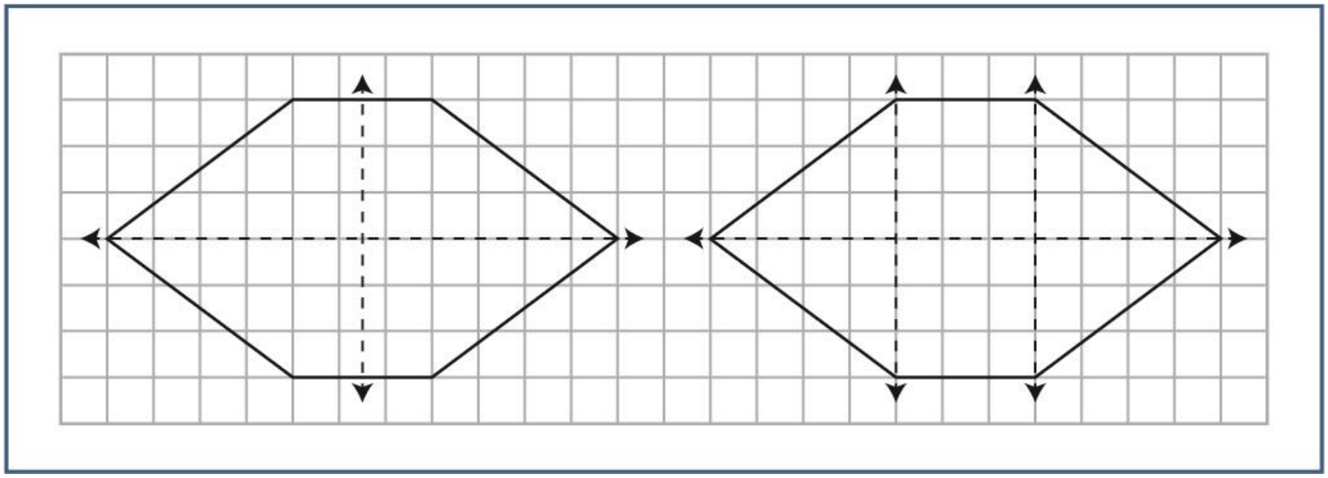
► Identifying and Addressing Common Misconceptions

- Students may mistake height for side length when finding the areas of triangles or parallelograms. Ask students to explain how to find the area of the following parallelogram.



Some students may give an answer of 30 square centimeters. Help students to transform the parallelogram into a rectangle. Calculate the area, both by counting the squares and by using the base and height of the rectangle, and use this to show that using the height of 4 rather than the side length of 5 gives the correct area. Give additional models to allow students to explore the idea that base and height must be perpendicular to one another.

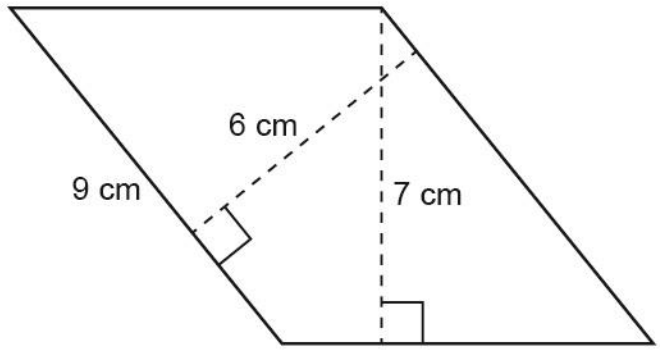
- Students may think that any decomposition of a shape can be used to find the area. Ask students to decompose the following hexagon in order to find its area. Two possible decompositions are shown.



Help students to observe that while both decompositions can be utilized to find the area of the hexagon, only the second breaks the shape into easily measurable parts.

- Students may confuse the concepts of perimeter and area. Ask students to explain the difference between perimeter and area. In addition to reminding students of mathematical definitions, relate the terms to everyday language to reinforce the meanings (e.g., a fence around the perimeter of a building or a play area).

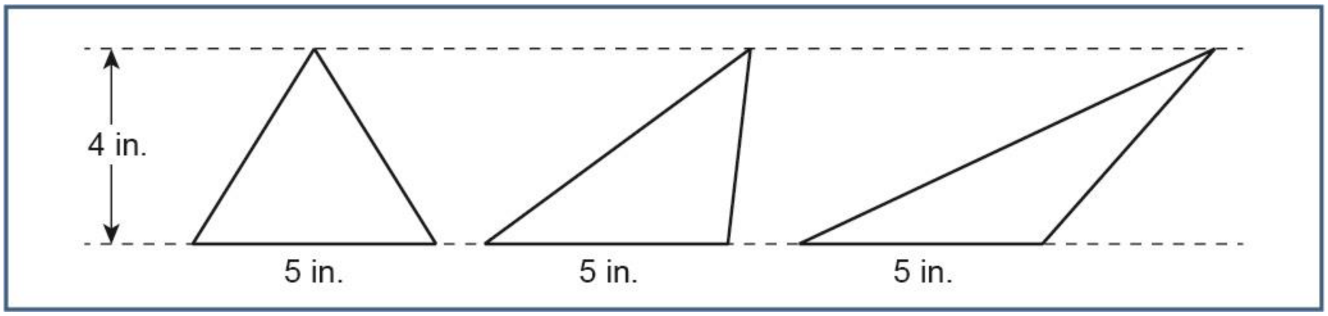
- Students may think that the height of a triangle or parallelogram is always measured vertically. Ask students to find the area of the following parallelogram.



Some students may give the area as 63 square centimeters. Ask students to explain how they determined the base and height of the figure. Then ask them to rotate the figure until the previously slanted sides are horizontal. Help students to understand that the base and the height must be determined as a pair and that, while the height can always be made to be vertical, it may not be oriented that way when presented.

► Enrichment

- Ask students to use peg boards and rubber bands to create polygons that fit specific characteristics. For example, ask students to create a polygon with exactly 6 sides and an area of 32 square units.
- Ask students to design multiple triangles or other polygons that have differing appearances but identical areas. For example, challenge students to create a triangle and a rectangle, both with areas of 10 square units.
- Ask students to explain why there is no triangle with a given base and a given area that has the largest perimeter. For example, ask students to find the triangle with a base of 5 inches and an area of 10 square inches that has the largest perimeter. Some possible triangles are shown.



Since any triangle with a base of 5 inches and a height of 4 inches will have an area of 10 square inches, the top two sides of the triangle can always be made longer by moving the top vertex of the triangle farther from the base.

► Key Terms

area, triangles, quadrilaterals, polygons, decomposition, composite, base, height

C-G.1.1.1 & C-G.1.1.2

► Reporting Category, Assessment Anchor, Anchor Descriptor

Geometry

Solve real-world and mathematical problems involving area, surface area, and volume.

Find area, surface area, and volume by applying formulas and using various strategies.

► Eligible Content

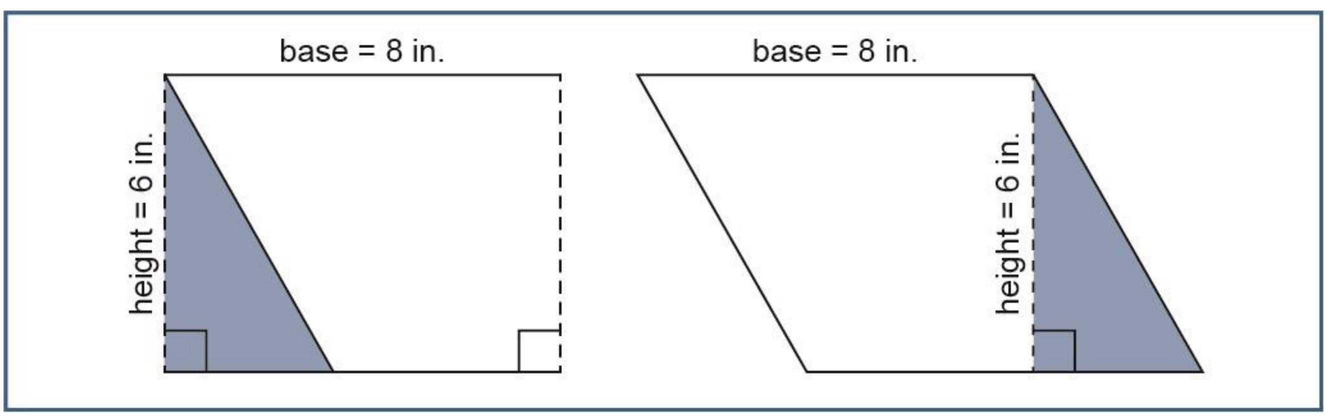
C-G.1.1.1 Determine the area of triangles and special quadrilaterals (i.e., square, rectangle, parallelogram, rhombus, and trapezoid). Formulas will be provided.

C-G.1.1.2 Determine the area of irregular or compound polygons.

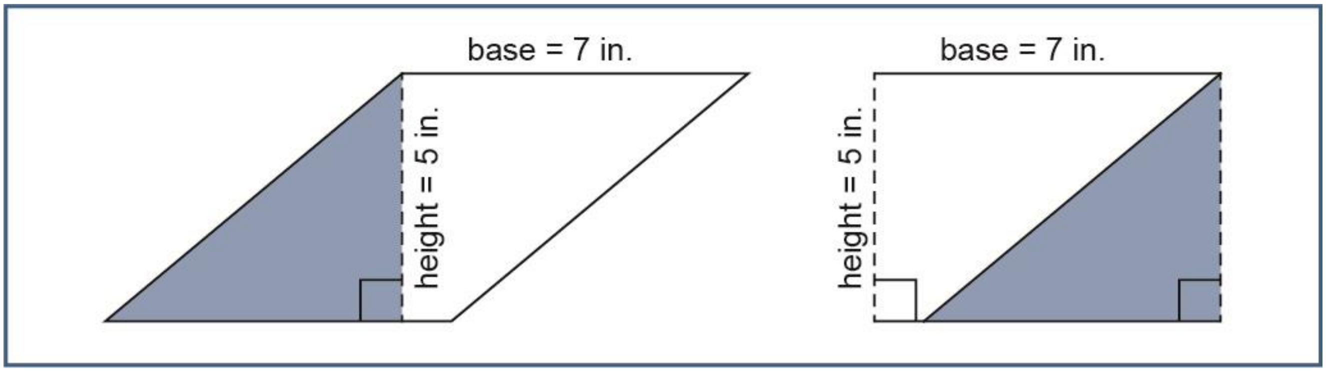
► Instructional Supports

Parallelograms

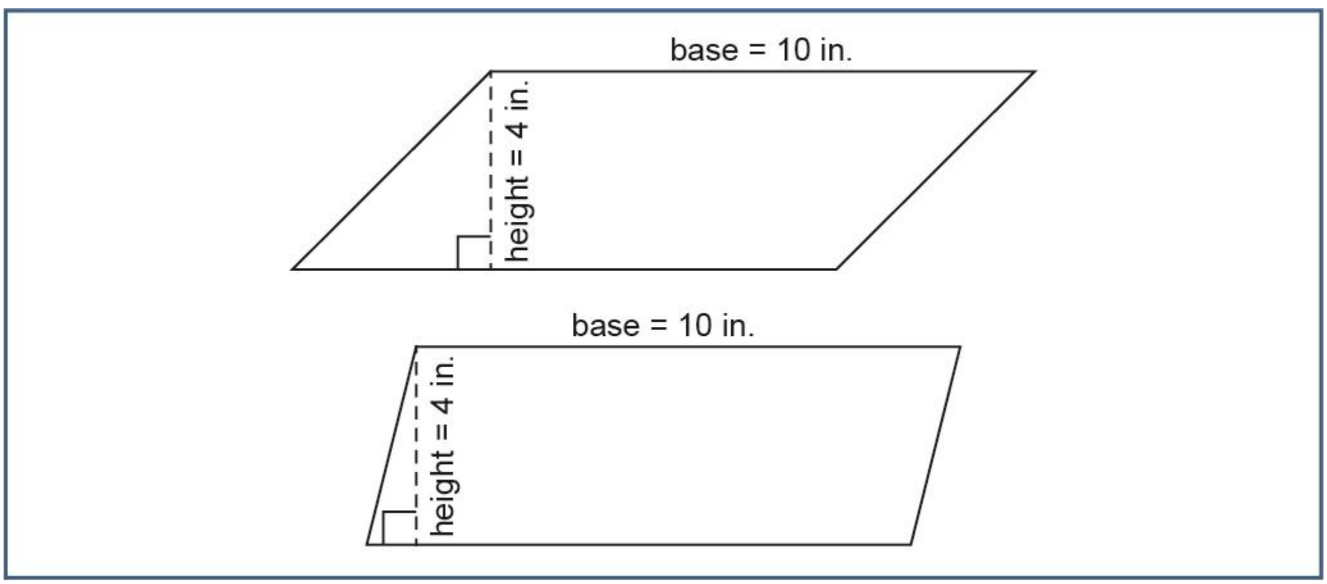
- Ask students to manipulate a model of a rectangle into a parallelogram with the same area by removing a triangular section from one end and placing it at the other end. For example, provide a rectangle with a base of 8 inches and a height of 6 inches. Ask students to draw a line to create a right triangle and move the shaded triangle to place it at the other end, creating a parallelogram with the same area of 48 square inches.



- Ask students to manipulate a model of a parallelogram into a rectangle in order to find the area of the parallelogram. For example, provide a parallelogram with a base of 7 inches and a height of 5 inches. Allow students to manipulate the triangular section of the figure formed by the dashed line to reposition the triangular piece to form a rectangle. The area of the rectangle and, consequently, the area of the parallelogram can be calculated as 35 square inches.

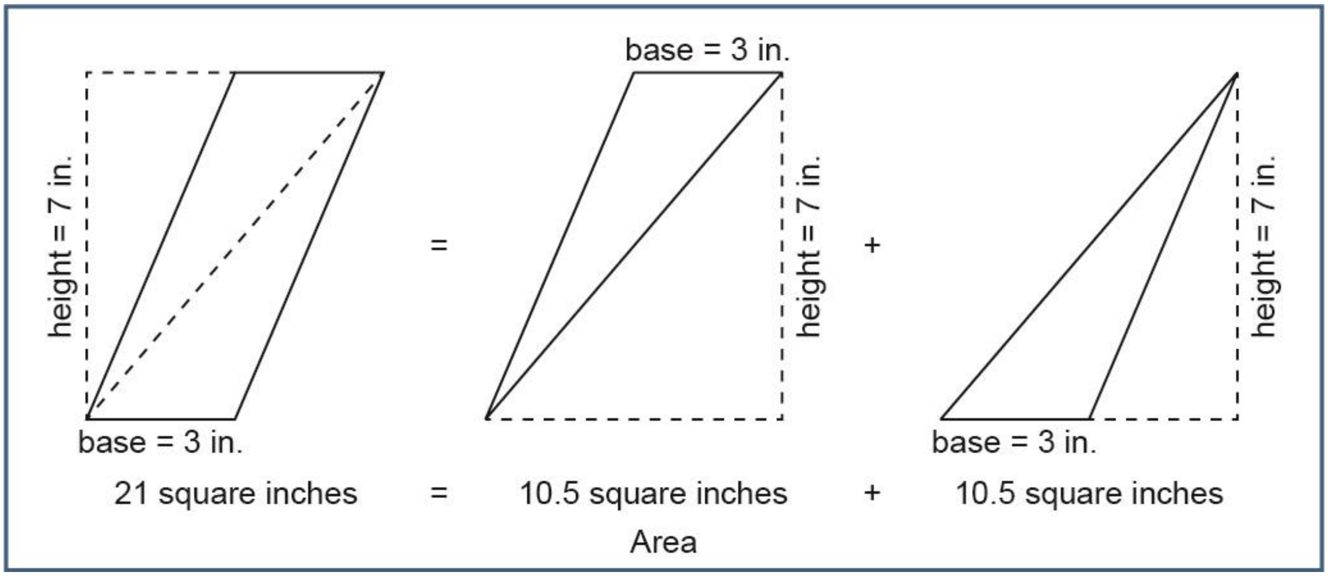


- Ask students to compare parallelograms and observe that even when parallelograms have sides with different lengths, if the bases and heights are equal, the areas are also equal. For example, provide two parallelograms with heights of 4 inches and bases of 10 inches. Discuss with students that, despite differing slant heights, the parallelograms have the same area of 40 square inches.

