A1.1.1.1.1

▶ Module, Assessment Anchor, Anchor Descriptor

Operations and Linear Equations & Inequalities

Operations with Real Numbers and Expressions

Represent and/or use numbers in equivalent forms (e.g., integers, fractions, decimals, percents, square roots, and exponents).

▶ Eligible Content

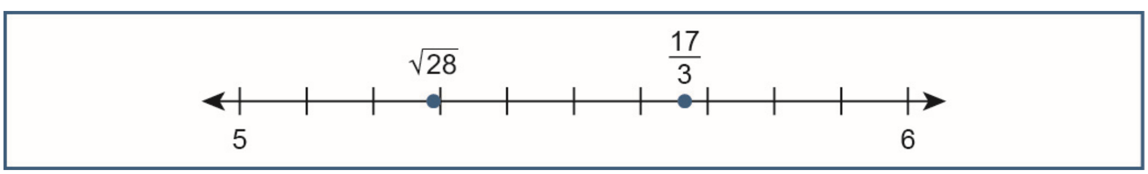
A1.1.1.1.1 Compare and/or order any real numbers.

Note: Rational and irrational may be mixed.

▶ Instructional Supports

- Ask students to compare two given real numbers. For example, give students the numbers $\sqrt{28}$ and $\frac{17}{3}$ to compare. Students can determine that $\sqrt{28}$ is between 5 and 6 because $(5)^2 < 28 < (6)^2$. Students can use trial and error to compare 28 with $(5.1)^2$, $(5.2)^2$, $(5.3)^2$, and so on up to $(5.9)^2$. By using that method, students can determine that $(5.2)^2 < 28 < (5.3)^2$, because $(5.2)^2 = 27.04$ and $(5.3)^2 = 28.09$. A calculator can be used to determine that $\sqrt{28} \approx 5.29$.

  Students can conclude that $\sqrt{28}$ falls between 5.2 and 5.3. Next, students should convert $\frac{17}{3}$ to a decimal, which yields $\frac{17}{3} = 5\frac{2}{3} = 5.\overline{6}$. Therefore, $\sqrt{28} < \frac{17}{3}$, as shown on the number line.



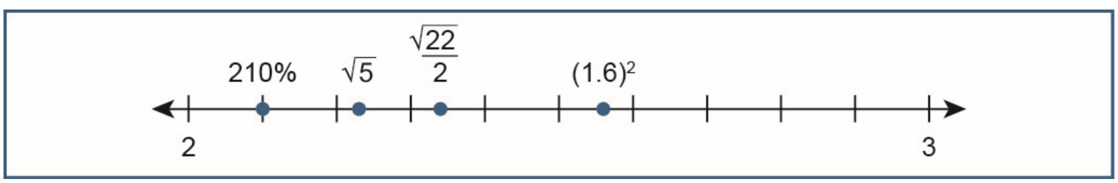
- Ask students to order a set of three or more real numbers from least to greatest or from greatest to least. For example, ask students to order the following list of numbers shown from least to greatest.

$$\sqrt{5},\ 210\%,\ (1.6)^2,\ \frac{\sqrt{22}}{2}$$

Students can start by determining decimal approximations of each number.

- Students can first determine that the decimal approximation of $\sqrt{5}$ is between 2 and 3 because $2^2 < 5 < 3^2$ Next, students can calculate $(2.1)^2$, $(2.2)^2$, and $(2.3)^2$ to determine that the value of $\sqrt{5}$ is between 2.2 and 2.3 because $(2.2)^2 < 5 < (2.3)^2$.
- Students can recall that a percentage means "parts per hundred." Therefore, $210\% = \frac{210}{100} = 2.10$.
- Students can determine that $(1.6)^2 = (1.6)(1.6) = 2.56$.
- Students can determine that $\frac{\sqrt{22}}{2}$ has an approximate decimal equivalence of 2.35. Students can start by determining that the value of $\sqrt{22}$ is between 4 and 5 because $4^2 < 22 < 5^2$. Further, by considering the tenths place, students can determine that the value of $\sqrt{22}$ is between 4.6 and 4.7 because $(4.6)^2 < 22 < (4.7)^2$. Because 22 is closer to $(4.7)^2 = 22.09$, students can approximate $\sqrt{22}$ as 4.69. Substituting this back into $\frac{\sqrt{22}}{2}$ results in $\frac{4.69}{2} = 2.345$. A calculator can be used to determine that $\frac{\sqrt{22}}{2} = 2.345$.

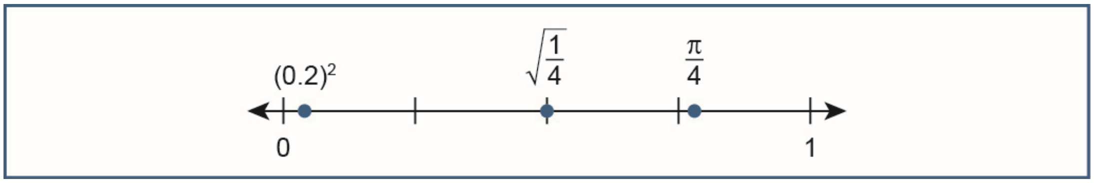
The number line shows the order of the decimal forms of the given numbers as 2.10, 2.24, 2.345, and 2.56 from least to greatest.



Therefore, the list of the given numbers from least to greatest is 210%, $\sqrt{5}$, $\frac{\sqrt{22}}{2}$, $(1.6)^2$.

- Ask students to plot a set of real numbers on a number line. For example, give students the numbers $(0.2)^2$, $\sqrt{\frac{1}{4}}$, and $\frac{\pi}{4}$ and ask them to plot the approximate values on a number line. Students can start by determining the decimal approximations of each number.
  - Students can determine that $(0.2)^2 = (0.2)(0.2) = 0.04$.
  - Students can determine that the rational number $\sqrt{\frac{1}{4}} = \frac{\sqrt{1}}{\sqrt{4}} = \frac{1}{2} = 0.5$.
  - Students can determine that the irrational number $\frac{\pi}{4} \approx 0.79$.

Therefore, students can determine the points should be plotted between 0 and 1 on a number line. Students can create a number line with values of 0, $\frac{1}{4}, \frac{1}{2}, \frac{3}{4},$ and 1, and approximate the locations on the number line of the decimal equivalents of each given number, as shown.



- Ask students to order from least to greatest a set of terms containing n and where $0 < n < 0.5$. For example, give the students the terms $2\sqrt{n}$, $\frac{n}{3}$, n^3, and $5\sqrt[3]{n}$.

  Because the given values for n can be any value greater than 0 but less than 0.5, students can use values such as 0.1 and 0.4 for n to determine a possible pattern. First, students can substitute 0.1 for n in each term to determine that $2\sqrt{0.1}$ is approximately 0.63, $\frac{0.1}{3}$ is equivalent to $0.0\overline{3}$, $(0.1)^3$ is equivalent to 0.001, and $5\sqrt[3]{0.1}$ is approximately 2.32. Because $0.001 < 0.0333 \ldots < 0.63 < 2.32$, the terms n^3, $\frac{n}{3}$, $2\sqrt{n}$, and $5\sqrt[3]{n}$ are ordered from least to greatest when $n = 0.1$. Students can then substitute 0.4 for n into each term to determine that $2\sqrt{0.4}$ is approximately 1.26, $\frac{0.4}{3}$ is equivalent to $0.1\overline{3}$, $(0.4)^3$ is equivalent to 0.064, and $5\sqrt[3]{0.4}$ is approximately 3.68. Therefore, the same order of n^3, $\frac{n}{3}$, $2\sqrt{n}$, and $5\sqrt[3]{n}$ represents the terms from least to greatest when $0 < n < 0.5$.

► Identifying and Addressing Common Misconceptions

- Students may use too few decimal values in a decimal approximation of a number when comparing or ordering several numbers. Ask students to compare $\frac{15}{4}$ and $\sqrt{15}$. Students may correctly approximate $\frac{15}{4}$ as 3.75 but incorrectly approximate $\sqrt{15}$ as 3.5 because $\sqrt{15}$ is between 3 and 4. This gives an incorrect answer of $\frac{15}{4} > \sqrt{15}$. Ask students to determine the two tenths that $\sqrt{15}$ is between by checking the tenths between 3 and 4. Students should determine that 15 is between $(3.8)^2$ and $(3.9)^2$. Therefore, $\frac{15}{4} < \sqrt{15}$.

- Students may approximate a square root by dividing the radicand by two. Ask students to approximate $\sqrt{52}$. Students who incorrectly approximate a square root may think that $\sqrt{52} = 26$. Remind students that the square root is the inverse of a square. So, $\sqrt{52}$ is the number that, when squared, results in 52. Since $7^2 = 49$ and $8^2 = 64$, the value of $\sqrt{52}$ is between 7 and 8.

- Students may believe a percentage is the number before the percent symbol when comparing a number to another number. Ask students to compare 1.13 and 98%. Students who have this misconception about percentages may respond that 1.13 < 98% because 1.13 < 98. Remind students that percent means "per hundred," so $98\% = \frac{98}{100} = 0.98$. Therefore, 1.13 > 98%.