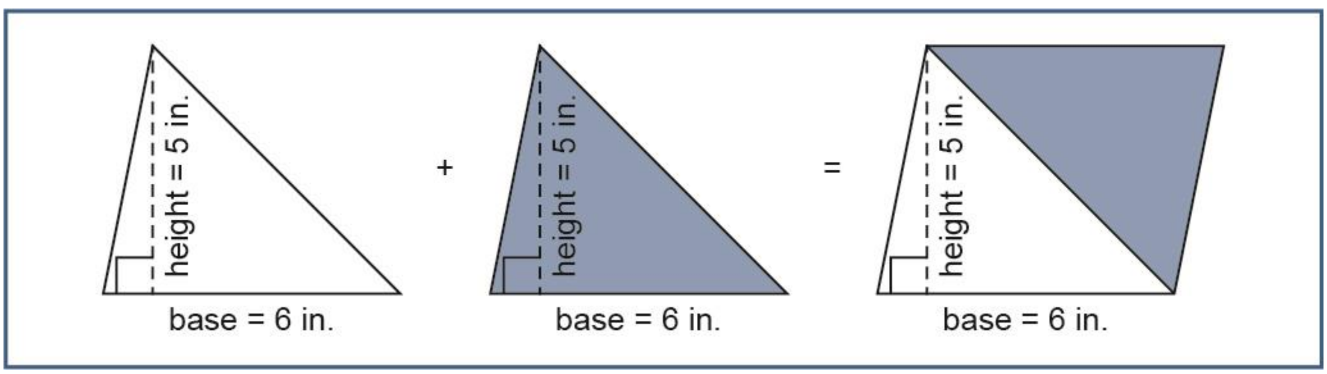


Triangles

- Ask students to decompose a model of a parallelogram into two triangles of equal size so that the area of each triangle measures half the area of the parallelogram. For example, provide a parallelogram with a base of 3 inches and a height of 7 inches that may be bisected to create two triangles, each with an area of 10.5 square inches.



- Give students a triangle and ask them to join copies of the triangle together to form a parallelogram that has twice the area of each individual triangle. For example, provide two triangles, each with a base of 6 inches and a height of 5 inches. Allow students to align the triangles along corresponding sides in order to determine that since the constructed parallelogram has an area of 60 square inches, each triangle must have an area of 30 square inches.



Composite Figures

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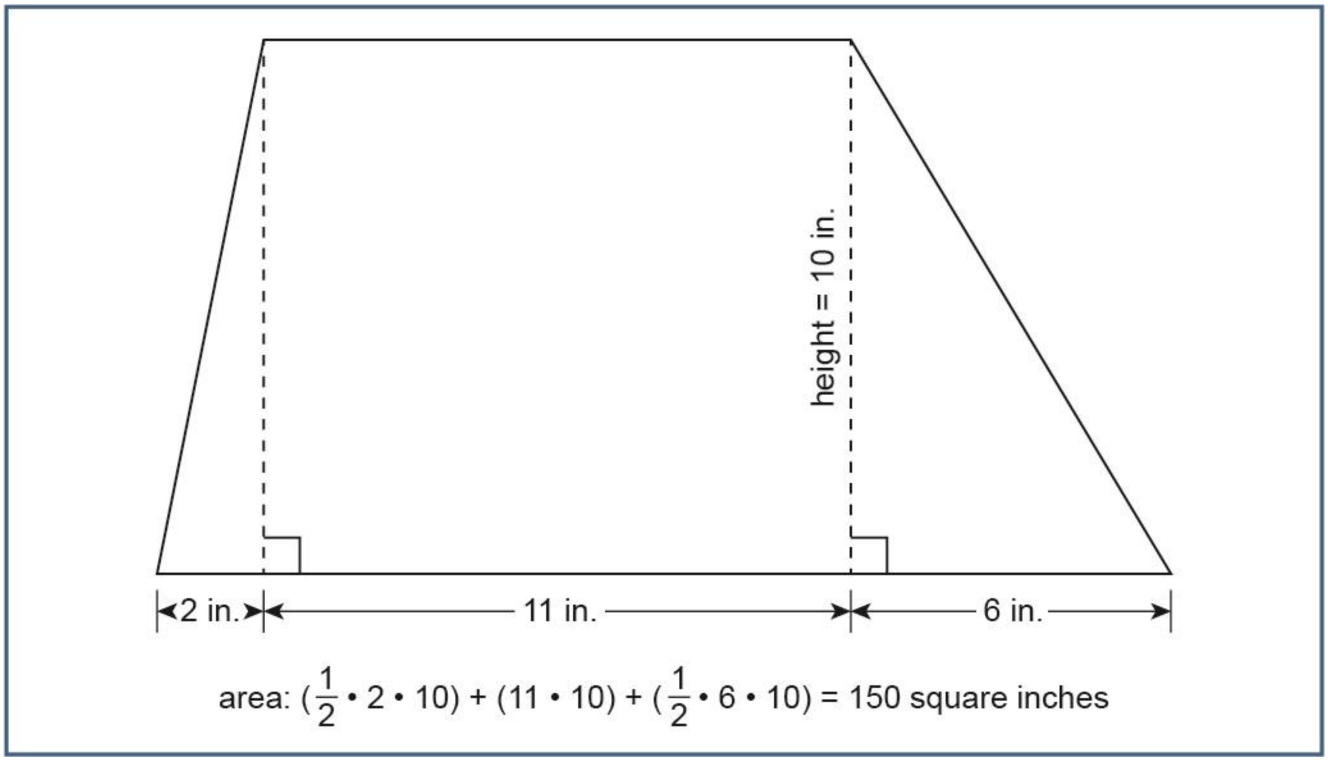

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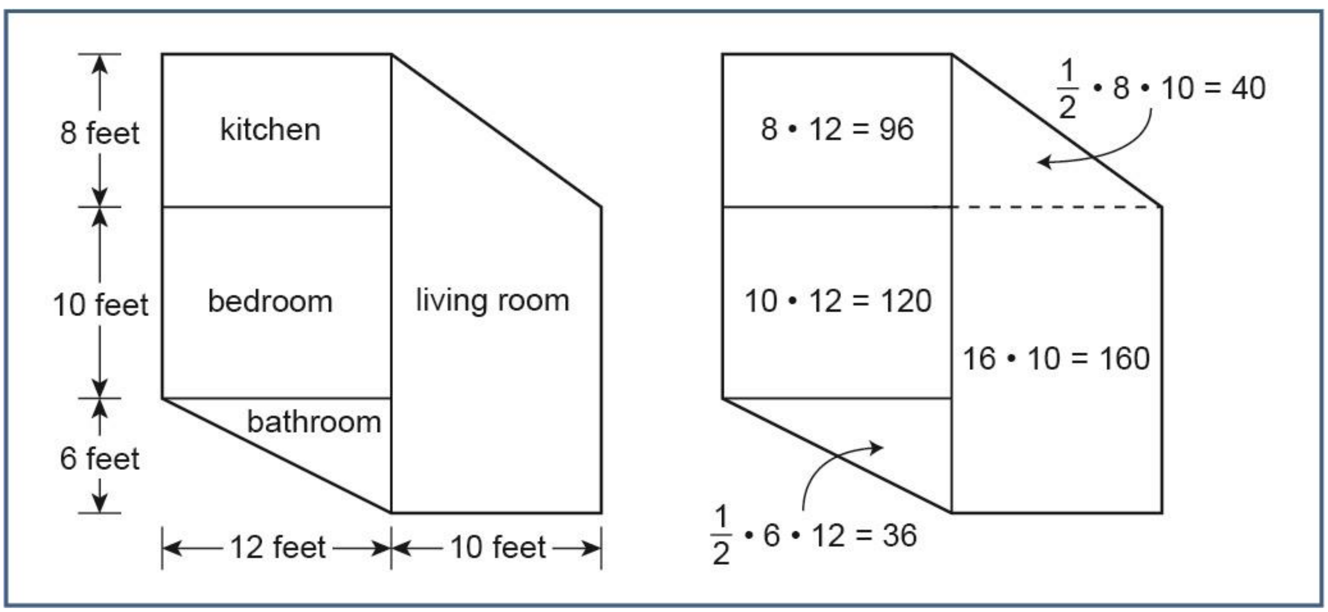

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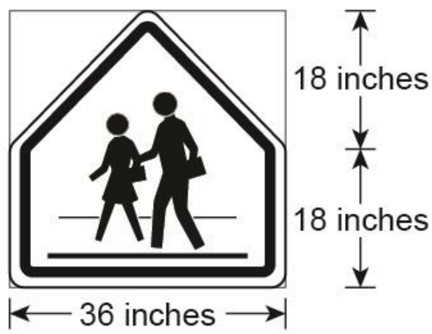

► Identifying and Addressing Common Misconceptions

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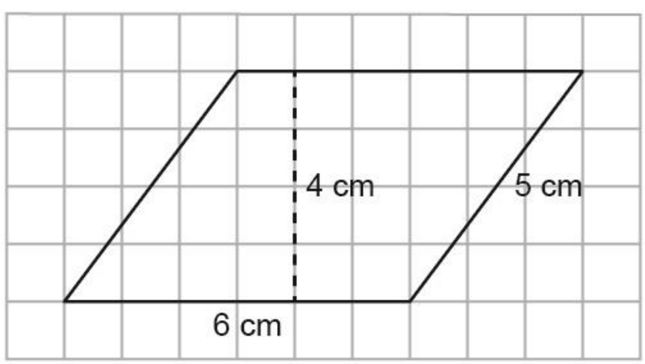

Some students may give an answer of 30 square centimeters. Help students to transform the parallelogram into a rectangle. Calculate the area, both by counting the squares and by using the base and height of the rectangle, and use this to show that using the height of 4 rather than the side length of 5 gives the correct area. Give additional models to allow students to explore the idea that base and height must be perpendicular to one another.

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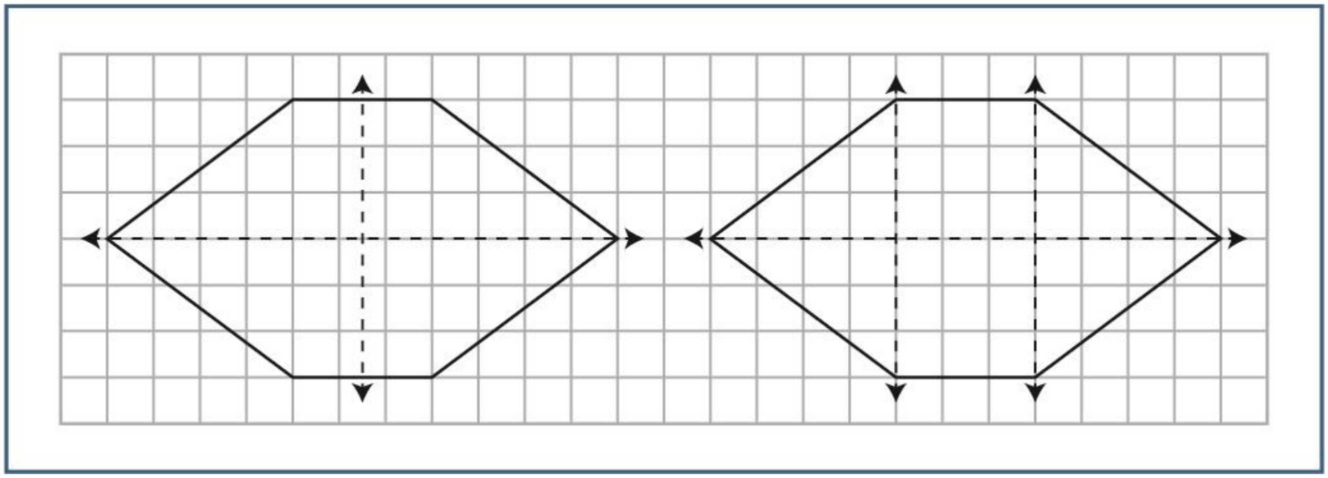

Help students to observe that while both decompositions can be utilized to find the area of the hexagon, only the second breaks the shape into easily measurable parts.

- Students may confuse the concepts of perimeter and area. Ask students to explain the difference between perimeter and area. In addition to reminding students of mathematical definitions, relate the terms to everyday language to reinforce the meanings (e.g., a fence around the perimeter of a building or a play area).

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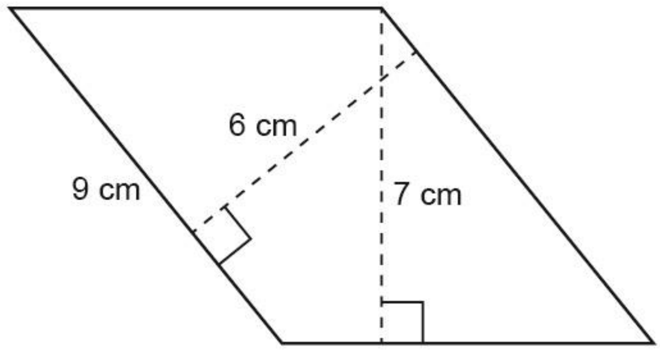

Some students may give the area as 63 square centimeters. Ask students to explain how they determined the base and height of the figure. Then ask them to rotate the figure until the previously slanted sides are horizontal. Help students to understand that the base and the height must be determined as a pair and that, while the height can always be made to be vertical, it may not be oriented that way when presented.

► Enrichment

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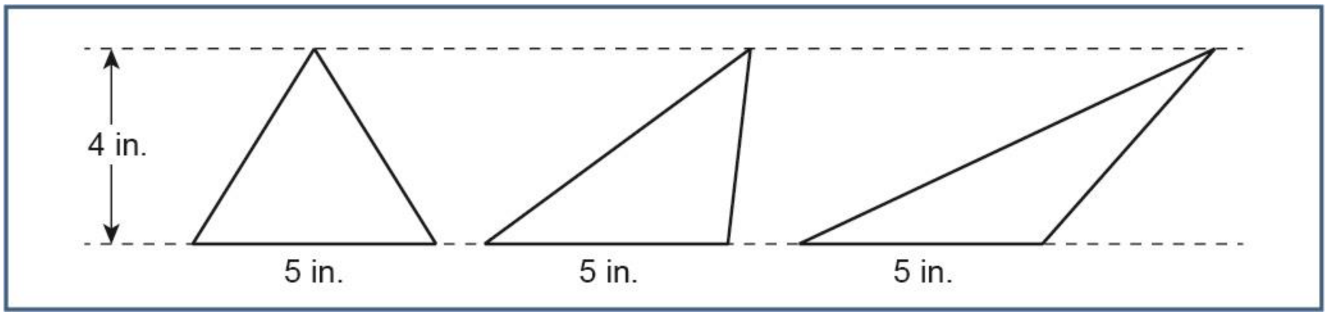
- Ask students to explain why there is no triangle with a given base and a given area that has the largest perimeter. For example, ask students to find the triangle with a base of 5 inches and an area of 10 square inches that has the largest perimeter. Some possible triangles are shown.