▶ Enrichment

- Ask students to determine a real number between two real numbers. For example, ask students to determine an irrational square root that falls between 6.2 and $\frac{23}{5}$. Students can use n to represent the number. Since n must fall between the two numbers, n^2 must fall between $(6.2)^2$ and $\left(\frac{23}{5}\right)^2$. Students can express $(6.2)^2$ as a decimal by evaluating $(6.2)(6.2) = 38.44$. Students can express $\left(\frac{23}{5}\right)^2$ as a decimal by evaluating $\left(\frac{23}{5}\right)^2 = (4.6)^2 = (4.6)(4.6) = 21.16$. Therefore, n^2 falls between 38.44 and 21.16. Because the number needs to be irrational, any nonperfect square between these two values will work. So, n^2 could be 24, 27, 35, or any other nonperfect square between the two values. This means that $\sqrt{24}$, $\sqrt{27}$, and $\sqrt{35}$ are all numbers that fall between 6.2 and $\frac{23}{5}$.
- Ask students to predict which irrational number falls between two numbers with close proximity. For example, ask students to determine an irrational number that falls between $\frac{38}{5}$ and $\frac{39}{5}$.
- Ask students to investigate the product of two irrational numbers that results in a given rational number. For example, ask students to determine two irrational numbers that have a product of 8.

▶ Key Terms

approximate, irrational, percent, radicand, rational, real, square root

A1.1.1.1.2

▶ Module, Assessment Anchor, Anchor Descriptor

Operations and Linear Equations & Inequalities

Operations with Real Numbers and Expressions

Represent and/or use numbers in equivalent forms (e.g., integers, fractions, decimals, percents, square roots, and exponents).

▶ Eligible Content

A1.1.1.1.2 Simplify square roots (e.g., $\sqrt{24} = 2\sqrt{6}$).

▶ Instructional Supports

- Ask students to simplify the square root of a perfect square number. For example, ask students to simplify $\sqrt{121}$. Students should check the squares of numbers to recognize that $(11)^2 = 121$. The square root of a number represents the positive number that, when squared, results in the radicand. Thus, $\sqrt{121} = 11$.
- Ask students to prove that $\sqrt{m} \bullet \sqrt{n} = \sqrt{mn}$. For example, give students $\sqrt{144}$ and $\sqrt{16} \bullet \sqrt{9}$. Ask students to describe the relationship between 144 and 16 • 9. Students should explain that 144 = (16)(9). Then, ask students to determine whether the two expressions given are equivalent. Students should determine that $\sqrt{144} = 12$ because $(12)^2 = 144$ and that $\sqrt{16} \bullet \sqrt{9} = (4)(3) = 12$. Therefore, the two expressions are equivalent. Have students repeat the process with other pairs, such as $\sqrt{100}$ and $\sqrt{4} \bullet \sqrt{25}$.
- Ask students to simplify square roots of nonperfect squares. For example, ask students to simplify $\sqrt{90}$. Students should check for perfect squares that are factors of 90. Students should determine that 9 is a factor of 90 and a perfect square and that 90 = (9)(10) or 90 = (9)(5)(2). Therefore, $\sqrt{90} = \sqrt{9} \bullet \sqrt{5} \bullet \sqrt{2}$. Since $\sqrt{9} = 3$, the number $\sqrt{90} = 3\sqrt{10}$.
- Ask students to solve problems involving square roots containing a variable. For example, ask students to simplify the expression $\sqrt{8x}$ when $x = 3$. Students should substitute 3 in for x to

get $\sqrt{(8)(3)} = \sqrt{24}$. Next, students should determine that $\sqrt{24}$ has a perfect square factor of 4 and that $\sqrt{24} = \sqrt{(4)(6)}$. Since $\sqrt{4} = 2$, the number $\sqrt{24} = 2\sqrt{6}$.

▶ Identifying and Addressing Common Misconceptions

- Students may simplify a square root by dividing the radicand by two. Ask students to simplify $\sqrt{80}$. Some students may divide by 2 to get 40. Remind students of the definition of a square root and ask them if (40)(40) = 80. Have students check for perfect square numbers that are factors of 80 to simplify. Students should determine that 16 is the greatest perfect square factor and that $\sqrt{80} = \sqrt{(16)(5)} = 4\sqrt{5}$. Students may only partially simplify a square root. Ask them to simplify $\sqrt{72}$. Students incorrectly simplifying a square root may simplify this to $\sqrt{(9)(8)} = 3\sqrt{8}$ or $\sqrt{(4)(18)} = 2\sqrt{18}$. Ask students to consider the number remaining inside the radical symbol and decide whether there are any other perfect square factors. Once students determine that there are additional perfect square factors, students should determine that $\sqrt{72} = 3\sqrt{(4)(2)} = 6\sqrt{2}$ or $\sqrt{72} = 2\sqrt{(9)(2)} = 6\sqrt{2}$.
- Students may add the coefficients of a square root instead of multiplying them. Ask students to simplify $5\sqrt{63}$. Students who incorrectly add the coefficients of a square root may respond with $5\sqrt{(9)(7)} = 8\sqrt{7}$ by adding the 5 and the 3 instead of multiplying them. Ask students what operation is used between the coefficient and the radical. If the student struggles to identify the operation, compare the relationship to other expressions with hidden multiplication, such as $2x$ or 3(4 + 7). Students should see that the 5 and the 3 are being multiplied, so the expression $5\sqrt{63} = 5\sqrt{(9)(7)} = 5 \bullet 3\sqrt{7} = 15\sqrt{7}$.

▶ Enrichment

- Ask students to simplify the product of square roots. For example, ask students to simplify $3\sqrt{10} \bullet 2\sqrt{35}$. Students should use the commutative property to write the equivalent expression of $3 \bullet 2\ \sqrt{10} \bullet \sqrt{35}$. The whole numbers and the radicals can each be multiplied together to get $6\sqrt{350}$. The students should look for the greatest perfect square factor of 350, which in this case is 25. Since $350 = 25 \bullet 14$, the expression can be rewritten as $6\sqrt{25} \bullet \sqrt{14} = 6 \bullet 5\sqrt{14}$. Therefore, $3\sqrt{10} \bullet 2\sqrt{35} = 30\sqrt{14}$.

- Ask students to add and subtract square roots. For example, ask students to simplify $\sqrt{45} + \sqrt{72} - \sqrt{80} + \sqrt{18}$. Students should first simplify each square root to get $3\sqrt{5} + 6\sqrt{2} - 4\sqrt{5} + 3\sqrt{2}$. Students should then add the like terms to get $3\sqrt{2} - \sqrt{5}$.
- Ask students to simplify the square root of a quotient of perfect square numbers. For example, ask students to simplify $\sqrt{\frac{9}{64}}$. Students can begin by rewriting the expression as $\frac{\sqrt{9}}{\sqrt{64}}$. Since $(3)^2 = 9$, $\sqrt{9} = 3$. Since $(8)^2 = 64$, $\sqrt{64} = 8$. Therefore, $\sqrt{\frac{9}{64}} = \frac{3}{8}$.

► Key Terms

nonperfect square, perfect square, radical, radical symbol, radicand, square root

A1.1.1.2.1

▶ Module, Assessment Anchor, Anchor Descriptor

Operations and Linear Equations & Inequalities

Operations with Real Numbers and Expressions

Apply number theory concepts to show relationships between real numbers in problem-solving settings.

▶ Eligible Content

A1.1.1.2.1 Find the Greatest Common Factor (GCF) and/or the Least Common Multiple (LCM) for sets of monomials.

▶ Instructional Supports

- Ask students to determine the greatest common factor of three numbers. For example, consider the numbers 18, 42, and 108. Students can begin by expressing each number as the product of its prime factors.

  $18 = 2 \cdot 3 \cdot 3$

  $42 = 2 \cdot 3 \cdot 7$

  $108 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$

  The common factors between these numbers are 2 and 3. Therefore, the greatest common factor of the three numbers is 6.

- Ask students to determine the greatest common factor for a set of monomials. For example, give students $8x^3y^4$ and $18xy^2$. Students should start by writing out each expression as the product of its prime factors.

  $8x^3y^4 = 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y$

  $18xy^2 = 2 \cdot 3 \cdot 3 \cdot x \cdot y \cdot y$

  Then, students should highlight the common factor of the coefficients and of the variables common in both monomials.

  $8x^3y^4 = 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y$

$$18xy^2 = 2 \bullet 3 \bullet 3 \bullet x \bullet y \bullet y$$

The common factors are $2(x)(y)(y)$. Students should multiply these factors to get the greatest common factor of $2 \bullet x \bullet y \bullet y = 2xy^2$.

- Ask students to factor out the greatest common factor of a polynomial expression. For example, ask students to factor out the greatest common factor of the terms in the expression $16m^3n^2 - 10m^4n$. Students can identify the common factors of each term in the expression. The greatest common factor of 16 and 10 is 2. The greatest common factor of m^3 and m^4 is m^3. The greatest common factor of n^2 and n is n. These factors are multiplied together to get the greatest common factor of the two monomials, which is $2m^3n$. Factoring $2m^3n$ out of the first term, $16m^3n^2$, results in $8n$. Factoring $2m^3n$ out of the second term, $10m^4n$, results in $5m$. Therefore, $16m^3n^2 - 10m^4n = 2m^3n(8n - 5m)$.

- Ask students to identify the least common multiple of a set of monomials. For example, give students $3x$, $5xy^2$, and $6y^3$. The least common multiple needs to be determined for each of the coefficients and for each of the variables x and y. To determine the least common multiple of the coefficients, students can list the multiples of 3, 5, and 6.

  3: 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, …
  5: 5, 10, 15, 20, 25, 30, 35, 40, …
  6: 6, 12, 18, 24, 30, 36, 40, …

  The first multiple the coefficients have in common is 30. Therefore, the least common multiple of the numbers is 30. For like variables, the one with the greatest exponent is the least common multiple of that variable. The greatest exponent of x in any of the monomials is x, so x is the least common multiple for x. The greatest exponent of y in a monomial is y^3, so y^3 is the least common multiple of y. Multiplying these values together gives a least common multiple of $30xy^3$.