Since any triangle with a base of 5 inches and a height of 4 inches will have an area of 10 square inches, the top two sides of the triangle can always be made longer by moving the top vertex of the triangle farther from the base.

► Key Terms

area, triangles, quadrilaterals, polygons, decomposition, composite, base, height

C-G.1.1.3

► Reporting Category, Assessment Anchor, Anchor Descriptor

Geometry

Solve real-world and mathematical problems involving area, surface area, and volume.

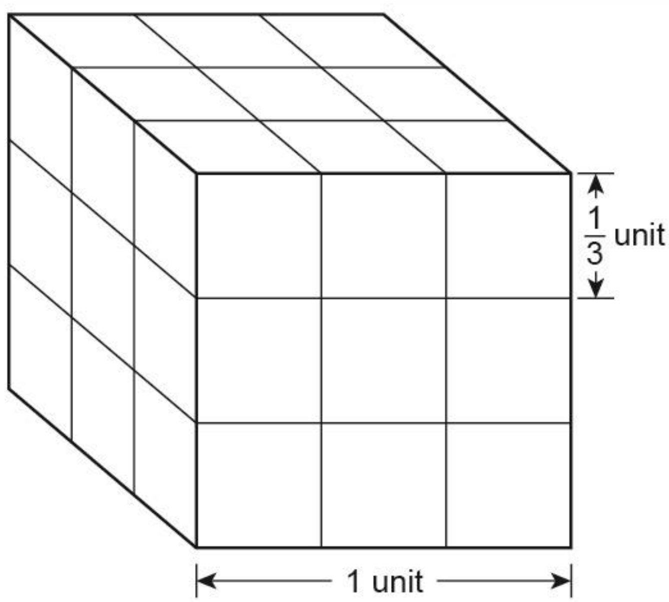
Find area, surface area, and volume by applying formulas and using various strategies.

► Eligible Content

C-G.1.1.3 Determine the volume of right rectangular prisms with fractional edge lengths. Formulas will be provided.

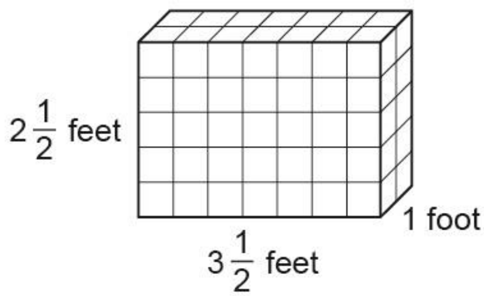
► Instructional Supports

- Ask students to find the volume of a cube with edge lengths that are unit fractions by filling a unit cube with the fractional cubes. For example, ask students to find the volume of a cube with edge lengths of $\frac{1}{3}$ unit by packing multiple copies into a unit cube.



The unit cube has edge lengths of 1 unit. Since the smaller, unit fraction cubes have edge lengths of $\frac{1}{3}$ unit, there are three of these unit fraction cubes packed in each direction of the unit cube, for a total of 27 cubes. Since the volume of the unit cube is 1 cubic unit, the volume of each of the smaller cubes is $\frac{1}{27}$ cubic units. Guide students to see that, in general, a cube with edge lengths of $\frac{1}{n}$ units has a volume of $\frac{1}{n^3}$ cubic units.

- Ask students to find the volume of a figure by packing it with smaller cubes. For example, ask students to find the volume of a box that is $2\frac{1}{2}$ feet by 1 foot by $3\frac{1}{2}$ feet by filling it with cubes that are each $\frac{1}{2}$ foot by $\frac{1}{2}$ foot by $\frac{1}{2}$ foot, as shown in the following diagram.

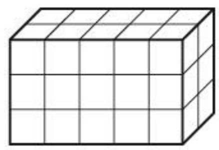


Students should find that each of the cubes has a volume of $\frac{1}{2^3}$, or $\frac{1}{8}$ cubic foot. When the box is packed with the smaller cubes, there are 2 cubes, 5 cubes, and 7 cubes along each edge of the box. Students should calculate that there are 2 • 5 • 7 = 70 cubes, in total, packed in the box. Since the volume of each individual cube is $\frac{1}{8}$ cubic foot and it takes 70 small cubes to fill the large box, students should determine that the volume of the large box is $70 \bullet \frac{1}{8} = \frac{70}{8}$, or $\frac{35}{4}$ cubic feet.

- Ask students to determine the size of a cube that can be used to fill a given rectangular prism and to determine the number of cubes that are needed to completely fill the prism. For example, give students a rectangular prism with a height of $1\frac{2}{3}$ inches, a length of $3\frac{1}{2}$ inches, and a width of 2 inches. Students may find that a cube with edge lengths of $\frac{1}{6}$ inch can be used to fill the rectangular prism without gaps or overlap. The prism would be 10 cubes tall, 21 cubes long, and

12 cubes wide. Therefore, 10 • 21 • 12 = 2,520 cubes are needed to fill the prism. (Note that since the cubes are not unit cubes, the volume of the prism is not 2,520 cubic inches.)

- Ask students to show and explain why the volume of a rectangular prism can be found by multiplying the length, width, and height. For example, give students the following figure packed full of cubes that have edge lengths of $\frac{1}{5}$ inch.



Ask students to find the number of cubes packed into the figure. Students should determine that there are 5 • 3 • 2 = 30 cubes in the figure. Students should also find that each cube has a volume of $\frac{1}{125}$ cubic inches. Therefore, the volume of the entire prism is $5 \cdot 3 \cdot 2 \cdot \frac{1}{125} = 5 \cdot 3 \cdot 2 \cdot \frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{5} = \frac{5}{5} \cdot \frac{3}{5} \cdot \frac{2}{5}$, which is the product of the length, width, and height of the figure. Remind students that this method is the same as for prisms with whole-number edge lengths.

- Ask students to find the volume of a rectangular prism with fractional edge lengths in context. For example, give students the following problem: "An apple orchard packages bags of apples into a box that is $23\frac{1}{2}$ inches long, $15\frac{1}{2}$ inches wide, and $9\frac{1}{4}$ inches high. What is the volume of this apple box?" Students should use the formula $V = lwh$ to write the equation $V = 23\frac{1}{2} \cdot 15\frac{1}{2} \cdot 9\frac{1}{4}$ and come up with a solution of $3369\frac{5}{16}$ cubic inches.

► Identifying and Addressing Common Misconceptions

- Some students may think that the numerical value of the volume must be greater than the numerical value of the length of any edge. Ask students whether the volume of a cube with edge lengths of $\frac{1}{4}$ inch is greater than or less than $\frac{1}{4}$ cubic inch. Some students may think the volume must be greater. Remind students that while whole-number multiplication results in a product equal to or larger than its factors, multiplying two (positive) numbers that are less than 1 results in a product that is smaller than either factor.

- Some students may multiply the corresponding edge lengths of a fractional cube and larger rectangular prism instead of dividing when finding how many fractional cubes fit into the larger prism. Ask students how many $\frac{1}{2}$-inch-by-$\frac{1}{2}$-inch-by-$\frac{1}{2}$-inch cubes can be packed into a cube that measures 4 inches by 4 inches by 4 inches. Some students may calculate $(4 \bullet \frac{1}{2}) \bullet (4 \bullet \frac{1}{2}) \bullet (4 \bullet \frac{1}{2})$ and give 8 cubes as an answer. Ask students to determine how many 3-inch sticks fit on a 21-inch line. Then ask students to explain why dividing $21 \div 3$ gives the correct answer in that situation. Help students to understand the parallels between the stick question and finding the number of $\frac{1}{2}$-inch cubes that fit along the edge of a 4-inch cube.

- Some students may add the edge lengths instead of multiplying when finding the volumes of rectangular prisms. Ask students to find the volume of a rectangular prism with edge lengths of $\frac{1}{5}$ unit, $\frac{3}{5}$ unit, and $\frac{4}{5}$ unit. Some students may give an answer of $\frac{8}{5}$ units. Remind students that just like with whole-number side lengths, the lengths of the edges are multiplied when finding the volume. So $\frac{1}{5} \bullet \frac{3}{5} \bullet \frac{4}{5} = \frac{12}{125}$ cubic units is the volume of the prism.

► Enrichment

- Ask students to find patterns in the size of unit fraction cubes that can pack a larger box. For example, the dimensions of the box are $7\frac{1}{2}$ inches by $3\frac{1}{5}$ inches by 5 inches. Unit cubes with edge lengths of $\frac{1}{10}$ inch can be packed into the box. In general, this edge length can be found by creating a unit fraction from the least common multiple of the denominators of the lengths from the larger box.

- Provide students with a box and manipulatives such as number cubes or sugar cubes. Ask them to use measuring tools to find the size of the smaller cubes and the maximum number of cubes that the box can accommodate. Then have them use the cubes to fill the box. Discuss why the actual numbers may or may not match the calculated number. Some possible reasons include minor irregularities in the cubes leading to gaps or having gaps because cubes would need to be cut in order to completely pack the box.

- Ask students to find the difference in the sizes of two boxes without finding the volume of the boxes. For example, ask students to find how many more cubes with edge lengths of $\frac{1}{4}$ fit into a box with the dimensions 2 × 3 × 2 than a box with the dimensions 2 × 2 × 2. The first box can be packed with an 8 × 12 × 8 array of cubes, and the second box can be packed with an 8 × 8 × 8 array of cubes. Students determine that the first box contains 4 more 8 × 8 layers of cubes than the second box.

► Key Terms

edge, prism, right rectangular prism, volume

C-G.1.1.4

► Reporting Category, Assessment Anchor, Anchor Descriptor

Geometry

Solve real-world and mathematical problems involving area, surface area, and volume.

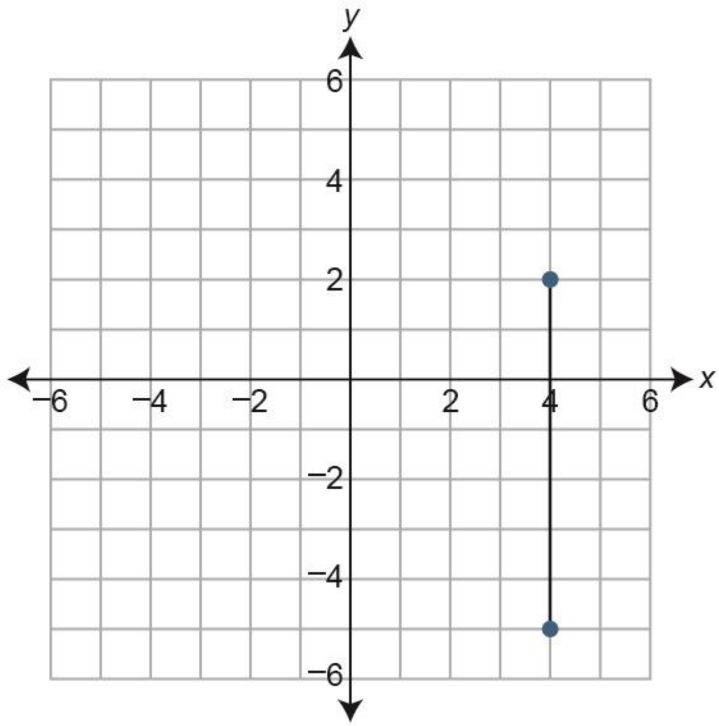
Find area, surface area, and volume by applying formulas and using various strategies.

► Eligible Content

C-G.1.1.4 Given coordinates for the vertices of a polygon in the plane, use the coordinates to find side lengths and area of the polygon (limited to triangles and special quadrilaterals). Formulas will be provided.

► Instructional Supports

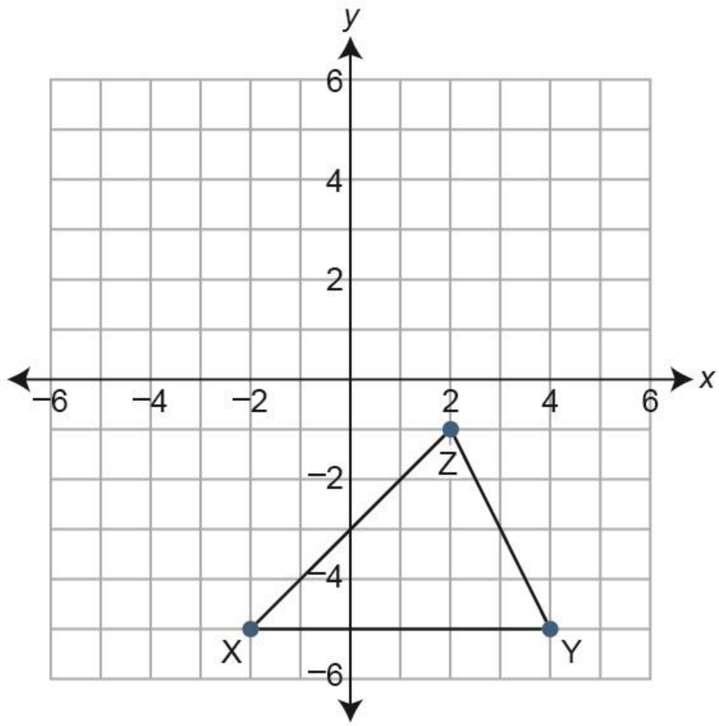
- Ask students to find the length of a line segment on a coordinate plane when the endpoints of the segment have the same first coordinate or the same second coordinate. For example, ask students to find the length of line segment BC of quadrilateral ABCD on a coordinate plane when B is located at (4, 2) and C is located at (4, –5).



Some possible methods are shown.