► Identifying and Addressing Common Misconceptions

- Students may not realize the greatest common factor can be one of the monomials. Ask students to determine the greatest common factor of $6x$ and $12x^2$. Some students may conclude that the greatest common factor is 1, 6, or x, not realizing that the entire $6x$ can be included as the factor. Ask students to express the two monomials as the product of their prime factors. Then, ask students to highlight the common factors and put them together to get the greatest common factor of $6x$. Remind students that an expression is a factor of itself and may be the greatest common factor.

- Students may divide exponents of variables when identifying the greatest common factor. Ask students to determine the greatest common factor of $20m^3n^2$ and $9m^6n^8$. Some students may conclude that the greatest common factor is m^2n^4. Ask students to write the two monomials as a product of their prime factors and highlight the common factors. Have the students compare the results with their answer.
- Some students may think that a monomial cannot have a multiple that involves a variable not in the original monomial. Ask students to determine the least common factor of $12a^3$ and $10a^2b^4$. Students with this misconception about monomials may answer $60a^3$. Ask students to consider whether $60a^3$ is a multiple of $10a^2b^4$. When they realize it isn't a multiple, ask them to describe what is missing. Then, ask students to consider whether $60a^3b^4$ is a multiple of $12a^3$. If they say it isn't, remind them that $12a^3$ can be multiplied by any variable, including b^4. Therefore, $60a^3b^4$ is a multiple of $12a^3$ because $12a^3 \bullet 5b^4 = 60a^3b^4$.

▶ Enrichment

- Ask students to determine a set of monomials that contains a given greatest common factor. For example, ask students to determine a set of monomials that contains a greatest common factor of $4x^2y^5$.
- Ask students to determine a set of monomials that contains a given least common multiple. For example, ask students to determine a set of monomials with a least common multiple of $4x^2y^5$.

▶ Key Terms

coefficient, greatest common factor, least common multiple, monomials, prime factors

A1.1.1.3.1

▶ Module, Assessment Anchor, Anchor Descriptor

Operations and Linear Equations & Inequalities

Operations with Real Numbers and Expressions

Use exponents, roots, and/or absolute values to solve problems.

▶ Eligible Content

A1.1.1.3.1 Simplify/evaluate expressions involving properties/laws of exponents, roots, and/or absolute values to solve problems.

Note: Exponents should be integers from −10 to 10.

▶ Instructional Supports

- Ask students to simplify expressions using the laws of exponents. For example, ask students to simplify $\frac{x^3y^{-2}}{(3x^2y^4)^2}$. Order of operations dictates that the terms within the parentheses are simplified first. Since the power of the parentheses represents the Power of a Power Rule, the outer exponent (2) is multiplied by all the exponents inside the parentheses. Students should recall that the 3 has an unwritten exponent of 1. Therefore, $\frac{x^3y^{-2}}{(3x^2y^4)^2} = \frac{x^3y^{-2}}{3^2x^4y^8}$. Next, students can apply the Quotient Rule, which subtracts the exponents that have the same bases in the numerator and denominator of a quotient, to get $\frac{x^3y^{-2}}{3^2x^4y^8} = \frac{x^{-1}y^{-10}}{3^2}$. Finally, the negative exponents can be moved to the denominator by using the Negative Exponent Rule. Since both variables in the numerator will move to the denominator, leaving nothing in the numerator, a 1 is placed there. The final answer is $\frac{1}{3^2x^3y^{10}}$ or $\frac{1}{9x^3y^{10}}$.
- Ask students to simplify square root expressions. For example, ask students to simplify $\sqrt{8x^4y^5}$. Students can begin by listing all the perfect square factors for each part of the expression. The number 4 is the greatest perfect square factor of 8. Thus, $\sqrt{8} = \sqrt{4} \bullet \sqrt{2}$. The expression x^4 is the greatest perfect square factor of x^4 because $(x^2)^2 = x^4$. The expression y^4 is the greatest perfect square factor

of y^5 because $(y^2)^2y = y^5$. Therefore, students can rewrite the entire expression as $\sqrt{4} \bullet \sqrt{2} \bullet \sqrt{x^4} \bullet \sqrt{y^4} \bullet \sqrt{y}$ and simplify it to $2x^2y^2\sqrt{2} \bullet \sqrt{y}$, which is equivalent to $2x^2y^2\sqrt{2y}$.

- Ask students to simplify cube root expressions. For example, ask students to simplify $\sqrt[3]{54x^5y^3}$. Students can begin by listing all the perfect cube factors of 54, x^5, and y^3. The number 27 is the greatest perfect cube factor of 54 because $3^3 \bullet 2 = 54$. The expression x^3 is the greatest perfect cube factor of x^5 because $x^3 \bullet x^2 = x^5$. Finally, the expression y^3 is the greatest perfect cube factor of y^3. Therefore, the expression can be rewritten as $\sqrt[3]{27} \bullet \sqrt[3]{2} \bullet \sqrt[3]{x^3} \bullet \sqrt[3]{x^2} \bullet \sqrt[3]{y^3}$. Simplifying each cube root of a perfect cube results in $3xy\sqrt[3]{2} \bullet \sqrt[3]{x^2}$, which is equivalent to $3xy\sqrt[3]{2x^2}$.

- Ask students to simplify absolute value expressions. For example, give students the expression $\frac{|4x-10x|}{|-3|}$. Students can begin by combining the like terms in $|4x - 10x|$ to get $|-6x|$, which results in the expression $\frac{|-6x|}{|-3|}$. Then, the absolute value of −6 and −3 can be determined. The absolute value of a number represents the distance that number is from 0, so $|-6| = 6$, and $|-3| = 3$. Thus, the expression can be simplified to $\frac{6|x|}{3}$, which is equivalent to $2|x|$.

- Ask students to evaluate algebraic expressions involving exponents, roots, and/or absolute values.
    - For example, ask students to evaluate $\sqrt[3]{2x^4y^3}$ when $x = 5$ and $y = 3$. Students can substitute the given values in for the variables to get $\sqrt[3]{2(5^4)(3^3)}$. This can be rewritten with perfect cube factors such as $\sqrt[3]{2} \bullet \sqrt[3]{5^3} \bullet \sqrt[3]{5} \bullet \sqrt[3]{3^3}$ for easier simplification. The expression simplifies to $(5)(3)\sqrt[3]{2}\bullet \sqrt[3]{5}$, which is equivalent to $15\sqrt[3]{10}$.
    - For another example, ask students to evaluate the expression $|3x - 4y| - |xy|$ when $x = -3$ and $y = -2$. Students can begin by substituting the values into the variables using parentheses. The expression is represented as $|3(-3) - 4(-2)| - |(-3)(-2)|$. Students should use order of operations to simplify each expression within absolute value bars to get $|-9 - (-8)| - |6|$, which equals $|-9 + 8| - |6|$ and then $|-1| - |6|$. Finally, the students should determine the absolute value of each number of the expression $|-1| - |6|$ to get $1 - 6 = -5$.

- Ask students to solve problems involving expressions with exponents, roots, and/or absolute value.
    - Give the students the following situation.

      If $\sqrt{3x} = 3\sqrt{2}$, what is the value of 2^x? Students can begin by reasoning through the value of x in the first relationship. The expression $3\sqrt{2}$ can be rewritten as $\sqrt{(3^2)(2)} = \sqrt{18}$. Since

$\sqrt{3x} = \sqrt{18}$, students can set it up as $3x = 18$, which can be used to determine that $x = 6$. Therefore, the value of 2^x when $x = 6$ is $2^6 = 64$.

- Give the students the following situation.

  The radius of circle A is $5\sqrt{6}$ inches. The diameter of circle B is $9\sqrt{7}$ inches. Which circle is the largest? To compare the values, students can begin by either comparing both lengths as radii or both lengths as diameters. The radius of the circle A can be doubled to determine the diameter: $2 \cdot 5\sqrt{6} = 10\sqrt{6}$ inches. Students should notice that the diameters can't be compared by inspection as they are written. Rewriting both diameters with the number entirely under the radicand will facilitate students in determining which diameter is greater.

  Circle A has a radius of $10\sqrt{6} = \sqrt{(10^2)(6)} = \sqrt{600}$.

  Circle B has a radius of $9\sqrt{7} = \sqrt{(9^2)(7)} = \sqrt{567}$.

  Therefore, circle A is the larger circle.

▶ Identifying and Addressing Common Misconceptions

- Students may not correctly follow the order of operations when evaluating an expression. Ask students to evaluate $\frac{(3x^2)^3}{2x}$. Some students may apply the Quotient Rule first to get $\left(\frac{3}{2}x\right)^3 = \frac{27x^3}{8}$. Ask students to recite the order of operations: parentheses, exponents, (before) multiplication/division. Then ask students to apply those steps to get $\frac{27x^6}{2x}$ and then $\frac{27x^5}{2}$.
- Students may simplify variables inside a root by taking the roots of the exponents on the variables. Ask students to simplify $\sqrt{x^5y^6}$. Students simplifying variables inside a root incorrectly may conclude that the root cannot be further simplified because 5 and 6 don't have any perfect square factors. Remind students of the Product Rule for Powers and ask them to use it to rewrite x^5 and y^6 in terms of squares. Students should get $\sqrt{(x^2)(x^2)(x)(y^2)(y^2)(y^2)}$, which can be simplified as $(x)(x)(y)(y)(y)\sqrt{x}$, which is equivalent to $x^2y^3\sqrt{x}$.
- Students may think that absolute value can be applied to each term inside of the absolute value bars. Ask students if the expression $|3x - y|$ can be simplified. Some students may simplify it to $3x + y$. Ask students to substitute $x = 3$ and $y = 7$ into both the original and (incorrectly) simplified expressions. Remind students to follow the order of operations when evaluating each expression. In doing so,

students will determine $|3(3) - 7| = |9 - 7| = 2$ and $3(3) + 7 = 9 + 7 = 16$. Students will see that the values of the expressions are not equal.