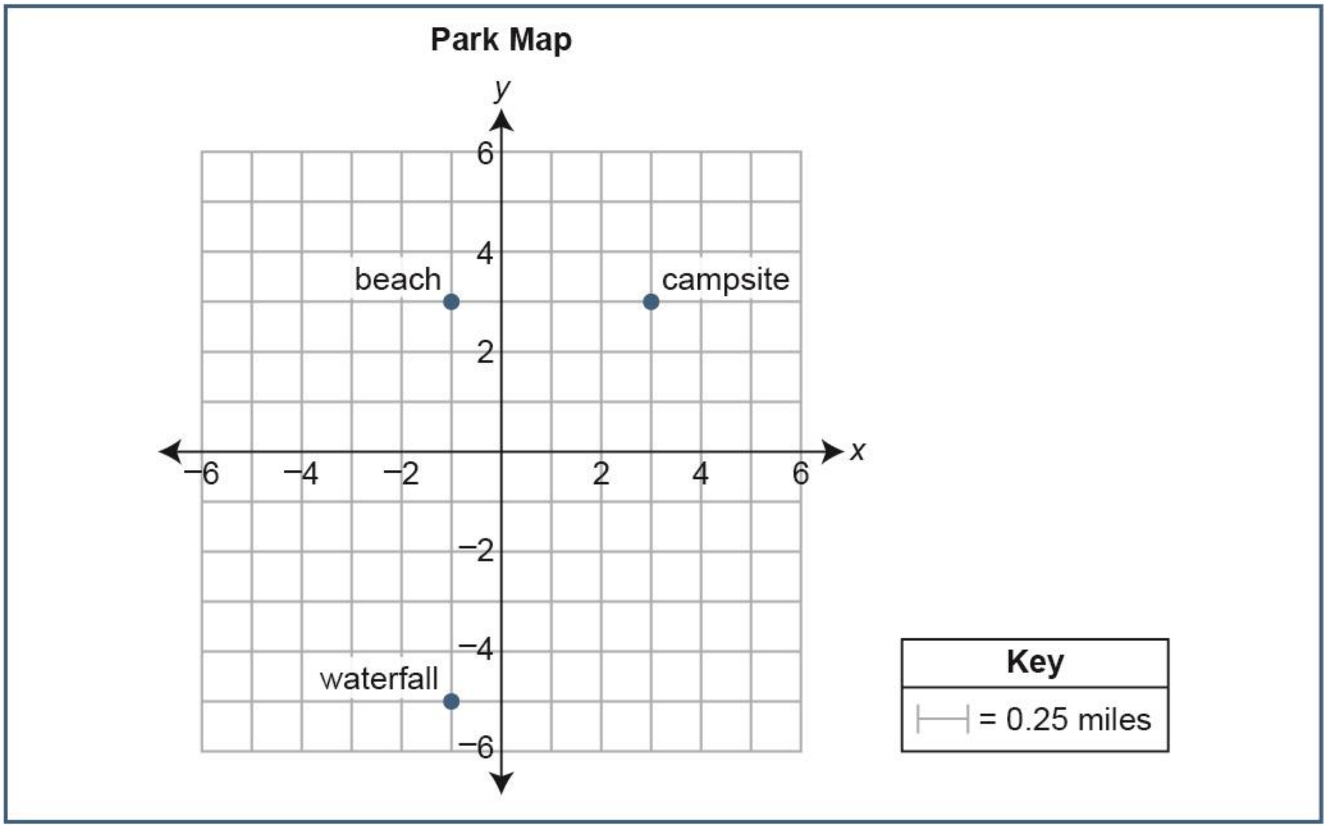
- Count the units on the coordinate plane to get a length of 7.
- The point (4, –5) is 5 units below the *x*-axis, and the point (4, 2) is 2 units above the *x*-axis. Therefore, the endpoints are 5 + 2 = 7 units apart.

- Ask students to find the length of a horizontal or vertical side of a figure drawn on a coordinate plane. For example, give students the triangle shown and ask them to find the length of side XY.



Students should find that because X and Y have the same *y*-coordinate, the distance can be counted to be 6 units.

- Give students the length of a segment of triangle FGH and the coordinates of one endpoint of the segment on a coordinate plane and ask them to find possible coordinates of the other endpoint. For example, ask students to find four possible locations of point H of line segment GH when G is located at (5, 9) and the length of line segment GH is 3 units. Students can find the location of H by increasing or decreasing the value of each coordinate of G by 3. Since 5 + 3 = 8, 5 – 3 = 2, 9 + 3 = 12, and 9 – 3 = 6, the possible locations of point H are (8, 9), (2, 9), (5, 12), and (5, 6), respectively. Have students plot the points on the coordinate plane to verify.
- Ask students to find a distance in a real-world problem in which locations are plotted on a coordinate plane. For example, give students the following map and ask them to find the shortest total distance hiked by Julie when she starts at her campsite, hikes to the beach, and then hikes to the waterfall.



Students may first determine that the campsite is located at (3, 3), the beach is located at (–1, 3), and the waterfall is located at (–1, –5). Then, students should find that the shortest distance between the campsite and the beach is a straight line that is 4 units and that the shortest distance between the beach and the waterfall is a straight line that is 8 units. On the map, Julie hikes a total of 12 units. Because the key on the map shows that each unit is equal to 0.25 miles, students should determine that Julie hikes a total of 12 • 0.25 = 3 miles.

► Identifying and Addressing Common Misconceptions

- Some students may think they can calculate the distance between two points when the *x*-coordinate of one point is the same as the *y*-coordinate of the other point by comparing the other coordinates. Ask students to find the distance between (7, 2) and (2, –3). Some students may notice that 7 and –3 are 10 units apart and give a distance of 10. Ask students to graph both points and then draw the shortest path between the points. Then ask students to use a ruler to compare the length of their path with 10 units on the grid. Students should notice that their path is shorter than 10 units.

► Enrichment

- Help students to explore the idea of "taxicab" distance, or the distance from the origin to a point on the coordinate grid when restricted to moving only in horizontal and vertical directions. (Imagine a taxicab driving on a grid of downtown city streets.) Make sure to point out that this is **not** the way to measure actual distance.
    - Ask students to plot all the points that are 8 units, in taxicab distance, from the origin, where the "taxicab" is allowed only to move up and to the right.
    - Ask students if there are different ways to arrive at the same point by traveling the same distance.
    - Ask students to make observations about all the points that are 8 units from the origin.

► Key Terms

attributes, axis, coordinate plane, coordinates, vertices

C-G.1.1.5 & C-G.1.1.6

► Reporting Category, Assessment Anchor, Anchor Descriptor

Geometry

Solve real-world and mathematical problems involving area, surface area, and volume.

Find area, surface area, and volume by applying formulas and using various strategies.

► Eligible Content

C-G.1.1.5 Represent three-dimensional figures using nets made up of rectangles and triangles.

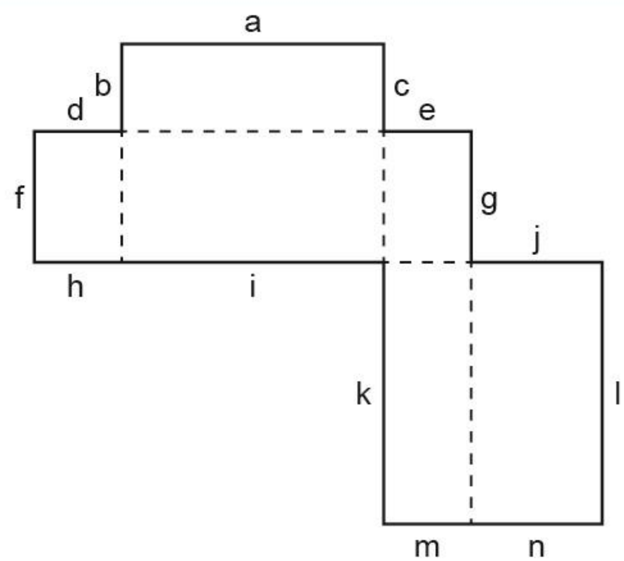
C-G.1.1.6 Determine the surface area of triangular and rectangular prisms (including cubes). Formulas will be provided.

► Instructional Supports

- Ask students to create nets out of three-dimensional objects. For example, ask students to
    - cut apart and unfold a cereal box to create a net or
    - wrap a box in newspaper and then cut the wrapping along the edges of the box to create a net.

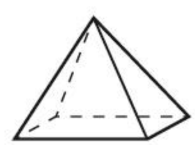
  Ask students to compare their nets and demonstrate that multiple different nets can be created for the same object.

- Give students a net that represents a three-dimensional figure and ask them to use colors to indicate which edge lengths will meet up when the net is refolded and connected to model the three-dimensional figure. For example, give students the following net. Ask students to use markers or highlighters to color-code edges that will connect when the net is folded to model a rectangular prism.

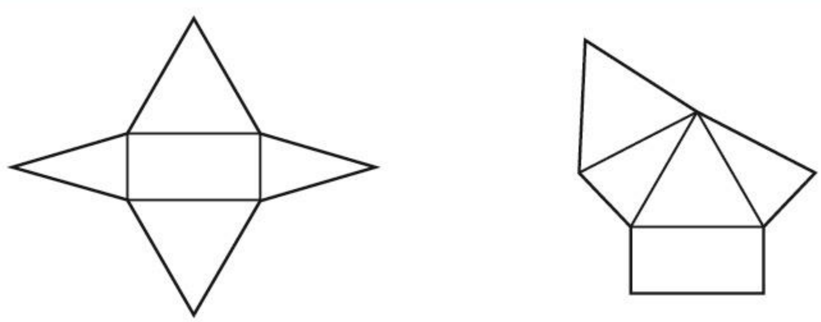


The pairs of edges that should have the same color are a and l, b and d, c and e, f and n, h and m, i and k, and g and j. Students can check their answers by cutting out the shape and folding it into a rectangular prism.

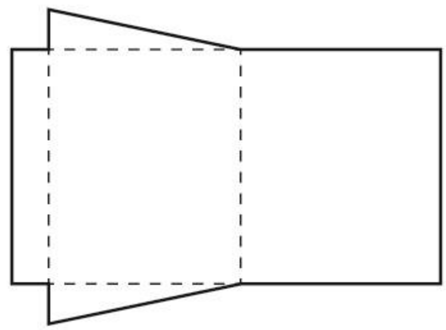
- Ask students to draw a net of a given three-dimensional figure. For example, ask students to draw a net of the figure shown.



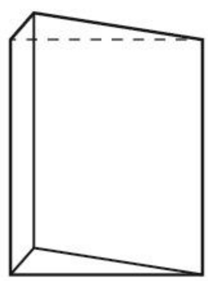
The pyramid has a rectangular base and four triangular sides. Some possible nets are shown.



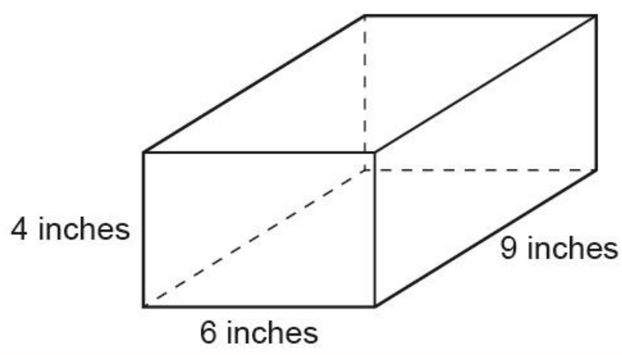
- Ask students to draw a three-dimensional figure given the net of the figure. For example, give students the following net and ask them to draw the three-dimensional figure it represents.



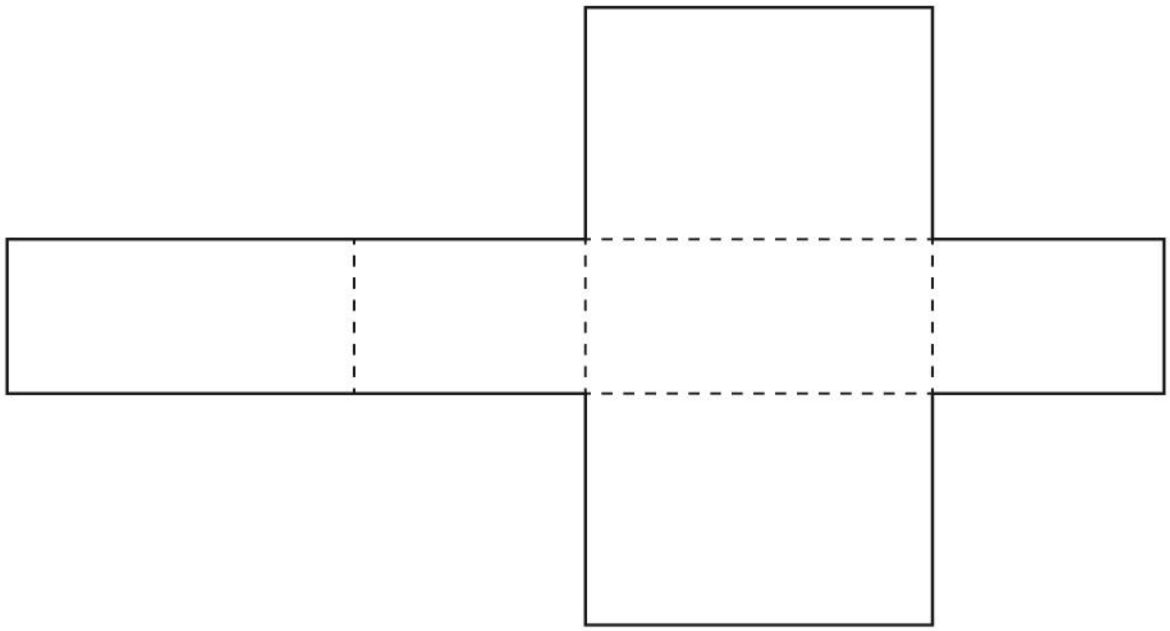
While there is only one correct figure, students should notice that it can be drawn with several different orientations. One example is shown.



- Give students a three-dimensional figure and ask them to use a net to find the surface area of the figure. For example, give students the following figure.

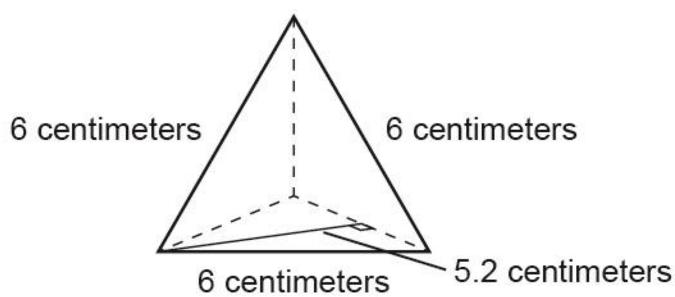


Ask students to create a net of the figure. A possible net is shown.



Ask students to find the surface area of the figure by finding the sum of the areas of each face of the prism. Also, since the figure is a rectangular prism, there are three pairs of faces of the prism that are the same size. Help students to see that the surface area of the figure can be found by calculating 2(4 • 6) + 2(4 • 9) + 2(6 • 9). Students should find that the surface area of the rectangular prism is 228 square inches.

- Ask students to find the surface area of a figure in a real-world situation. For example, ask students to find the amount of wrapping paper, in square centimeters, needed to cover the gift box shown, which is composed of four triangles of the same size and shape.



Some students may find it helpful to draw a net. The surface area can be found by calculating 4(0.5 • 6 • 5.2). Therefore, students should determine that 62.4 square centimeters of paper is needed to cover the gift box.

► Identifying and Addressing Common Misconceptions

- Some students may think there is only one way to draw the net of a figure. Give students a rectangular prism and ask them to draw multiple different nets of the figure. Some students may be able to create only one net. Ask students to cut out their net and reassemble it into the original figure. Then ask them to recut it along different edges to create a different net.
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- Students may use cubic units instead of square units when labeling surface area because they are working with three-dimensional figures. Ask students to find the surface area of a three-dimensional figure using a net. Some students may give their answer in cubic units. Ask students to identify each shape in the net as two-dimensional or three-dimensional. Help students recognize that the surface of a three-dimensional figure is made up of flat, two-dimensional shapes. Therefore, surface area is connected to area and should be labeled in square units.

► Enrichment

- Ask students to draw a net of a three-dimensional figure that includes shapes other than triangles and rectangles. Where applicable, ask students to calculate the surface area of the figure.
- Ask students to draw a three-dimensional figure represented by a given net that includes shapes other than triangles and rectangles.

- Ask students to draw a net to represent a composite three-dimensional figure. Where applicable, ask students to calculate the surface area of the figure.

► Key Terms

net, surface area, three-dimensional

C-G.1.1.5 & C-G.1.1.6

► Reporting Category, Assessment Anchor, Anchor Descriptor

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Solve real-world and mathematical problems involving area, surface area, and volume.