▶ Enrichment

- Ask students to write an expression involving roots, exponents, and/or absolute value that has a given value when certain values are substituted in. For example, ask students to write a cubed root expression that has a value of 35 when $x = 3$ and $y = 2$.
- Ask students to simplify quotients of roots. For example, ask students to simplify the expression $\frac{\sqrt[3]{8x^2y^5}}{\sqrt[3]{16x^{10}y^3}}$.
- Ask students to create tables for simple root, radical, or absolute value equations and to use the table to sketch a graph of the function. For example, ask students to create a table representing $y = \sqrt[3]{x}$. Encourage students to think through which numbers are best to substitute in for x, such as 1, 8, 27, and 64. Then ask students to plot the points on a coordinate plane.

▶ Key Terms

absolute value, exponent, power, product, quotient, radical, radicand, root

A1.1.1.4.1

▶ Module, Assessment Anchor, Anchor Descriptor

Operations and Linear Equations & Inequalities

Operations with Real Numbers and Expressions

Use estimation strategies in problem-solving situations.

▶ Eligible Content

A1.1.1.4.1 Use estimation to solve problems.

▶ Instructional Supports

- Ask students to solve a numeric estimation problem involving intervals of numbers. For example, give students the following problem: "Julissa and Li are buying yarn to make a blanket. Each ball of yarn costs between \$7.25 and \$12.30. They need to buy between 7 and 9 balls of yarn. They will split the total cost evenly. About how much money will each person spend on yarn?" Two options for students to approach solving the problem are shown.
    - One option is for students to determine that the average cost of a ball of yarn is \$9.78 and Julissa and Li will need an average of 8 balls of yarn. Students can then multiply those numbers together to get the total cost and divide by 2 to get the cost per person. This gives an estimated price per person of \$39.12.
    - Another option is for students to determine the least amount that could be spent and the greatest amount and then average those prices together. The least amount that could be spent is \$7.25 per ball of yarn for 7 balls of yarn, divided between two people. This is $(7.25 \bullet 7) \div 2 = \$25.38$. Similarly, the greatest amount that could be spent is \$12.30 per ball of yarn on 9 balls of yarn, divided by 2. This is represented by $(12.30 \bullet 9) \div 2 = \$55.35$. The average of these two outcomes would be a reasonable estimate. Therefore, another estimate is $(25.38 + 55.35) \div 2 = \$40.37$.
- Ask students to solve estimation problems involving one variable. For example, give students the following situation.

Arturo's dog eats between 200 and 250 grams of food for breakfast and between 300 and 325 grams of food for dinner. Arturo buys a bag of dog food that contains x kilograms of dog food. About how many days will the bag of dog food feed Arturo's dog, in terms of x?" Students can determine the average of the least amount and the greatest amount that Arturo's dog eats per day. The dog eats a minimum of 200 + 300 = 500 grams of food per day and a maximum of 250 + 325 = 575 grams of food per day. Averaging these results gives an average of (500 + 575) ÷ 2 = 537.5 grams of dog food each day. The amount of food in the bag is given in kilograms, so this will need to be converted to grams. There are 1,000 grams in each kilogram, so the bag contains 1,000x grams of dog food. To determine how many days the bag will last, students should divide the total mass of the bag of dog food by the daily amount, which gives 1,000x ÷ 537.5 ≈ 1.86x. Therefore, a bag with x kilograms of food can be expected to last around 1.86x days.

- Ask students to solve estimation problems involving one variable and make specific estimations by substituting values into the expression. For example, give students the following situation.

  Zane wants to estimate the minimum number of trees in a forest. He estimates the forest is 3.5 to 4 miles long and 2 to 2.5 miles wide. Write an expression to represent the approximate number of trees in a forest if he counts x trees in a 20-foot-by-20-foot section of the land. Then use the expression to predict the number of trees in the forest if he counted 10 trees in the 20-foot-by-20-foot section. Students can begin by determining the minimum estimated area of the forest by using 3.5 miles long and 2 miles wide. Units should be converted from miles to feet so they are consistent throughout. Since 1 mile is equivalent to 5,280 feet, 3.5 miles = 5,280 • 3.5 = 18,480 feet. Similarly, 2 miles = 5,280 • 2 = 10,560 feet. Therefore, the minimum approximation of the area of the forest is 18,500 • 10,600 = 196,100,000 square feet. The area of the section in which Zane counted trees is 20 • 20 = 400 square feet. Dividing 196,100,00 by 400 reveals that there are 490,250 in a 20-foot-by-20-foot sections of the entire forest. Since there are x trees in each of the 490,250 sections, the forest has a predicted minimum of 490,250x trees in it. Substituting x = 10 results in an estimated minimum of 4,902,500 trees in the forest.

► Identifying and Addressing Common Misconceptions

- Some students may use the lower bound or upper bound of an interval of values when the average is needed. Ask students to estimate the number of goals a soccer player will make this season if the player appears in 15–20 games and scores 1–2 goals per game. Students mistakenly using the lower or upper bound of an interval of values may multiply 15 games by 1 goal per game and calculate an answer of 15 goals. Remind students that when looking for a prediction, a measure of center is a good predictor, as opposed to using the lower or upper bound. In this case, the only measure of center we can use is the average, so a better prediction is 17.5 games times 1.5 goals per game, which results in 26.25 goals. This number rounds to a predicted 26 goals.

- Some students may guess values for variables or they may just omit variables entirely instead of leaving them in the expressions. Ask students to create an estimate for the price of an office party for x people, if \$20–\$50 is spent on decorations and \$5–\$10 per person is spent on food. Students using a variable incorrectly or mistakenly omitting one may give an answer of \$110 by guessing that 10 people will be invited to the party. For example, students might estimate \$35 spent on decorations as the average of \$20 and \$50; they might also estimate \$7.50 • 10 as the average of \$5 and \$10 times the 10 guests. Remind students that because an unknown is in the problem, the variable should stay in the answer, because it represents all the different numbers of people who attend the party instead of only 10 people. After students revise their answer to $(35 + 7.5x)$ dollars, have them substitute different values for x to represent the cost for different numbers of people.

► Enrichment

- Ask students to write an estimation problem involving intervals of numbers given a context and an expression modeling the estimation. For example, ask students to write an estimation problem involving the cost of ice cream for a picnic, where the total cost is $(212 + 4.6x)$ dollars.
- Ask students to draw comparisons using estimations for two different scenarios.

► Key Terms

approximate, average, estimate, interval, lower bound, minimum, maximum, upper bound, variable

A1.1.1.5.1

▶ Module, Assessment Anchor, Anchor Descriptor

Operations and Linear Equations & Inequalities

Operations with Real Numbers and Expressions

Simplify expressions involving polynomials.

▶ Eligible Content

A1.1.1.5.1 Add, subtract, and/or multiply polynomial expressions (express answers in simplest form).

Note: Nothing larger than a binomial multiplied by a trinomial.

▶ Instructional Supports

- Ask students to add, subtract, or multiply polynomial expressions. Several examples are provided.
    - Ask students to simplify $(5x^2 + 3x - 8) + (2x^2 - 4x + 1)$. Since there are no numbers (positive or negative) to distribute to either polynomial, the parentheses can be removed. Students can then use the commutative property to put like terms next to one another, from the greatest to the least powers of exponents: $5x^2 + 2x^2 + 3x - 4x - 8 + 1$. Students should then combine like terms. When combining like terms, the coefficients/constants are either added or subtracted while the variables remain the same: $5x^2 + 2x^2 = 7x^2$, $3x - 4x = -x$, and $-8 + 1 = -7$. Therefore, the expression is equivalent to $7x^2 + -x + -7$ or $7x^2 - x - 7$.
    - Ask students to simplify $(7x^3 - 2x^2 + 5x - 3) - (6x^3 - 4x + 3)$. Since $(6x^3 - 4x + 3)$ is being subtracted from $(7x^3 - 2x^2 + 5x - 3)$, the parentheses cannot be removed without first applying the minus sign to each term in $(6x^3 - 4x + 3)$. This will cause every term in the polynomial to have a sign change, so $-(6x^3 - 4x + 3) = -6x^3 + 4x - 3$. Students can now combine like terms by adding or subtracting the coefficients/constants while the variables remain the same: $7x^3 - 6x^3 = x^3$, $5x + 4x = 9x$, and $-3 - 3 = -6$. The term $-2x^2$ remains the same, because there is no other like term. Therefore, the given expression is equivalent to $x^3 + -2x^2 + 9x + -6$ or $x^3 - 2x^2 + 9x - 6$.
    - Ask students to simplify $(9x + 1)(2x - 7)$. Both terms of $(2x - 7)$ will need to be multiplied by both terms of $(9x + 1)$. The $9x$ will be multiplied by both the $2x$ and the -7 and the 1 will also

be multiplied by both the $2x$ and the -7. This means that $(9x + 1)(2x - 7)$ yields $(9x)(2x) + (9x)(-7) + (1)(2x) + (1)(-7)$. This results in $18x^2 - 63x + 2x - 7$; the students can then combine the like terms of $-63x$ and $2x$ as $-61x$. Therefore, the given expression is equivalent to $18x^2 - 61x - 7$.

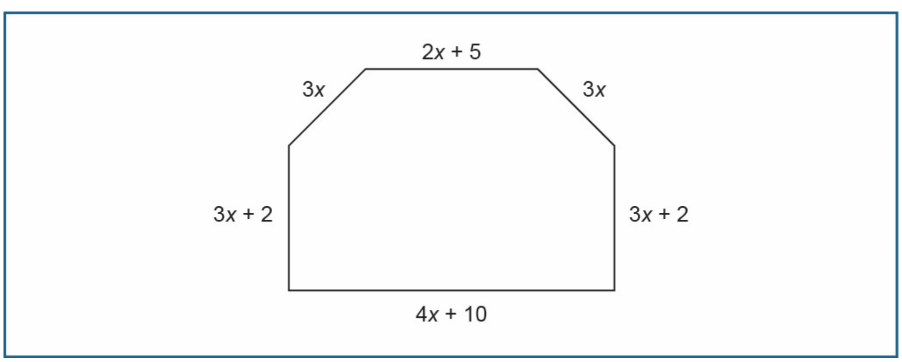
- Ask students to simplify a polynomial expression that involves two or more operations with polynomials. For example, ask students to simplify $(3y - 2)(4y + 1) - 5(2y + 7)$. The first product, $(3y - 2)(4y + 1)$, can be multiplied as $3y(4y + 1) + -2(4y + 1) = 12y^2 + 3y - 8y - 2 = 12y^2 - 5y - 2$. Similarly, $-5(2y + 7) = -10y - 35$. Combining these results gives the expression $12y^2 - 5y - 2 - 10y - 35$. Finally, students can group and combine like terms to get $12y^2 - 5y - 10y - 2 - 35 = 12y^2 - 15y - 37$.

- Ask students to determine a missing coefficient in a polynomial expression involving addition, subtraction, or multiplication. For example, ask students to determine the value of n in the equation $(2x^2 + 5)(3x^3 - nx + 1) = 6x^5 + 3x^3 + 2x^2 - 30x + 5$. To determine the value of n, first simplify the left side of the equation by distributing:

$$\begin{aligned}(2x^2 + 5)(3x^3 - nx + 1) &= 2x^2(3x^3 - nx + 1) + 5(3x^3 - nx + 1)\\ &= 6x^5 - 2nx^3 + 2x^2 + 15x^3 - 5nx + 5\\ &= 6x^5 - 2nx^3 + 15x^3 + 2x^2 - 5nx + 5\end{aligned}$$

  Since both sides of the equation must be equal, students can first determine that the x^3 terms of $-2nx^3 + 15x^3$ must be equal to $3x^3$. All 3 terms can be divided by x^3 to set the expressions equal, leaving $-2n + 15 = 3$, which yields $n = 6$. This same process can be used with the x terms of $-5nx$ being equal to $-30x$. The equation $-5n = -30$ yields the same result: $n = 6$.

- Ask students to write a simplified polynomial expression to model a real-world situation. For example, ask students to represent the perimeter, in units, of the following figure as a fully simplified polynomial.



The perimeter of a figure can be determined by adding all the side lengths. Therefore, the perimeter, in units, is $(2x + 5) + (3x) + (3x + 2) + (4x + 10) + (3x + 2) + (3x)$. The students can rearrange the expression by grouping like terms: $2x + 3x + 3x + 4x + 3x + 3x + 5 + 2 + 10 + 2$. Combining like terms yields $(18x + 19)$ units.

▶ Identifying and Addressing Common Misconceptions

- Students may add a variable's exponents when combining like terms. Ask students to add $(7x^3 + 9x^2)$ and $(2x^3 + 8x^2)$. Some students may respond with a sum of $9x^6 + 17x^4$. Remind students that each power of a variable is a different term, and the variable part of a term remains the same; only the coefficients change. Have students verbally describe the terms as they combine them. Students could respond with "7 x-cubed plus 2 more x-cubed gives a total of 9 x-cubed."
- Students may not multiply all terms when multiplying polynomials. Ask students to simplify $(3x + 9)(2x + 4)$. Some students may respond with $6x^2 + 36$. Remind students that the $3x$ should be distributed to both terms of $(2x + 4)$, not just to the $2x$. Similarly, the 9 should be distributed to both terms of $(2x + 4)$, not just to the 4. Have students rewrite the expression as $3x(2x + 4) + 9(2x + 4)$ and then distribute.
- Students may not subtract every term when subtracting a polynomial. Ask students to simplify $(7x^2 – 4x) – (3x^2 + 8x)$. Students may subtract the $3x^2$ term and add the $8x$ term, resulting in $7x^2 – 4x – 3x^2 + 8x = 4x^2 + 4x$. Remind students that both terms of $(3x^2 + 8x)$ are being subtracted, which means the subtraction symbol needs to apply to both terms. Have students draw arrows to represent the distribution of the factor of $–1$ to both the $3x^2$ and $8x$ terms.

▶ Enrichment

- Ask students to multiply a trinomial by a trinomial. For example, ask students to multiply $(5x^2 + 6x – 8)(2x^2 – 3x + 1)$.
- Ask students to determine two polynomials that add, subtract, or multiply to result in a given polynomial. For example, ask students to determine two binomials that result in the sum of $8x^3 + 5x – 11$.

▶ Key Terms

add, binomial, coefficient, constant, exponent, factor, like terms, polynomial, power, subtract, trinomial, variable

A1.1.1.5.2

▶ Module, Assessment Anchor, Anchor Descriptor

Operations and Linear Equations & Inequalities

Operations with Real Numbers and Expressions

Simplify expressions involving polynomials.

▶ Eligible Content

A1.1.1.5.2 Factor algebraic expressions, including difference of squares and trinomials.

Note: Trinomials are limited to the form $ax^2 + bx + c$ where a is equal to 1 after factoring out all monomial factors.

▶ Instructional Supports

- Ask students to describe the relationship between the factored form and the standard form of a trinomial with a leading coefficient of 1. A leading coefficient of 1 means that the term with the highest power in a polynomial has a coefficient of 1. For example, ask students to compare the equivalent expressions $(x + 7)(x + 3)$ and the trinomial $x^2 + 10x + 21$. Students should note that the sum of the numbers 7 and 3 is equal to the coefficient of the x term, and the product of 7 and 3 is equal to the constant 21. Ask students to multiply $(x + 7)(x + 3)$ as $x^2 + 3x + 7x + 21$. The 7 and 3 are both multiplied by x and the like terms combine to form the middle term. The 7 and 3 are also multiplied to form the constant.
- Ask students to factor a trinomial with a leading coefficient of 1.
  - For example, ask students to factor $x^2 + 5x + 6$. Students should recall the relationship between the numbers in the factored form and the numbers in the standard form, as described in the previous example. Therefore, students need to determine the pair of numbers needed to fill into $(x + \_\_)(x + \_\_)$ that will produce a sum of 5 and a product of 6. Because both the sum and the product are positive, the two numbers must be positive. Students can begin by checking the numbers 1 and 6; $(1)(6) = 6$, but $1 + 6 = 7$, so these numbers are not the solution. Students can then try 3 and 2; $(3)(2) = 6$ and $3 + 2 = 5$, so these are the numbers to use. Therefore, the factored form of the trinomial $x^2 + 5x + 6$ is $(x + 3)(x + 2)$ or $(x + 2)(x + 3)$.

- For another example, ask students to factor $x^2 + 8x - 9$. This time, students should look for a pair of numbers that have a product of −9 and a sum of 8. Since multiplying the numbers needs to result in a negative number, one of the numbers must be negative. Since the numbers need to have a sum that is positive, the factor with the greater absolute value must also be positive. Students can try different combinations, as shown in the table.

Number Pair	Product	Sum
−3 and 3	−9	0
3 and −3	−9	0
−9 and 1	−9	−8
9 and −1	−9	8

  The numbers in the last row meet the requirements, so they are used in the factored form. Therefore, the expression $x^2 + 8x - 9$ is equivalent to $(x + 9)(x - 1)$ or $(x - 1)(x + 9)$.

- Ask students to factor $x^2 - 7x + 12$. Again, students should look for a pair of numbers that produce a product of 12 and a sum of −7. Since the constant is positive and the coefficient of the x term is negative, both numbers must be negative. Students should check the following factor pairs of 12, when both numbers are negative.

  −12 and −1: sum to −13

  −3 and −4: sum to −7

  −6 and −2: sum to −8

  Since −3 and −4 meet the requirements, they are used in the factored form $(x + -3)(x + -4)$ or $(x - 3)(x - 4)$. The order of the two factors can be switched.

- Ask students to factor an expression that requires factoring out the greatest common factor. For example, ask students to factor $2x^3 - 4x^2 - 6x$. Students should note that the leading coefficient is not 1, so this trinomial cannot be factored using the previous methods. However, students should notice that each of the terms has at least 1 x and that each of the coefficients is a multiple of 2.

Students should determine that $2x$ can be factored out from each term and rewrite the expression as $2x(x^2 - 2x - 3)$. Students should notice that the trinomial $(x^2 - 2x - 3)$ has a leading coefficient of 1. Students should now look for a pair of numbers that produce a product of −3 and a sum of −2. The only two pairs of numbers that produce a product of −3 when multiplied are −3 and 1 OR 3 and −1. Of these numbers, −3 and 1 have a sum of −2. Therefore, the final factored form of the given expression is $2x(x - 3)(x + 1)$.