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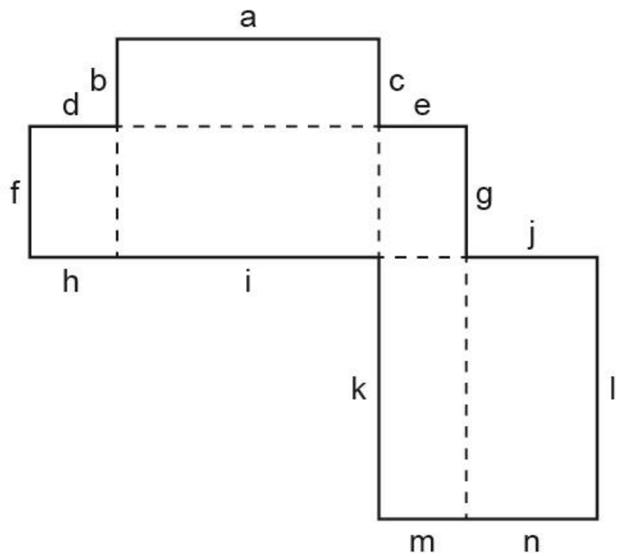

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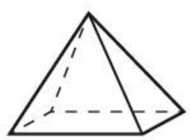

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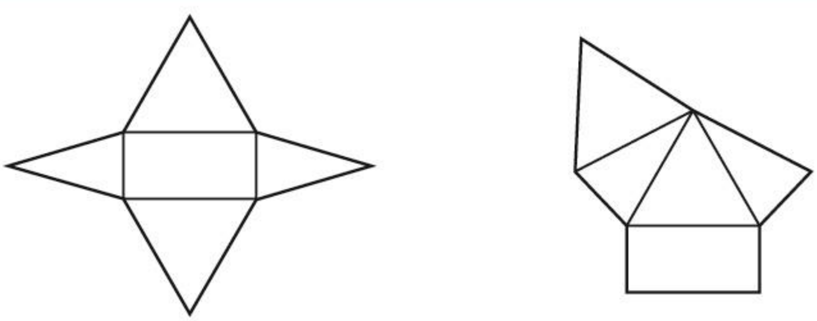

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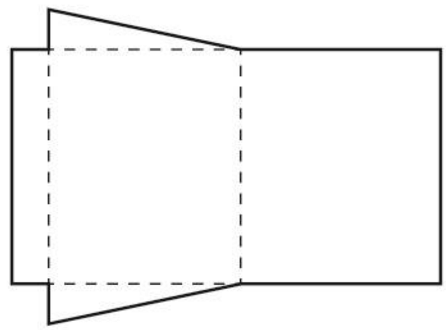

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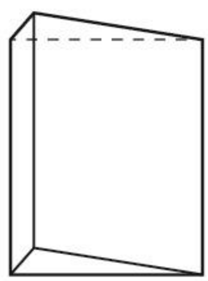

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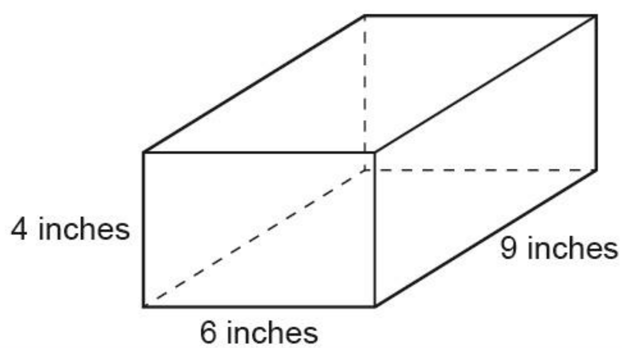

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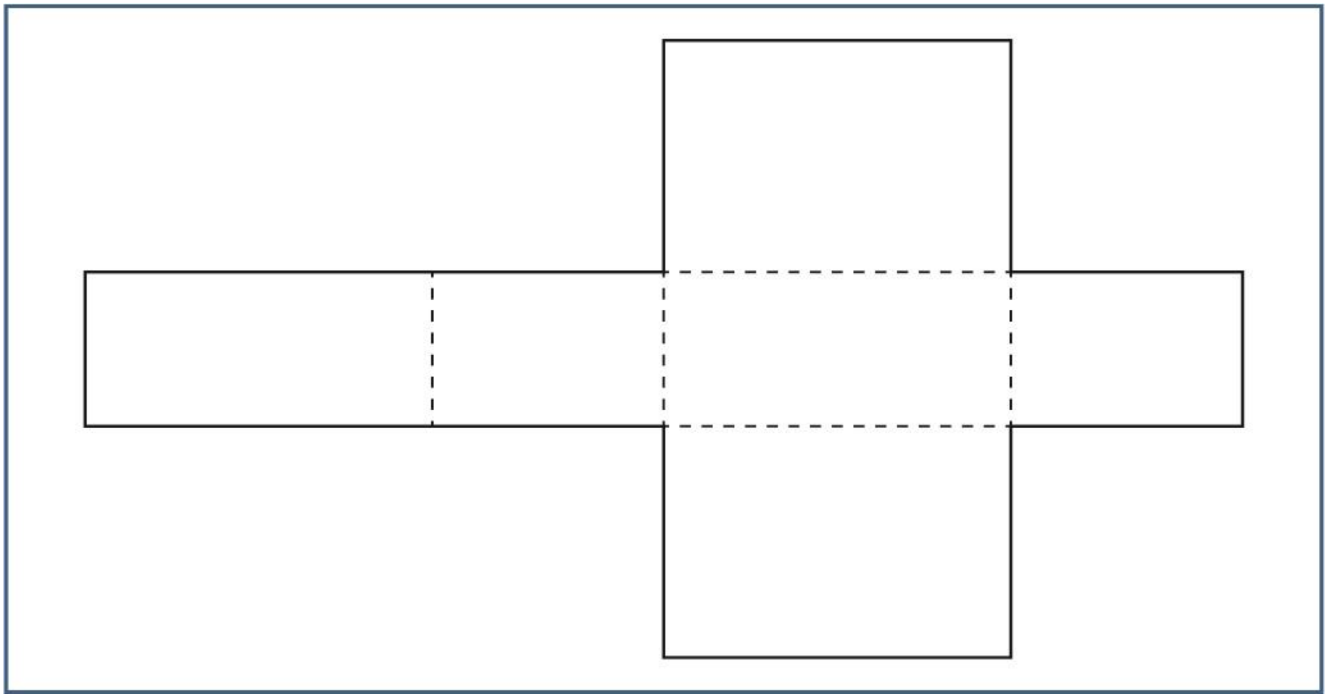

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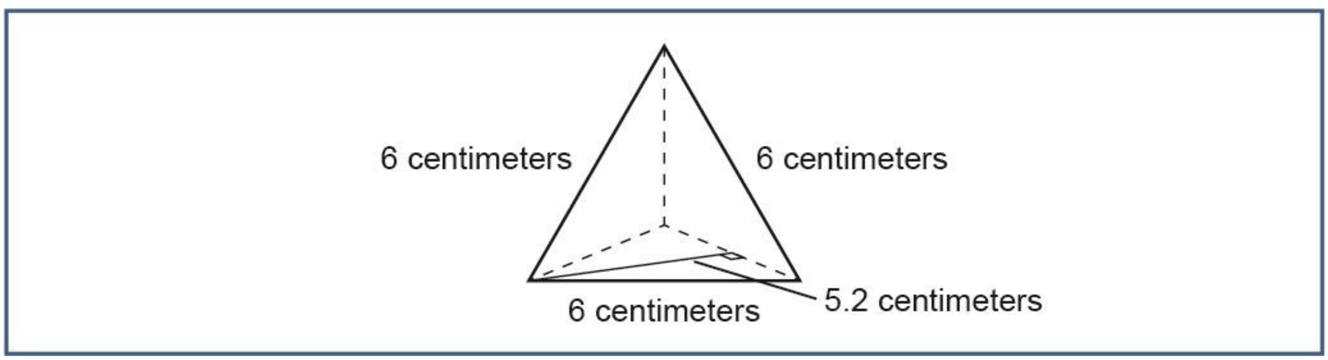

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- Ask students to draw a three-dimensional figure represented by a given net that includes shapes other than triangles and rectangles.

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► Key Terms

net, surface area, three-dimensional

D-S.1.1.1

► Reporting Category, Assessment Anchor, Anchor Descriptor

Statistics and Probability

Demonstrate understanding of statistical variability by summarizing and describing distributions.

Display, analyze, and summarize numerical data sets in relation to their context.

► Eligible Content

D-S.1.1.1 Display numerical data in plots on a number line, including line plots, histograms, and box-and-whisker plots.

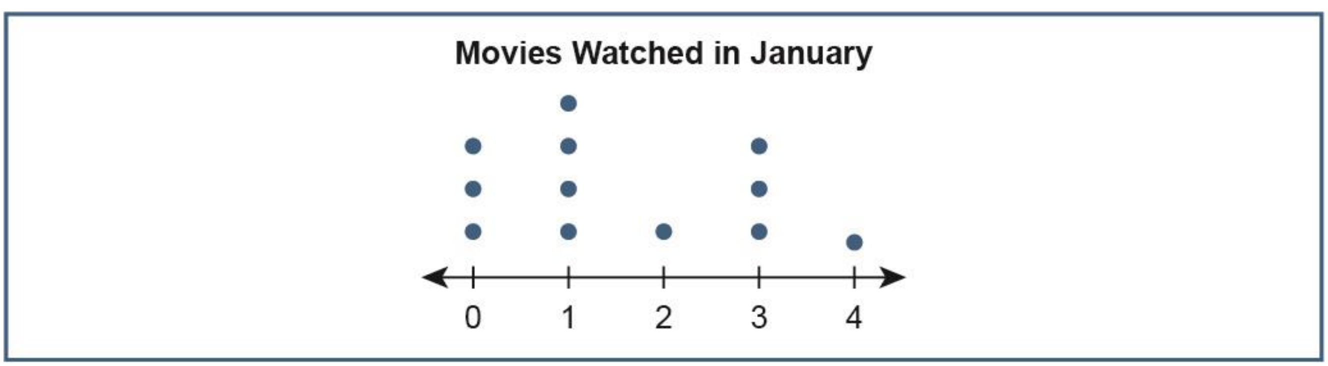
► Instructional Supports

Dot plots

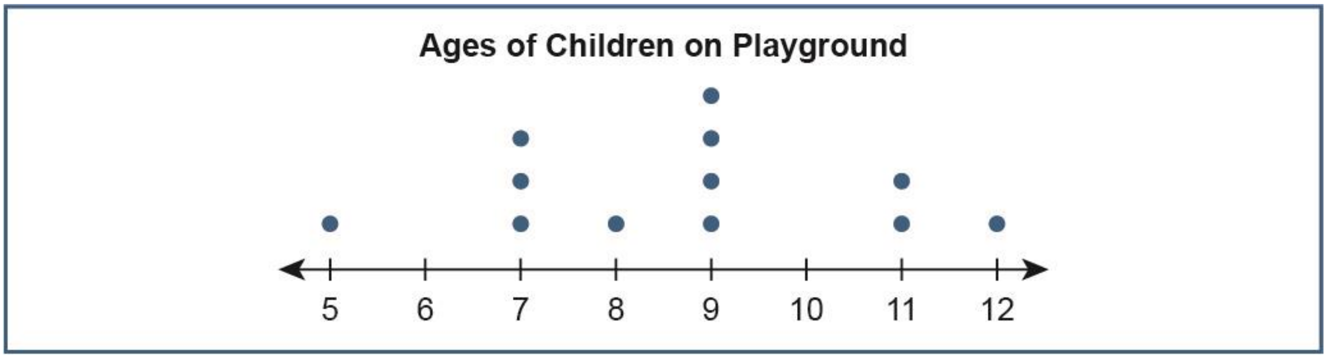
- Ask students to represent a data set by making a dot plot. For example, the data set shown represents the numbers of movies watched by 12 different students during the month of January.

  {1, 0, 1, 3, 2, 4, 3, 0, 1, 0, 1, 3}

  Encourage students to account for all the data when making the dot plot by arranging the data in numerical order, crossing off each value after it is plotted, and verifying that the number of dots on the dot plot is the same as the number of values in the data set. The dot plot shown represents the data set.

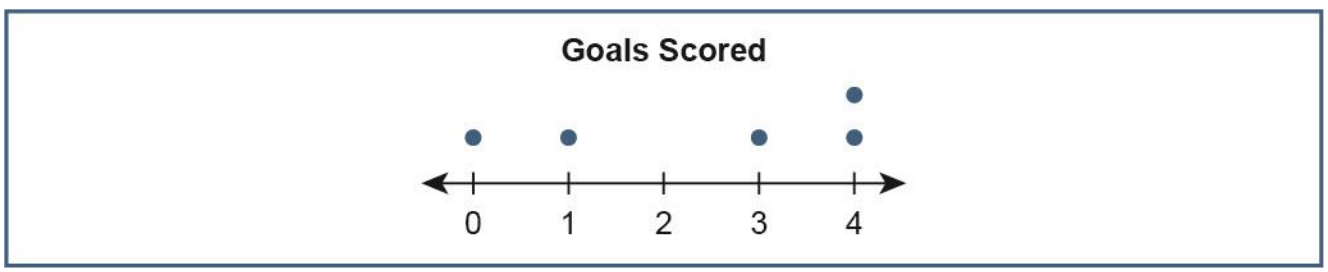


- Asks students to identify a data set that is displayed in a dot plot and answer questions about the data. For example, the dot plot shown displays the ages of children on a playground.



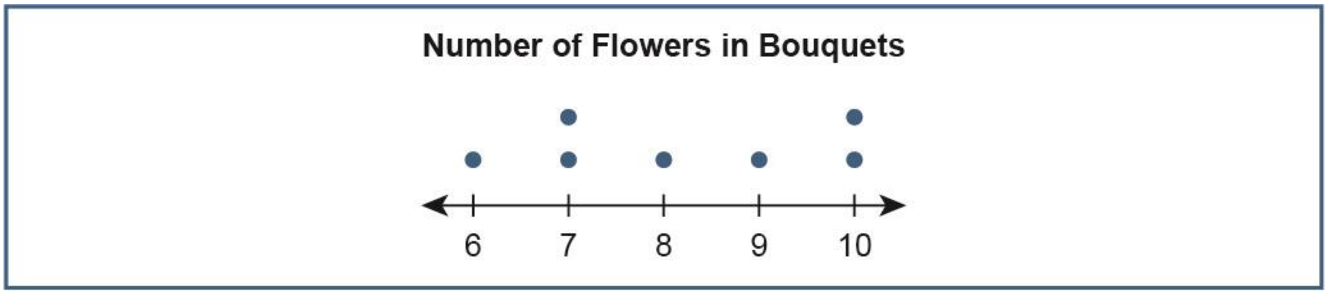
Students should identify the data set {5, 7, 7, 7, 8, 9, 9, 9, 9, 11, 11, 12}. Explain to students that the values in data sets will not necessarily appear in order from least to greatest. Possible questions to ask students about the data set are "What are the ages of the youngest and oldest children on the playground?" and "What is the most common age on the playground?"

- Ask students to determine the mean of a data set that is displayed in a dot plot. For example, the dot plot shown displays the numbers of goals scored by a soccer team during the first five games of the season.

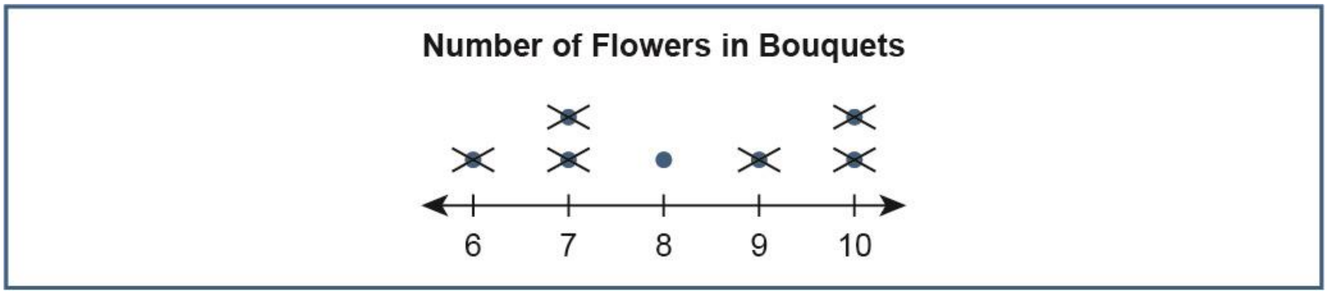


The corresponding data set is {0, 1, 3, 4, 4} and the mean can be calculated by the expression $\frac{0+1+3+4+4}{5}$. Therefore, the mean number of goals scored is $\frac{12}{5}$ or 2.4.

- Ask students to determine the median of a data set that is displayed in a dot plot. For example, the dot plot shown displays the numbers of flowers in seven different bouquets.



The corresponding data set is {6, 7, 7, 8, 9, 10, 10}. The fourth value is the middle value because there are seven values ordered from least to greatest. Therefore, the median of the data set is 8. Students may also determine the median of the data set by crossing off the first and seventh values on the dot plot, then the second and sixth values, and finally the third and fifth values, as shown on the following dot plot.



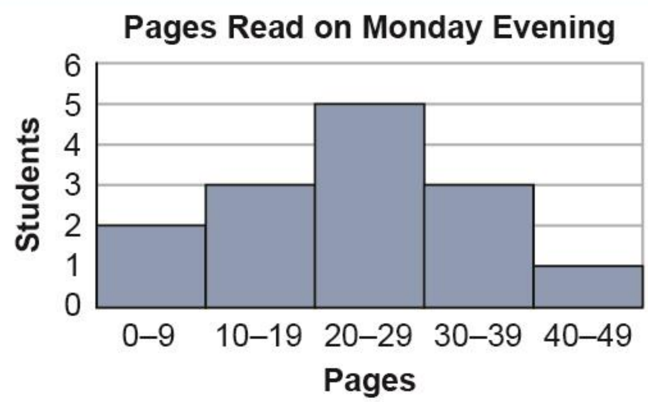
Only the middle value, 8, remains.

Histograms

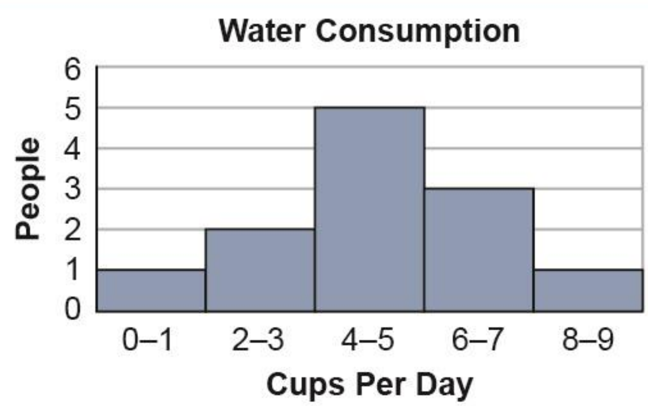
- Ask students to create a histogram to represent a given data set. For example, the data set shown represents the numbers of pages some sixth-graders read on Monday evening.

{5, 8, 12, 15, 18, 20, 20, 25, 27, 28, 30, 35, 36, 42}

Discuss what might be an appropriate interval or range for each bin in the histogram. In this case, five intervals of ten (0−9, 10−19, 20−29, 30−39, 40−49) would be appropriate for displaying the data. If the bin widths are too large, the value of the data display can be lost because it no longer displays the distribution of the data in the specified intervals. Discuss and determine appropriate labels and where they should be located on the histogram. An example of a completed histogram for this data set is shown.



- Ask students to create a possible data set that could be represented by a given histogram. For example, the histogram shown represents the numbers of cups of water consumed per day by 12 adults.



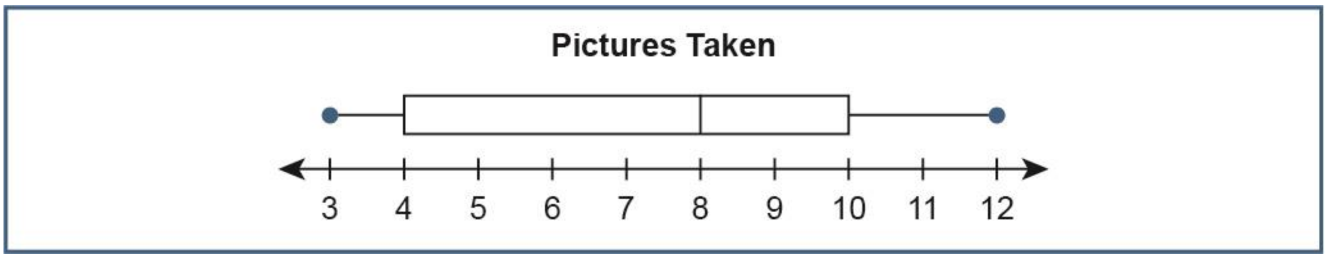
First, ask students to note the number of values within each of the bins: one in 0−1, two in 2−3, five in 4−5, three in 6−7, and one in 8−9. Then, ask students to create an example set of data that could be represented by the histogram. Ask several students to share their data sets, noting that there are multiple data sets that can be represented by the histogram. One possible data set is {1, 3, 3, 4, 4, 5, 5, 5, 6, 7, 7, 9}. Another possible data set is {1, 2, 2, 4, 4, 4, 4, 5, 7, 7, 7, 8}.

Box plots

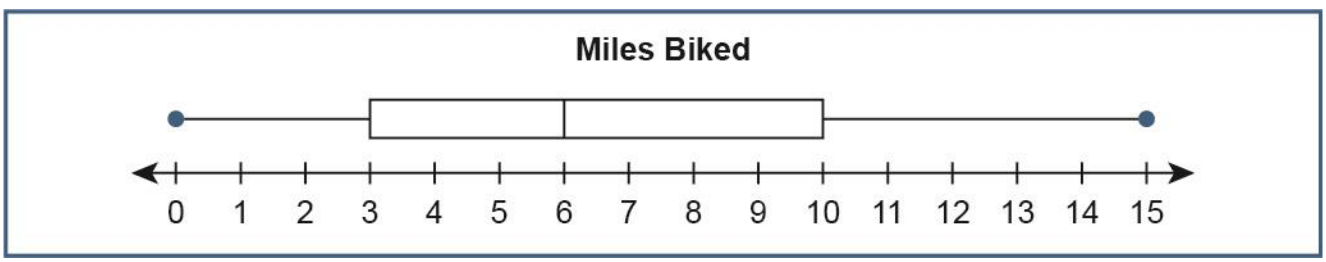
- Ask students to create a box plot to represent a given data set. For example, the data set shown represents the numbers of pictures taken by 10 students on a field trip.

  {3, 3, 4, 5, 7, 9, 9, 10, 11, 12}

  Students should begin by determining five pieces of information: the least value, the greatest value, the median, the first quartile, and the third quartile. In this case, 3 is the least value and 12 is the greatest value. The median of the data is equal to the mean of 7 and 9, which is 8. The first quartile, which is the median of the lower half, is 4. The third quartile, which is the median of the upper half, is 10. Tell students to use these five values to create a box plot such as the following box plot.



- Ask students to create a possible data set that could be represented by a given box plot. For example, the box plot shown represents the number of miles biked by 11 people on Sunday.



  Given the box plot, students can initially determine that the median of the data must be 6 because it is the middle value with five points to its left and five points to its right when the data are ordered from least to greatest. In addition, the minimum value must be 0 because that is the point that is farthest to the left. The maximum value must be 15 because that is the point that is farthest to the right. The first quartile must be 3 because there are exactly five values in the lower half of the data. The third quartile must be 10 because there are exactly five values in the upper half of the data. Therefore, the values shown must be designated as the following parts of the data set.

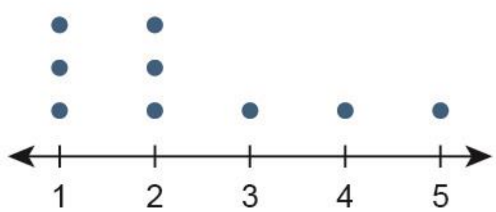
{0,	___,	3,	___,	___,	6,	___,	___,	10,	___,	15}
min		1st quartile			median			3rd quartile		max

There are many different possibilities for the remaining values. One possible complete data set is {0, 1, 3, 4, 5, 6, 8, 8, 10, 12, 15}. Ask students to compare their data set with a partner and discuss why both data sets can be represented by the box plot.

► Identifying and Addressing Common Misconceptions

- Some students may attempt to determine the median of a data set that is displayed on a dot plot by finding the middle number on the number line rather than the middle value of the data set. Give students the following dot plot that is skewed and ask "What is the median of this data set?"

Number of Flowers in Bouquets



Some students may give an answer of 3. To add further clarification, ask students to list all the values of the dots in numerical order and locate the middle value. When displayed as {1, 1, 1, 2, 2, 2, 3, 4, 5}, students should be able to identify that 2 is the median.

- Some students may think that the first interval of a histogram must start at 1. Give students a statistical question that could include a response of 0 and ask them to determine intervals that may be appropriate to use in a histogram. Some students may choose to start the first interval at 1. Ask students to explain how they would indicate responses of 0. Remind students that all bins of a histogram should have equal interval spans. Students should see that responses of 0 should not be in their own bin and the value 0 should be included in the first interval.

- Some students may think that the five values necessary to construct a box plot for a data set must all be numbers included in the data set. Ask students to identify the quartiles of {2, 4, 5, 8, 9, 11, 12, 14}. Some students may choose only numbers from the data set. Ask students to explain how they found the median of the data set. Remind them that the median is not always a value included in the data set. Because quartiles are the medians of the upper and lower halves of the data, the quartiles, similarly, are not always values included in the data set. The minimum and maximum, however, are always values that appear in the data set.

► Enrichment

- Ask students to create a data set that, when represented by a box plot, results in a box plot having only one "whisker." Then, ask students to create a data set that, when represented by a box plot, results in a box plot without any "whiskers." Help students to recognize that they need a data set in which either the minimum value and the first quartile are equal, the maximum value and the third quartile are equal, or both.
- Help students use dot plots to visually illustrate the mean of a data set. Give students a dot plot where the mean of the data is a whole number and ask them to calculate the mean. Then, ask them to move the dot that represents the least value one unit to the right and the dot that represents the greatest value one unit to the left and recalculate the mean. Have students repeat the previous step until all the dots are in a single column. Ask students to make observations about patterns that they notice.
- Ask students to discuss which representation is the most useful for a given data set and why. For example, a dot plot would not be as useful for showing a set of data with a large spread of values because if the scale gets too large, it becomes difficult to see where each dot lies. As an additional example, a box plot would not be as useful for showing a data set that is bimodal or has large gaps in the data.
- Give students a histogram and ask them to determine a data set with the greatest possible mean (or median) and another data set with the least possible mean (or median) that could be represented by the histogram.
- Give students a box plot and ask them to determine a range of values for the mean of the possible data sets.

► Key Terms

box plot, data, dot plot, frequency, histogram, maximum, minimum, outlier, range, spread, statistical question, variation

D-S.1.1.2

► Reporting Category, Assessment Anchor, Anchor Descriptor

Statistics and Probability

Demonstrate understanding of statistical variability by summarizing and describing distributions.

Display, analyze, and summarize numerical data sets in relation to their context.

► Eligible Content

D-S.1.1.2 Determine quantitative measures of center (e.g., median, mean, and/or mode) and variability (e.g., range, interquartile range, and/or mean absolute deviation).

► Instructional Supports

Quartiles and interquartile range

- Ask students to determine the quartiles of a data set and describe what they indicate. For example, the following data set represents the number of chess games played by 12 members of the chess club during the winter break.

  {3, 4, 4, 6, 7, 8, 10, 12, 12, 14, 14, 15}

  The median of the data set is 9, which indicates that half (6 members) of the chess club played fewer than 9 games while the other half (6 members) played more than 9 games. The median of the lower half of the data is 5, which indicates that a quarter (3 members) of the chess club played fewer than 5 games while another quarter (3 members) played more than 5 games but fewer than 9 games. The median of the upper half of the data is 13, which indicates that a quarter (3 members) of the chess club played more than 13 games while another quarter (3 members) played fewer than 13 games but more than 9 games. The median of the lower half of the data (5), the median of all the data (9), and the median of the upper half of the data (13) represent the first, second, and third quartiles, respectively. These three values divide the data into fourths as shown.

3, 4, 4, 6, 7, 8, 10, 12, 12, 14, 14, 15

5 9 13

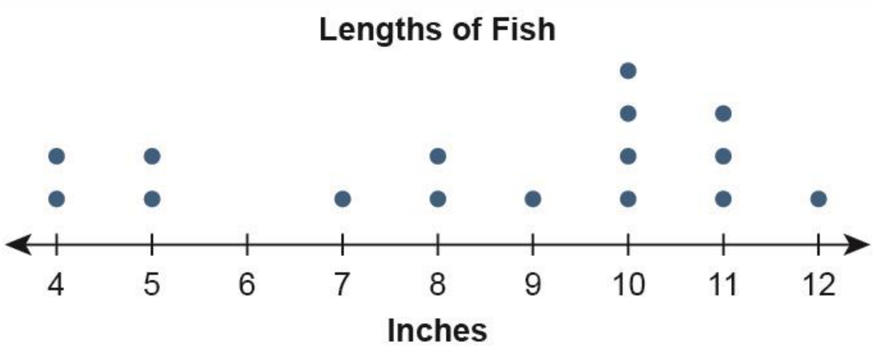
Note that if the median or any of the quartiles is a value in the data set, each of the four groups may not be exactly one-fourth of the data. An example is shown.

2, 5, 6, 6, 8, 9, 9, 9, 10, 14

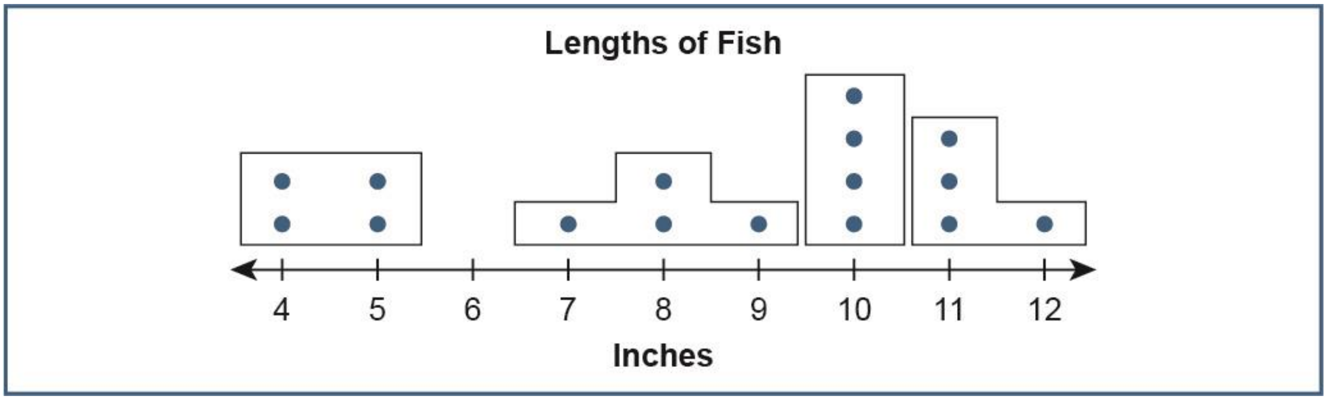
6 8.5 9

There are ten data values, but the median and quartiles divide the data into groups of size two.