- Ask students to compare an expression with its factored form. For example, ask students to compare $4x^2 - 81$ and $(2x + 9)(2x - 9)$. Students should note that the first term in each factor is the square root of $4x^2$ and that the second term in each factor is the square root of 81. Students should also note that one factor has a plus sign and the other has a minus sign. Ask students to multiply $(2x + 9)(2x - 9)$, which results in $4x^2 - 18x + 18x - 81$. Students should note that the x terms will result in $0x$, which is consistent with the given expression of $4x^2 - 81$ not having a middle term. The $2x$ and $2x$ are multiplied to form $4x^2$. The −9 and 9 are multiplied to form the constant −81. Give students additional examples to help them develop the rule for factoring expressions that are differences of squares.
- Ask students to factor the expression $9x^2 - 16$. Students should note that both terms in the difference are perfect squares, so the rule for difference of squares can be applied. The square root of $9x^2$ is $3x$, and the square root of 16 is 4. Therefore, the factored form is $(3x - 4)(3x + 4)$.

- Ask students to finish factoring a trinomial with one factor given and/or to solve for an unknown by using factoring techniques. For example, ask students to determine the value of b in the trinomial $x^2 + bx - 12$ when $(x - 6)$ is a factor of the trinomial. Students should begin by noting that if $(x - 6)$ is a factor, −6 must be a factor of −12, with 2 being the other factor. This means that the other factor of the expression is $(x + 2)$, and the factored form is $(x - 6)(x + 2)$. Students can multiply the factors to determine the value of b for the coefficient of x. For example, $(x - 6)(x + 2) = x(x + 2) - 6(x + 2) = x^2 + 2x - 6x - 12$, which results in $x^2 - 4x - 12$. Therefore, the value of b is the coefficient of the x term, −4.

► Identifying and Addressing Common Misconceptions

- Students may mix up the signs in their factor pairs when factoring a trinomial that has one or two negative values. Ask students to factor $x^2 - 7x - 8$. Students mixing up signs may respond with $(x - 1)(x + 8)$. Ask students to multiply those factors and compare the results to the original trinomial.

The students should notice that the x terms of $-1x$ and $8x$ have a sum of $7x$ instead of $-7x$. Direct them to the beginning to determine which pair of numbers that has a product of –8 and a sum of 7.

- Students may attempt to factor a trinomial by using two numbers with a product equal to the constant of the trinomial and a sum equal to the coefficient of its x term but without first dividing out a common factor to get a leading coefficient of 1. Ask students to factor $3x^2 - 32x + 60$. Students factoring trinomials incorrectly may respond with $(x - 30)(x - 2)$ because $(-30)(-2) = 60$ and $-30 + -2 = -32$. Remind students that the strategy that uses products and sums applies only when the leading coefficient of the trinomial is 1. Ask students whether it is possible to get a leading coefficient of 1 in this trinomial. Students should try to factor out a 3 from the trinomial $3x^2 - 32x + 60$.

- Students may think a difference of squares cannot be factored because they've tried to determine the square root of the negative constant instead of just the constant. Ask students to factor $x^2 - 16$. Some students may believe this isn't possible because the square root of -16 doesn't exist. Remind students that it is the **difference** of two squares, so the negative sign is represented in the difference, which is between the two squares. In this case, the squares are x^2 and 16, not x^2 and -16.

► Enrichment

- Ask students to write all the quadratic trinomials that can be factored with a given constant value. For example, ask students to write all the quadratic trinomials with a leading coefficient of 1 and a constant of 8 that can be factored. Challenge students to do this without writing the expression in factored form.
- Ask students to factor the sum and difference of cubes, given the formulas.
- Ask students to explore factoring trinomials with a leading coefficient not equal to 1. Ask students to compare the relationship between $(2x + 5)(3x + 2)$ and $6x^2 + 19x + 10$. Students should note that while the numbers 5 and 2 produce a product of 10, they do not have a sum of 19. This is because the relationship is slightly different when the leading coefficient is not 1. Give students several of these trinomials in their factored forms and ask them to explore the relationships. Then, give students their own trinomial to factor.

► Key Terms

binomial, coefficient, constant, difference of squares, expression, factor, factored form, leading coefficient, monomial, product, square root, standard form, sum, term, trinomial

A1.1.1.5.3

▶ Module, Assessment Anchor, Anchor Descriptor

Operations and Linear Equations & Inequalities

Operations with Real Numbers and Expressions

Simplify expressions involving polynomials.

▶ Eligible Content

A1.1.1.5.3 Simplify/reduce a rational algebraic expression.

▶ Instructional Supports

- Ask students to simplify a rational expression formed by two monomials. For example, ask students to simplify $\frac{5m^3n^2}{15mn^2}$ when $m \neq 0$ and $n \neq 0$. Restricted values such as $m \neq 0$ and $n \neq 0$ are the numbers that make the denominator of a rational expression or equation equal to zero, which is an undefined operation. These values are excluded from the domain of an expression because they would make the expression invalid. Because 5 and 15 are both multiples of 5, a factor of 5 can be reduced out of each number, which simplifies $\frac{5}{15}$ to $\frac{1}{3}$. The Exponent Quotient Rule states that the exponents are subtracted when dividing like bases. This means that $\frac{m^3}{m} = \frac{m^2}{1}$ and $\frac{n^2}{n^2} = \frac{n^0}{1}$. The Zero Exponent Rule states that any number or variable to the 0th power is equal to 1, so $\frac{n^0}{1} = 1$. Therefore, $\frac{5m^3n^2}{15mn^2} = \frac{m^2}{3}$.
- Ask students to simplify a rational algebraic expression that is in factored form. For example, ask students to simplify $\frac{3x(2x-5)(x+1)}{6x^2(x+1)(x+2)}$ when $x \neq -2, -1, 0$. Since the numerator and the denominator are both in factored form, the rational expression is ready to be simplified where possible. Since the factor $(x + 1)$ is in both the numerator and the denominator, that factor can be canceled, leaving $\frac{3x(2x-5)}{6x^2(x+2)}$. Note that the order of the factors does not matter, since $(x + 1)$ is the third factor in the numerator but the second factor in the denominator. While $3x$ and $6x^2$ are not exactly the same, they can be simplified, because $6x^2 = (3x)(2x)$. This means that $3x$ is a factor in the numerator and the

denominator and can be canceled out, leaving $\frac{2x-5}{2x(x+2)}$. There are no other common factors remaining; therefore, $\frac{2x-5}{2x(x+2)}$ is the completely simplified form of the given rational expression.

- Ask students to simplify a rational algebraic expression that first requires factoring.
  - For example, ask students to simplify the expression $\frac{x^2+9x+18}{x^3+3x^2}$ when $x \neq -3$ and 0. Rational expressions can be simplified only when they are in factored form, as only factors can be canceled out. With the expression written as it is, nothing can be canceled. However, both the numerator and the denominator can be factored. The trinomial $x^2 + 9x + 18$ can be factored as $(x + 6)(x + 3)$ because $(6)(3) = 18$ and $6 + 3 = 9$. The terms in the denominator, x^3 and $3x^2$, have a common factor of x^2, which will result in the factored form of $x^2(x + 3)$. So, $\frac{x^2+9x+18}{x^3+3x^2} = \frac{(x+6)(x+3)}{x^2(x+3)}$. Since $(x + 3)$ is a factor in both the numerator and the denominator, it can be canceled out, resulting in $\frac{x+6}{x^2}$. While it may appear that an x can be reduced from x and x^2, it cannot be canceled, because the x in the numerator is not a factor.
  - For another example, ask students to simplify the expression $\frac{x(x-2)-15}{2x^2-50}$ when $x \neq \pm 5$. Rational expressions can be simplified only when the numerator and the denominator are both written as a product of factors. This means the first step is to factor the numerator and the denominator, as shown:

$$x(x-2) - 15 = x^2 - 2x - 15 = (x-5)(x+3)$$

$$2x^2 - 50 = 2(x^2 - 25) = 2(x-5)(x+5)$$

    Substituting these factored forms into the expression results in $\frac{(x-5)(x+3)}{2(x-5)(x+5)}$. Now any common factors can be canceled out. Since $(x - 5)$ is in both the numerator and the denominator, that factor can be canceled out to leave $\frac{x+3}{2(x+5)}$. Since there are no remaining common factors, the expression $\frac{x+3}{2(x+5)}$ is the completely simplified form of the given expression.

- Ask students to identify a missing value in a rational algebraic expression. For example, ask students to determine the value of n in $\frac{x^2+nx-3}{x^2-1}$ when $x \neq \pm 1$, given that the expression can be completely simplified to $\frac{x+3}{x+1}$. Since the denominator is a difference of squares with factors $(x - 1)$ and $(x + 1)$, then $\frac{x^2+nx-3}{x^2-1} = \frac{x^2+nx-3}{(x-1)(x+1)}$. Since we were given $\frac{x+3}{x+1}$ as the completely simplified form, and the factor of $(x - 1)$ is no longer in the denominator, the numerator must have also had a factor of $(x + 1)$, with both factors then being canceled. The factor $(x + 3)$ remains in the numerator. The expression must have been $\frac{(x+3)(x-1)}{(x-1)(x+1)}$. The numerator, $(x + 3)(x - 1)$, is equivalent to $x^2 + nx - 3$ when multiplied. Since $(x + 3)(x - 1) = x^2 - x + 3x - 3 = x^2 + 2x - 3$, the coefficient of the x terms is equivalent and $n = 2$.

▶ Identifying and Addressing Common Misconceptions

- Students may cancel out terms instead of factors. Ask students to simplify $\frac{x^2+5x+4}{x^2-16}$ when $x \neq \pm 4$. Students who mistakenly cancel out the x^2 terms and reduce the 4 and the 16 to 1 over 4 to get $\frac{5x+1}{-4}$. Remind students that only multiplied factors can be canceled out; since this expression is not written as a product of factors, nothing can be simplified. Ask students to factor the numerator and the denominator first and then cancel out the common factors.

- Students may cancel out parts of factors instead of the entire factor. Ask students to simplify $\frac{(x-8)(x+3)}{(x-8)(x+6)}$ when $x \neq -6$ and 8. Some students may correctly cancel the $(x - 8)$ factors but then incorrectly reduce the 3 and the 6 to a 1 and a 2 to get $\frac{x+1}{x+2}$. Remind students that only entire factors can be reduced, not parts of factors. Ask students if they canceled an entire factor when they reduced 3 and 6. When students realize they reduced parts that can't be reduced, they should also realize that $\frac{x+3}{x+6}$ does not have any common factors that can be canceled.

- Students may not completely simplify a rational algebraic expression. Ask students to reduce $\frac{(2x+6)(x-4)}{(x+3)(x-4)}$ when $x \neq -3$ and 4. Some students may think that $\frac{2x+6}{x+3}$ is completely simplified. Remind students to always double-check that the expressions in the numerator and the denominator are completely factored. Students should notice that a 2 can be factored out of the expression $2x + 6$ in the numerator, which yields $2(x + 3)$ and allows the expression to be reduced even further. Therefore, $\frac{2(x+3)}{(x+3)} = 2$.

► Enrichment

- Ask students to create a rational algebraic expression that simplifies to a given expression. For example, ask students to write a rational expression including restricted values with a trinomial in the numerator and another in the denominator. The expression should simplify to $x + 4$.
- Ask students to multiply or divide rational algebraic expressions. For example, ask students to simplify $\frac{3x(x^2 - 16)}{9x^3 + 27x^2} \div \frac{x^2 - 2x - 24}{x^2 + 10x + 21}$ when $x = -4, -3, 0$, and 6.
- Ask students to identify and explain the restricted values for a rational algebraic expression. For example, ask students to complete the restricted values for the expression $\frac{x^2 - 49}{3x^3 + 27x^2 + 42x}$ when $x \neq$ __. Note that the restricted values must be listed even if that factor ends up being "canceled out."

► Key Terms

cancel out, common factor, completely simplified, denominator, exponent base, exponent quotient rule, factor, factored form, numerator, rational algebraic expression, reduce, restricted values, term, zero exponent rule

A1.1.2.1.1

▶ Module, Assessment Anchor, Anchor Descriptor

Operations and Linear Equations & Inequalities

Linear Equations

Write, solve, and/or graph linear equations using various methods.

▶ Eligible Content

A1.1.2.1.1 Write, solve, and/or apply a linear equation (including problem situations).

▶ Instructional Supports

- Ask students to solve a multistep linear equation. For example, ask students to solve the equation $\frac{1}{2}(3x-5)=2x-\frac{1}{4}$. The steps for solving for *x*, along with the justification for each step, are provided. (Note justification for each step in solving an equation is emphasized in standard A1.1.2.1.2. The justifications are given here to provide guidance in solving the equation.)

$\frac{1}{2}(3x-5)=2x-\frac{1}{4}$	*Given*
$\frac{3}{2}x-\frac{5}{2}=2x-\frac{1}{4}$	*Distributive property*
$\frac{3}{2}x-\frac{5}{2}+(-2x)=2x-\frac{1}{4}+(-2x)$	*Addition property of equality*
$-\frac{1}{2}x-\frac{5}{2}=-\frac{1}{4}$	*Combining like terms*
$-\frac{1}{2}x-\frac{5}{2}+\frac{5}{2}=-\frac{1}{4}+\frac{5}{2}$	*Addition property of equality*
$-\frac{1}{2}x=\frac{9}{4}$	*Combining like terms*
$-\frac{2}{1}\left(-\frac{1}{2}x\right)=-\frac{2}{1}\left(\frac{9}{4}\right)$	*Multiplication property of equality*
$x=-\frac{18}{4}$	*Multiplicative inverse and simplifying*

$x = -\frac{9}{2}$ Reducing the fraction

- Ask students to use an equation to solve a real-world problem. For example, give students the following situation.

  The number of pages, y, Leighton has remaining to read of her book after x hours of reading is modeled by the equation $y = 400 – 25x$. Leighton has 275 pages remaining to read of her book. How many hours has she read? Students can determine that the number 275 is the value for y since it represents the number of pages remaining. Students should substitute 275 into the equation for y to get the equation $275 = 400 – 25x$. Next, students should add −400 to both sides of the equation and simplify to get $-125 = -25x$. Multiplying both sides of the equation by $-\frac{1}{25}$ and simplifying will result in $5 = x$. Therefore, Leighton has read for 5 hours.

- Ask students to write a linear equation that models a real-world situation.
  - For example, give students the following situation.

    At a pizza shop, a cheese pizza costs x dollars. Additional toppings can be added for $1.50 each. A group of friends pay $55.50 for 1 cheese pizza, 2 one-topping pizzas, and 3 two-topping pizzas. Write an equation that can be used to solve for x, the cost of a cheese pizza. The cost of a cheese pizza is x. The cost of the two one-topping pizzas is $2(x + 1.50)$, since there are 2 cheese pizzas plus the extra fee of $1.50 for the 1 extra topping on each pizza. The cost of the 3 two-topping pizzas is $3(x + 3)$ because there are 3 cheese pizzas plus the extra fee of $3 for the two extra toppings on each pizza. All these pizzas cost $55.50 in total. The following equation represents this situation: $x + 2(x + 1.50) + 3(x + 3) = 55.50$.
  - For another example, give students the following situation.

    Bryn is driving on a road trip. She has already driven 50 miles. She will drive 65 miles per hour on the highway for x hours. Write an equation to represent the total distance, y, in miles Bryn has traveled after x hours of driving on the highway. Students should determine that the distance Bryn travels depends on the number of hours she has been driving. Bryn has already traveled 50 miles. The expression $65x$ represents the additional miles she will drive on the highway each hour. The total distance (y) is equal to the sum of 50 and $65x$, and the equation that represents this situation is $y = 65x + 50$.

- Ask students to write and solve a linear equation in a real-world situation. For example, give students the following situation.

Carlos paid \$100 for two new pairs of shoes. He used a 20%-off coupon for each pair of shoes, and there was no tax on the shoes. The price of one of the pairs of shoes was \$48 before the coupon was applied. What was the price, x, for the other pair of shoes, before the coupon was applied? Students should determine that Carlos paid 80% of the price of each pair of shoes because of the 20%-off coupon. Carlos paid 0.8(48) for the pair of shoes priced at \$48 and $0.8(x)$ for the other pair of shoes. The total price for the two pairs of shoes was \$100. Students can represent this situation with the equation $0.8(48) + 0.8x = 100$. Next, students can simplify the equation as $38.4 + 0.8x = 100$. Students should add −38.4 to both sides of the equation, resulting in $0.8x = 61.6$. Students should then multiply both sides of the equation by $\frac{1}{.8}$ or $\frac{10}{8}$, which yields $x = \$77$. Therefore, the other pair of shoes cost \$77 before the coupon was applied.

► Identifying and Addressing Common Misconceptions

- Some students may combine variables across the equal sign. Ask students to solve the equation $5x + 8 = 2x + 14$. Students mistakenly combining variables across the equal sign may add $5x + 2x$ and $14 + 8$ to get $7x = 22$. Remind students that terms cannot be combined across the equal sign. Ask students how they could move the $2x$ from the right side of the equation to the other side of the equation. Help students see that subtracting $2x$ (or adding $-2x$) moves the variable from the right side of the equation to the left side and adding -8 moves the constants from the left side of the equation to the right side.

- Some students may multiply (or divide) both sides of an equation by a number to isolate the variable without applying the factor to every term. Ask students to solve the equation $3x + 8 = 12$. Students who misunderstand how to isolate a variable may incorrectly multiply both sides of the equation by $\frac{1}{3}$ to get $x + 8 = 4$. Remind the students that any number being multiplied to both sides of the equation must be distributed to each term. Draw parentheses around the two expressions on both sides of the equation before multiplying by the $\frac{1}{3}$ to illustrate this point.

► Enrichment

- Ask students to determine the value of x that makes two expressions equal. For example, give students the following situation.

  Nora is comparing the total cost, y, to rent a bike for x hours at two bike rental companies. The total cost at Company A is modeled by the equation $y = 20x$. The total cost at Company B is modeled by the equation $y = 15x + 80$. For how many hours do the bikes need to be rented so that the total cost is the same at both companies?

- Ask students to write a word problem with a given variable that models a given equation. For example, ask students to write a word problem that involves the price of ice cream with x scoops and is represented by the equation $4(2x + 3) = 36$.

► Key Terms

coefficient, constant, distributive property, equation, expression, like terms, variable

A1.1.2.1.2

▶ Module, Assessment Anchor, Anchor Descriptor

Operations and Linear Equations & Inequalities

Linear Equations

Write, solve, and/or graph linear equations using various methods.

▶ Eligible Content

A1.1.2.1.2 Use and/or identify an algebraic property to justify any step in an equation-solving process.

Note: Linear equations only.