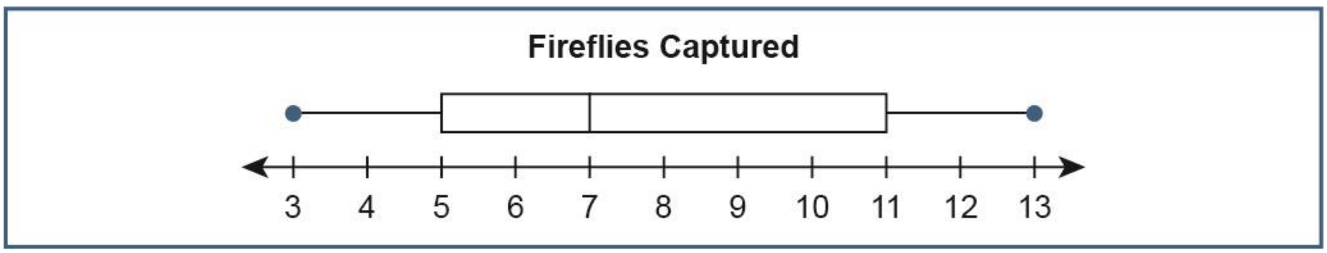
- Provide students with a dot plot and ask them to determine the quartiles and use them to divide the data into approximate fourths. For example, the data set below shows the lengths, in inches, of 16 different fish caught in a pond.



Because there are 16 data points, the median of the data is equal to the mean of the eighth and ninth values. This indicates that the median is 9.5 because $\frac{9+10}{2} = 9.5$ and that half the data is below 9.5 inches and half the data is above 9.5 inches. The median of the lower half of the data is 6 because 5 and 7 are the fourth and fifth values in the data set and $\frac{5+7}{2} = 6$. The median of the upper half of the data is 10.5 because 10 and 11 are the fourth and fifth values of that half (or the twelfth and thirteenth values of the entire set) and $\frac{10+11}{2} = 10.5$. The median and quartiles can be used to divide the data into fourths as shown: {4, 4, 5, 5,}, {7, 8, 8, 9}, {10, 10, 10, 10}, and {11, 11, 11, 12}.



- Provide students with a box plot and ask them to identify the interquartile range and explain what it means within a context. For example, the following box plot represents the number of fireflies captured by 12 different campers.



Students should be able to find the difference of the third and first quartile to identify that the interquartile range is 6. That means the middle 50% of the data (or number of fireflies captured) has a spread of 6.

Mean absolute deviation

- Give students a data set with context, ask them to calculate the mean absolute deviation, and ask them to explain what it indicates within the context. For example, this data set represents the widths, in inches, of 8 different picture frames: {8, 10, 13, 14, 16, 17, 19, 23}. Students should first determine that the mean is equal to $\frac{8+10+13+14+16+17+19+23}{8}$, or 15. Students should next determine the distance each value in the data set is from the mean.

$$15 - 8 = 7$$
$$15 - 10 = 5$$
$$15 - 13 = 2$$
$$15 - 14 = 1$$

$16 - 15 = 1$

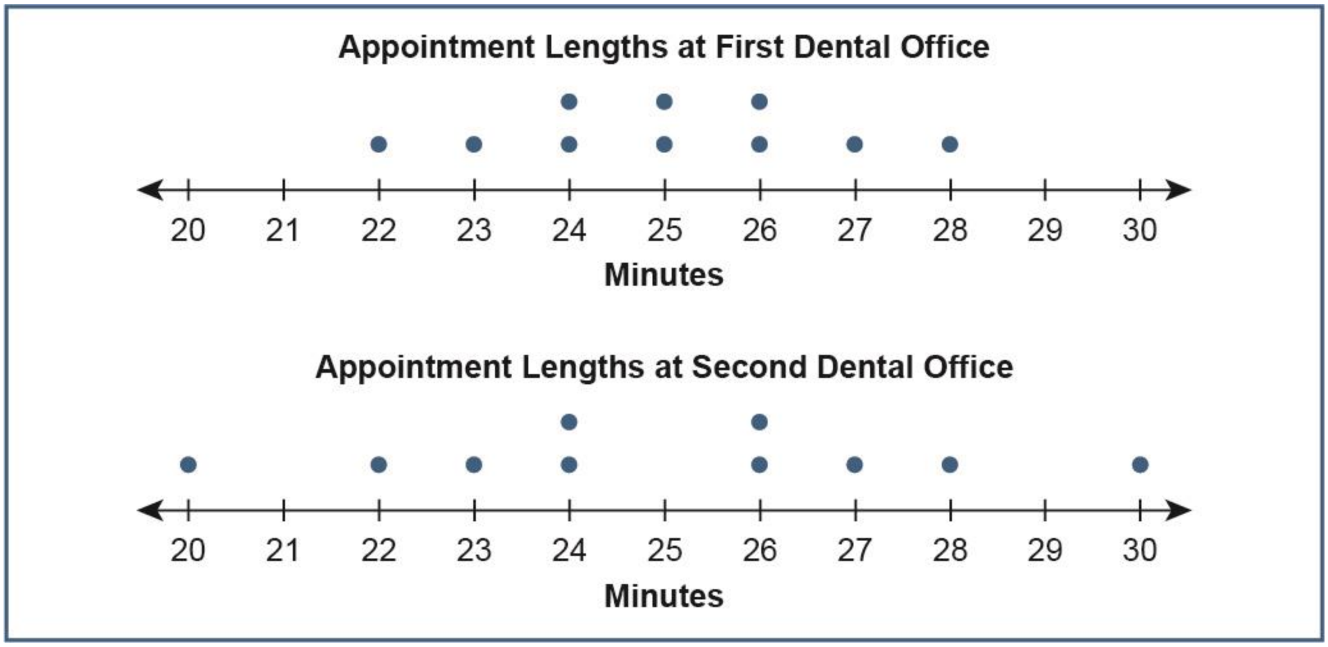
$17 - 15 = 2$

$19 - 15 = 4$

$23 - 15 = 8$

Finally, students can determine that the mean of those distances is $\frac{7+5+2+1+1+2+4+8}{8}$, or 3.75. This indicates that, on average, the widths of the picture frames are 3.75 inches away from the mean of 15 inches.

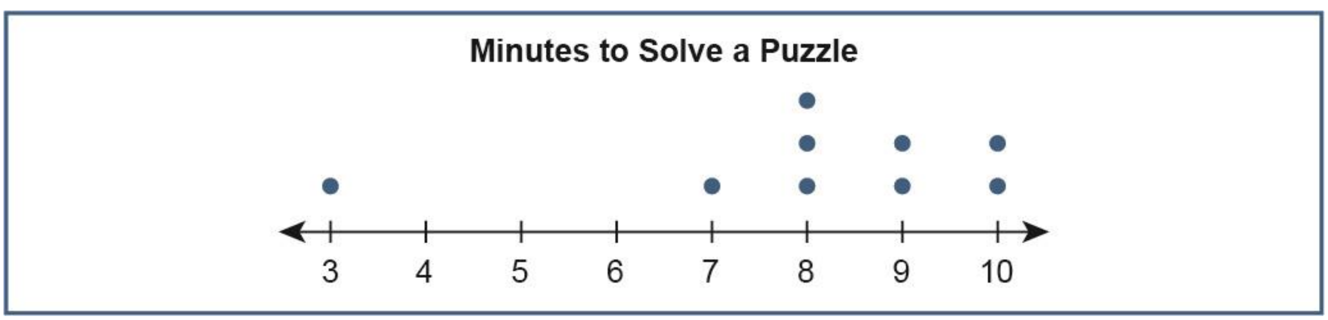
- Provide students with a large number line on the floor and ask them to use it to visualize data and determine and interpret the mean absolute deviation of a data set. For example, begin with the data set {1, 3, 6, 9, 11}, which has a mean of 6. Ask five students to represent this data by standing at each of the 5 locations on the number line. Ask each of the five students to indicate how many units they are away from the mean, then instruct the class to calculate the mean of those distances to determine the mean absolute deviation. The result of 3.2 indicates that, on average, each student is 3.2 units away from the mean. Now repeat the process with the data set {4, 5, 6, 7, 8}. Note that although the data set has the same mean, the mean absolute deviation is 1.2, which indicates that, on average, each student is 1.2 units from the mean. Ask students to explain which set of data was more "crowded" or "cramped" and how that crowdedness relates to the mean absolute deviation.
- Provide students with two data sets that are displayed on dot plots and ask them to calculate, compare, and explain the mean absolute deviations. For example, the dot plots shown represent the appointment lengths, in minutes, at two different dental offices.



Students should first observe that the mean of both data sets is 25. However, the mean absolute deviations of the data sets are different. In the first set, the mean absolute deviation is 1.4. In the second set, the mean absolute deviation is 2.4. Students should be able to articulate that although both dental offices had the same mean appointment lengths, the appointments in the first office were, on average, closer to the mean than the appointments in the second office.

Outliers

- Ask students to identify any outliers in a given data set. For example, give students the following dot plot showing the number of minutes nine students spent completing a puzzle.



The data point of 3 is an outlier because it is significantly lower than the other eight data points. In context, the outlier implies that one student completed the puzzle much faster than all the other students.

- Give students a data set that includes an outlier and ask them to calculate and compare measures of center with and without the outlier. For example, the following data set represents the number of points scored by a basketball player in 10 different games.

  {1, 16, 16, 16, 17, 17, 17, 18, 20, 21}

  The mean of the data set is 15.9 and the median is 17. When the outlier of 1 is removed, the mean of the data set is about 17.6 and the median remains 17. Students should observe that outliers tend to have a greater effect on the mean and range than on the median. In this case, the outlier has a great effect on the mean because the mean is smaller than all but one value in the data set. Because of this, the mean in this case is not a good representation of the data—it represents the outlier much more strongly than it represents any other individual data value. More generally, the median is usually a better representation of a data set that contains outliers.

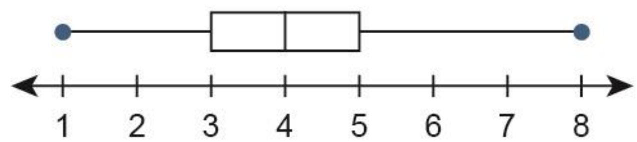
- Provide students with a data set that contains outliers and ask them to compare measures of variation, with and without the outliers, and discuss which measure of variation is more representative of the data. For example, the following data set represents the numbers of words in 10 different poems.

  {80, 84, 88, 90, 94, 100, 110, 115, 130, 488}

  The interquartile range is 27, the range is 408, and the mean absolute deviation is about 70. When the outlier of 488 is removed, the interquartile range is 26.5, the range is 50, and the mean absolute deviation is about 13. Students should notice that the outlier substantially affects the range and the mean absolute deviation but not the interquartile range. Therefore, the interquartile range is a more representative measure of variation for the data than either of the other two measures.

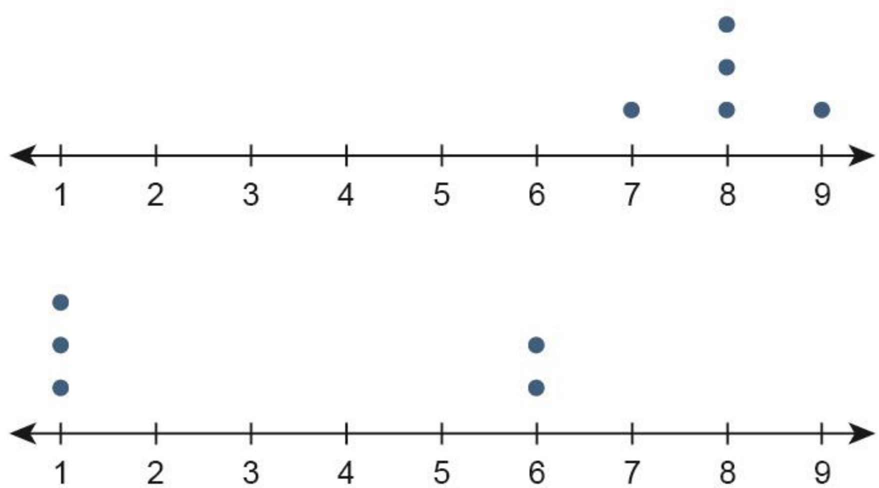
► Identifying and Addressing Common Misconceptions

- Some students may attempt to identify and interpret quartiles by dividing number lines in box plots into fourths rather than examining the location of the box itself. Give students the following box plot and ask them to identify the quartiles.



Some students may identify the quartiles at 2.5, 4.5, and 6.5. Emphasize that the number line does not represent the data set itself, but rather it shows the locations of the data values. Ask them to extend the number line out to 20 and consider whether the extended number line changes the data set and the quartiles.

- Some students may assume that the value of the interquartile range of a data set must be half the value of the range. Give students the data set {1, 3, 5, 6, 8, 10, 12, 14, 15, 20, 31} and ask them to explain how to calculate the interquartile range. Some students may give an interquartile range of 15 because it is half of the range, 30. Ask students to circle the numbers in the middle of the data set and recall that the interquartile range is the range of the middle 50% of the data. Then ask them to replace the 31 in the data set with 131. Ask them to consider whether the replacement changes the range and whether it affects the middle 50% of the data.
- Some students may assume that a data set with a greater mean than another data set must have a greater mean absolute deviation than the other data set. Give students the following line plots and ask them which data set has a greater mean absolute deviation.



Some students may think the first plot has a greater mean absolute deviation because the values in the data set are larger. Ask students to describe what the mean absolute deviation indicates about a data set, then have them calculate the mean absolute deviation for both sets of data.

▶ Enrichment

- Ask students to create a data set in which the number of values greater than the third quartile is much less than 25% of the number of values in the data set. One possible response is {1, 2, 3, 4, 5, 5, 5, 5, 5, 5, 5, 6}. Both the median and the third quartile are 5, and there is only one value in the data set greater than 5.

1, 2, 3, | 4, 5, 5, | 5, 5, 5, | 5, 5, 6

Q1 Median Q3

However, the medians and quartiles can still divide the data set into four equal portions.

- Give students a data set and ask them to discuss the consequences of using the mean or the median to describe it. For example, the data set {1, 1, 1, 2, 2, 3, 3, 5, 18} shows the number of hours students read each week. In what ways is the mean or the median more accurate, and in what ways is each measure misleading?
- Ask students to explain whether it would be possible for the mean absolute deviation of a data set to be greater than its mean.

▶ Key Terms

interquartile range (IQR), mean, mean absolute deviation (MAD), measure of center, measure of variability, median, outlier, skew

D-S.1.1.3

► Reporting Category, Assessment Anchor, Anchor Descriptor

Statistics and Probability

Demonstrate understanding of statistical variability by summarizing and describing distributions.

Display, analyze, and summarize numerical data sets in relation to their context.

► Eligible Content

D-S.1.1.3 Describe any overall pattern and any deviations from the overall pattern with reference to the context in which the data were gathered.

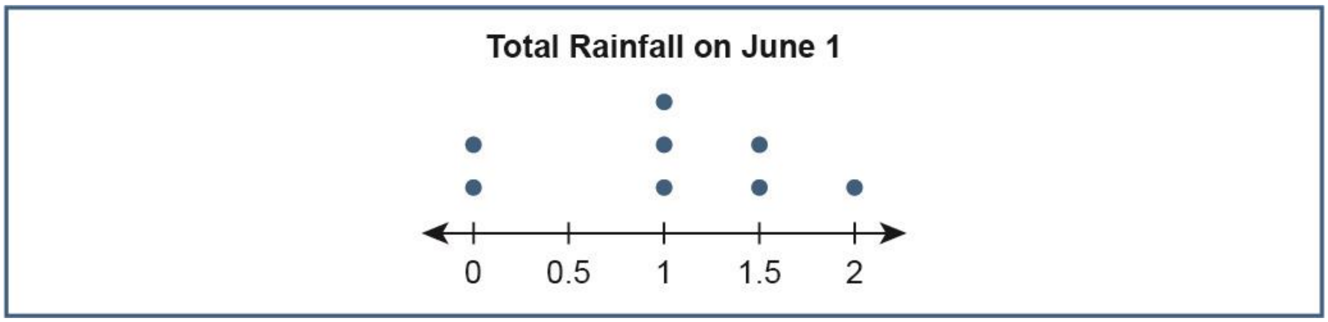
► Instructional Supports

- Ask students to discuss different ways of measuring an attribute and how the measurement method may affect comparisons of those measurements. Some examples are shown.
  - Ask students to discuss ways to measure a student's running speed. Measuring the student's average speed over 50 yards may give a different result from measuring the student's average speed over 100 yards or 200 yards. Alternately, the student's speed could be measured at every point in a 50-yard run to determine the speed.
  - Ask students to discuss ways to measure which state gets the most rainfall over a given month. Since rainfall could not feasibly be measured at every square inch of the state, measurements would need to be taken at specific locations. The number of locations and how they are chosen could affect outcomes.
- Ask students to identify factors that may affect measurements in various measurement situations. Some examples are shown.
  - Ask students to consider differing methods for measuring heights of flowers in a flowerbed. The heights of flowers in the spring are often shorter than their heights later in the summer when the plants are mature. Therefore, the time of day, month, or year may

be a significant factor, and measurements taken at one time may not always be comparable to measurements taken at a different time.

- Ask students to consider a survey of people's moods. The results of surveys that are conducted while people are at a movie theater may be different from the results of surveys that are conducted while people are at the post office. The location or circumstance may be a significant factor, and measurements taken in different locations or circumstances may not always be comparable.

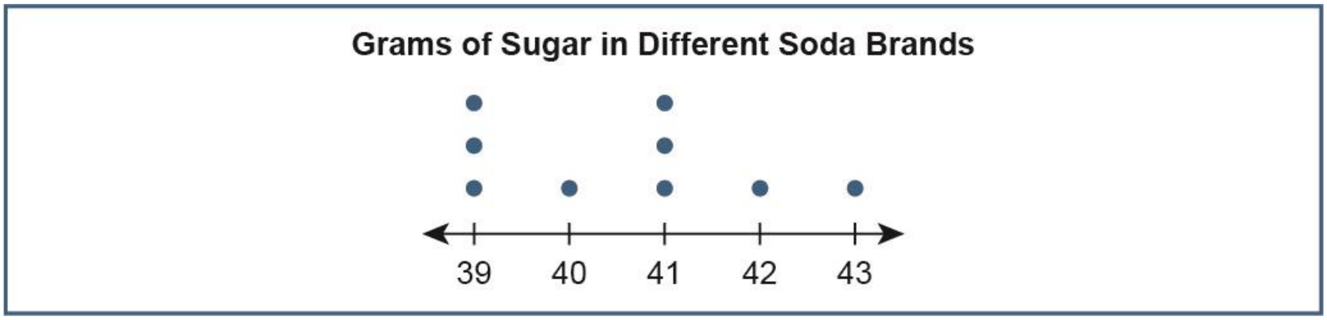
- Ask students to explain the limitations of calculating measures of center and spread or drawing conclusions when units are not included with a data set. For example, provide students with the following dot plot that shows the total rainfall on June 1 in eight different locations.



After studying the dot plot, students should be able to explain that although it provides enough information to determine the mean, median, and range of the data, those measures of center and spread are not as useful as they would be if a unit were indicated. In this context, the unit of measurement might have been either centimeters or inches.

▶ Identifying and Addressing Common Misconceptions

- Some students may not recognize the need for more than one unit of measurement. Give students the following dot plot and ask them to identify an important missing piece of information.



Some students may not recognize that although the graph includes the unit, grams, it does not include the serving or container size. Therefore, it is unclear whether the number of grams for each observation corresponds to the same quantity of soda.

► Enrichment

- Give students a measurement situation and ask them to discuss the pros and cons of two different units of measurement that could be used. For example, when collecting data about reading habits, ask students to discuss the benefits of measuring reading based on the number of pages read and the benefits of measuring reading based on the amount of time spent reading. The number of pages read may be misleading if the pages involved have many pictures or if the reading does not allow for the inclusion of reading done on a screen. On the other hand, measuring time does not necessarily account for the actual amount being read during that time.
- Ask students to consider and explain why a company or advertisement might intentionally withhold information about a unit of measurement or how something is measured.
- Ask students to discuss ways of measuring attributes that are not directly measurable. For example, ask students to discuss ways of measuring how funny a joke is. Students may suggest telling the joke to 20 people and counting how many people laugh at it, how loudly, and for how long. Some factors that may need to be considered include the person who is delivering the joke and the mood that the listeners are in.

► Key Terms

attribute, data, unit of measurement

D-S.1.1.4

► Reporting Category, Assessment Anchor, Anchor Descriptor

Statistics and Probability

Demonstrate understanding of statistical variability by summarizing and describing distributions.

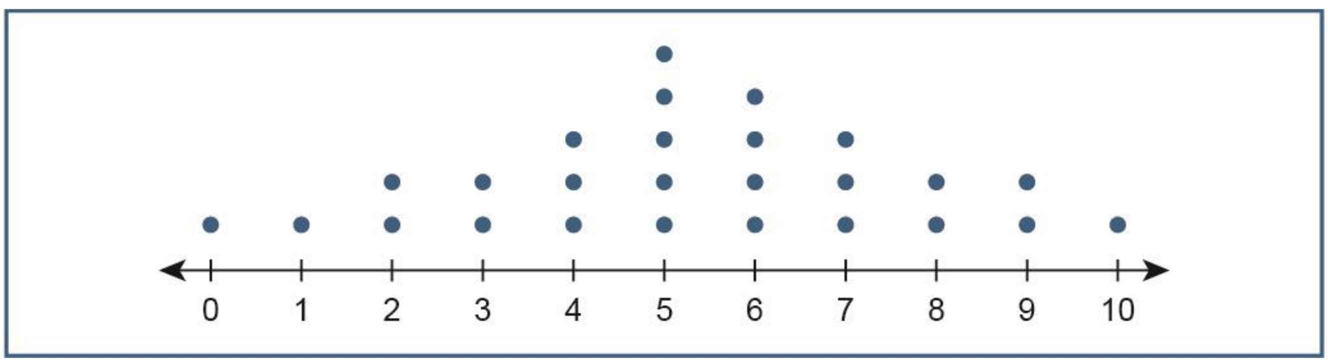
Display, analyze, and summarize numerical data sets in relation to their context.

► Eligible Content

D-S.1.1.4 Relate the choice of measures of center and variability to the shape of the data distribution and the context in which the data were gathered.

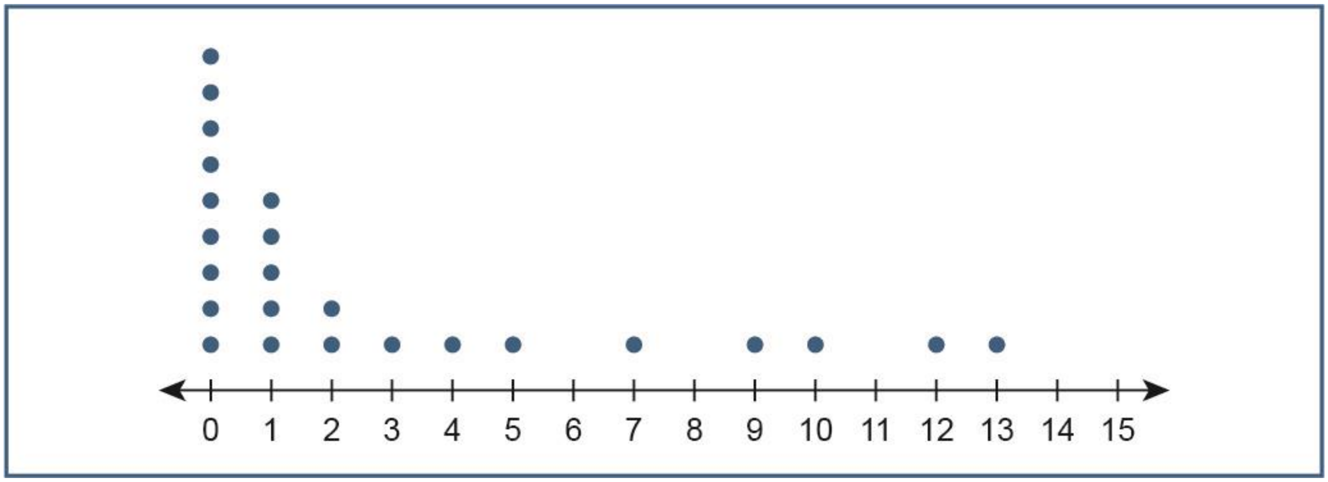
► Instructional Supports

- Ask students to explore how the shape of a distribution affects the mean and the median. Two examples are shown.
  - Give students the following data set.



The distribution is fairly symmetric. The mean is about 5.3 and the median is 5. Both values seem to accurately reflect the center of the distribution.

- Give students the following data set.



The distribution is skewed. The mean is 3 and the median is 1. There is a gap between the mean and median. In addition, the mean does not seem to reflect the majority of the data. There is only one data point that matches the mean, and the mean is not that close to the biggest cluster of points. The median, on the other hand, has five data points that match it and is next to the biggest cluster of points.

In general, if the distribution is symmetric, the median and mean are close together and both are accurate representations of the center. If the distribution is skewed, the median and mean are farther apart.

- Ask students to describe the benefits of using the mean as a measure of center and compare them to the benefits of using the median as a measure of center when analyzing data sets. When the distribution is symmetric, the difference between the mean and median is fairly small. However, the mean is usually preferred because it takes into account all data values. When the distribution is skewed or has outliers, the median is usually preferred because it better matches the largest clusters of data values. However, the mean is still sometimes preferred if, based on context, the rare, extreme values are important to consider.
- Give students a data set with context and ask them to determine whether the mean or median is a better measure of center given the context. Some examples are shown.

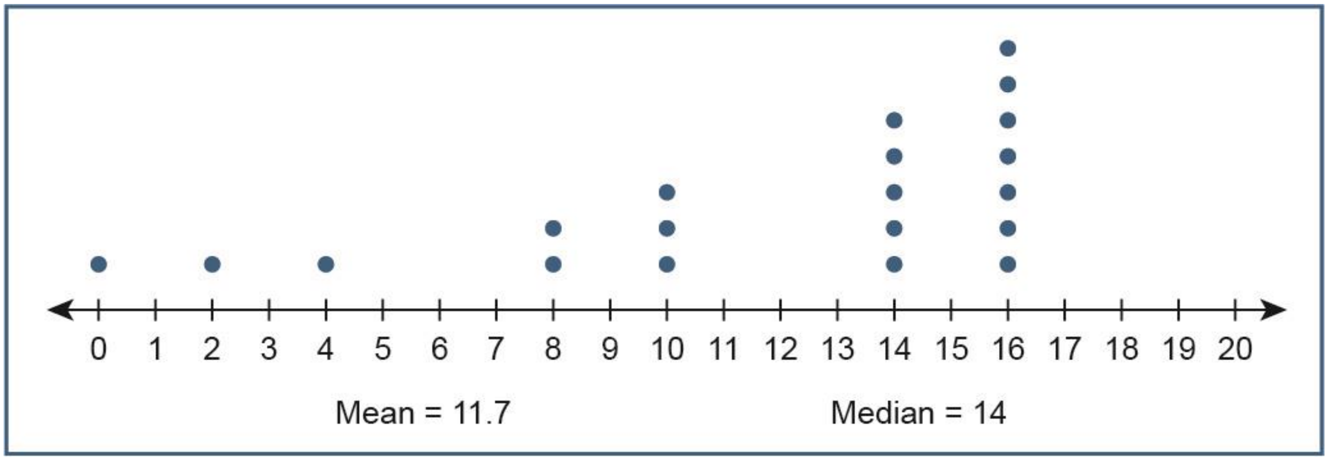
- Give students the following data, {3, 3, 4, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 8, 9, 35, 38, 55}, showing the number of minutes 20 museum visitors spent viewing a display. The mean of the data set is 10.7 and the median is 5. Students determine that the median is more representative of the data because the distribution is not symmetric. The mean suggests that the average museum visitor spent more than 10 minutes viewing the display even though only 3 of the 20 visitors spent that amount of time.

- Give students the following data set, {4, 5, 5, 6, 6, 6, 8, 8, 10, 13, 15, 20}, showing the amount of gasoline, in thousands of gallons, a gas station sold each month last year. The mean of the data set is about 8.83 and the median is 7. Although the distribution is not symmetric, the mean is the more appropriate choice for a measure of center. The extreme values should be considered because, unless the given distribution is not typical (e.g., last year was an unusual year), future years are likely to have a similar distribution.

- Ask students to explain which measure of variability is most appropriate for different data distributions. The mean absolute deviation is calculated using the mean. Therefore, it tends to be most appropriate in the same situations that the mean is most appropriate. Likewise, since the interquartile range is calculated using the median, it tends to be most appropriate in the same situations that the median is most appropriate. The range is almost never the most appropriate measure of spread because it has the same downside as the median in that it does not take into account all data values, and it has the same downside as the mean in that it is not always very representative of the data in the presence of asymmetry.

► Identifying and Addressing Common Misconceptions

- Some students may think that the mean is a more appropriate measure of center in a skewed distribution because it is closer to the center of the set of values in the data. Ask students to determine whether the mean or the median is a better representation of the data shown and why.



Some students may say that the mean is better because it is closer to the center of the dot plot. Ask students how many data points are closer to the mean and how many are closer to the median. Help students recognize that even though the mean is closer to the center of the numbers on the number line, the median is closer to the majority of the data values.

► Enrichment

- Give students a context and ask them to discuss whether the data set is likely to be skewed or symmetric and which measures of center and variability are most appropriate.
- Ask students to explain why a distribution that is skewed to the right has a mean that is greater than the median and why a distribution that is skewed to the left has a mean that is less than the median.

► Key Terms

mean, measure of center, measure of variability, median, mode, outlier