▶ Instructional Supports

- Ask students to identify the algebraic properties used in each step of the process when solving a linear equation. For example, ask students to identify the property used when the given equation $3x + \frac{4}{3} - 2x = 2x - 8 - 2x$ is transformed to $3x - 2x + \frac{4}{3} = 2x - 2x - 8$. Each side of the equation has the same $-2x$ but the values are presented in a different order. Therefore, the commutative property of addition was applied.
- Ask students to apply a given algebraic property to a linear equation. For example, ask students to apply the distributive property to $\frac{1}{2}(3x - 4) = 2x + 3x + 7$. The distributive property multiplies a number outside the parentheses by each term inside the parentheses. In this case, the $\frac{1}{2}$ can be distributed to both $3x$ and 4. This yields $\left(\frac{1}{2}\right)(3x) - \left(\frac{1}{2}\right)(4) = 2x + 3x + 7$ or $\frac{3}{2}x - 2 = 2x + 3x + 7$.
- Ask students to identify the missing step and the supporting algebraic property in a partially solved linear equation. For example, the equation shown is not complete, ask students to identify the missing step and the supporting algebraic property in the following steps.

$-\frac{4}{5}x + 22 = \frac{1}{5}x + 24$	*Given*
________	________
$-x + 22 = 24$	*Combining like terms*
$-x + 22 + (-22) = 24 + (-22)$	*Addition Property of Equality*

In the missing step, the $\frac{1}{5}x$ from the right side of the equation is moved to the left side of the equation and combined with the $-\frac{4}{5}x$ from the left side. The only way to move an entire variable term from one side of an equation is to add its additive inverse to both sides. In this case, the additive inverse is $-\frac{1}{5}x$. Therefore, the missing step is $-\frac{4}{5}x + 22 + \left(-\frac{1}{5}x\right) = \frac{1}{5}x + 24 + \left(-\frac{1}{5}x\right)$ because of the addition property of equality.

- Ask students to solve a linear equation and justify each step with the algebraic property used. For example, give students the equation $9 - 8x = 3 - 5x$.

$9 - 8x = 3 - 5x$	*Given*
$9 - 8x + 5x = 3 - 5x + 5x$	*Addition Property of Equality*
$9 - 3x = 3$	*Combining like terms*
$9 - 3x - 9 = 3 - 9$	*Addition Property of Equality*
$9 - 9 - 3x = 3 - 9$	*Commutative Property of Addition*
$-3x = -6$	*Combining like terms*
$-\frac{1}{3}(-3x) = -\frac{1}{3}(-6)$	*Multiplication Property of Equality*
$x = 2$	*Multiplicative inverse and simplifying*

Students can verify their answer by substituting 2 back into the original equation.

► Identifying and Addressing Common Misconceptions

- Students may confuse the additive inverse property and the addition property of equality. Point out to students that the addition property of equality involves the equal sign since equality and equal signify identical concepts. Therefore, when the same thing is added to both sides of an equation, the addition property of equality is being used. Give students a set of equations and the equivalent equations after the additive inverse property or addition property of equality has been applied. Have students sort each problem by which property has been applied. Ask students to draw conclusions about the structure of the equations when each property has been applied. Students may note, for instance, that the additive inverse property simplifies two numbers or two variables with opposite signs into 0 and that the addition property of equality causes the same number to be added to both sides of an equation. If needed, the process can be repeated with the multiplicative inverse property and the multiplication property of equality.

- Students may combine more than one step and ignore an algebraic property when justifying each step in an equation. Ask students to state the property used for each step when solving the equation $6(3x + 4) = 9x - 8$. Students who have this misconception may perform the first step as $18x + 24 - 9x = -8$ and justify the step with the distributive property. Ask students to instead distribute the 6 and compare their results to $18x + 24 - 9x$. When they notice that the two expressions are not equivalent, ask them where the $-9x$ came from. Help them understand that adding $-9x$ to both sides of the equation is another step that requires another property.

▶ Enrichment

- Ask students to write an equation that has a given solution and uses a given set of properties to solve. For example, ask students to write an equation that has a solution of $x = -\frac{1}{2}$ and requires the use of the distributive property, multiplication property of equality, and addition property of equality.
- Ask students to solve an equation in two different ways, justifying each step with the algebraic property used.

▶ Key Terms

addition property of equality, associative property, coefficient, combine like terms, commutative property, constant, distributive property, equation, expression, identity property of addition, inverse property, multiplication property of equality, multiplicative property, term, variable

A1.1.2.1.3

► Module, Assessment Anchor, Anchor Descriptor

Operations and Linear Equations & Inequalities

Linear Equations

Write, solve, and/or graph linear equations using various methods.

► Eligible Content

A1.1.2.1.3 Interpret solutions to problems in the context of the problem situation.

Note: Linear equations only.

► Instructional Supports

- Ask students to interpret the meaning of a solution provided to a one-variable equation that models a real-world situation. For example, give students the following situation.

  Joss spends \$117 on equipment to start a donut stand. She spends \$0.30 to make each donut and sells each donut for \$2.25. The equation $117 + 0.3x = 2.25x$ represents the situation. What does the solution $x = 60$ represent? Students should start by identifying what each side of the equation represents. $117 + 0.3x$ represents the cost to make x donuts, and $2.25x$ represents the money earned for selling x donuts. Therefore, the solution represents the result when the expenses equal the money earned. The solution $x = 60$ means that when 60 donuts are sold, Joss will break even.

- Ask students to interpret the meaning of provided solutions to a two-variable equation that models a real-world situation. For example, give students the following situation.

  Ms. Sanchez will spend \$75 on snacks for her class. The relationship between the number of ounces of pretzels, x, and the number of ounces of nuts, y, can be modeled by the equation $3x + 3.5y = 75$. One solution is (9.6, 13.2). What does (9.6, 13.2) represent in the context of this situation? Students should note that the values of the ordered pair are in the form (x, y). Students can therefore substitute each value into the equation to determine the ounces of pretzels and the ounces of nuts. The solution (9.6, 13.2) means that Ms. Sanchez can buy 9.6 ounces of pretzels and 13.2 ounces of nuts for \$75.

- Ask students to determine if a statement about a potential solution to a one- or two-variable equation is true. For example, give students the following situation.

  Harrison has y pages of a book left to read after x hours of reading. The equation $y = 212 - 27x$ models the situation. Harrison claims he will have 5 pages left to read after 77 hours of reading. Determine whether Harrison's claim is true. Students should start by identifying which variables the 5 and 77 correspond to. Since there will be 5 pages left to read, $y = 5$. Since he will have read for 77 hours, $x = 77$. Substituting these values into the equation yields $5 = 212 - 27(77)$. This simplifies to $5 = -1{,}867$, which is not true. Therefore, Harrison's claim is not correct. Students may notice that switching the variables so that $x = 5$ and $y = 77$ yields a true equation, so the statement should be that after reading for 5 hours, Harrison will have 77 pages left to read.

► Identifying and Addressing Common Misconceptions

- Some students may assign a random meaning to the variable when describing a solution to a one-variable equation. Ask students to describe the meaning of the solution $x = 4$ in the equation $5 - 1.25x = 0$ that models the number of hours it will take for a 5-gallon bucket with a slow leak to empty. Some students may answer that there will be 4 gallons of water in the bucket. Have students highlight the parts of the equation with the corresponding parts of the description in the same color. This means that students can highlight "5" and "5-gallon bucket" in one color and "0" and "empty" in another color. Next, ask students what key information is still not highlighted. They should notice that "hours" is not highlighted, so it must go with x. Therefore, it will take 4 hours for the bucket to empty.
- Some students may swap the variables when describing a solution to a two-variable equation. For instance, use the equation $5(x + y) = 100$ to model the number of \$5 game tickets, x, and \$5 ride tickets, y, that Bella buys at a fair if she spends a total of \$100. Ask students to describe the meaning of the solution (8, 12) in this situation. Students who mistakenly swap the variables may answer that Bella could have bought 8 ride tickets and 12 game tickets. Have students write the coordinates of ordered pairs as (x, y) and then substitute the meaning of each variable into the ordered pair. Students should write (game tickets, ride tickets). Have the students compare this with the given point of (8, 12). Students should notice that their answer is incorrect and revise it to match the correct variables.
- Some students may think that a two-variable equation can have only a single solution. For instance, use the equation $5x + 6y = 62$ to model the pounds of grass seed that costs \$5 per pound, x, and weed

killer that costs \$6 per pound, y, that Theo can buy for a total of \$62. Ask students to describe the meaning of the solution (10, 2) in this situation. Students who mistakenly believe that only a single solution is possible may respond that Theo must have bought 10 pounds of grass seed and 2 pounds of weed killer. Ask students if (4, 7) is also a solution, and if so, what it means. After students notice that 4 pounds of grass seed and 7 pounds of weed killer is also an option, ask them if anything distinguishes one as the better solution. Help students see that any positive point on the line is a reasonable solution, so there are many combinations of ways Theo could purchase grass seed and weed killer.

► Enrichment

- Ask students to compare solutions from two related equations. For example, give students the following situation.

  Reese has \$14 in her savings account and saves \$6 each week. Jesse has \$36 in his savings account and saves \$5 each week. Each of them wants to save \$86. The equation modeling this situation for Reese is $14 + 6x = 86$, and the equation for Jesse is $36 + 5x = 86$. The solution to Reese's equation is $x = 12$ and the solution to Jesse's equation is $x = 10$. Interpret and compare the solutions to the two equations.

► Key Terms

context, interpret, linear equation, substitute, solution, variable

A1.1.2.2.1

► Module, Assessment Anchor, Anchor Descriptor

Operations and Linear Equations & Inequalities

Linear Equations

Write, solve, and/or graph systems of linear equations using various methods.

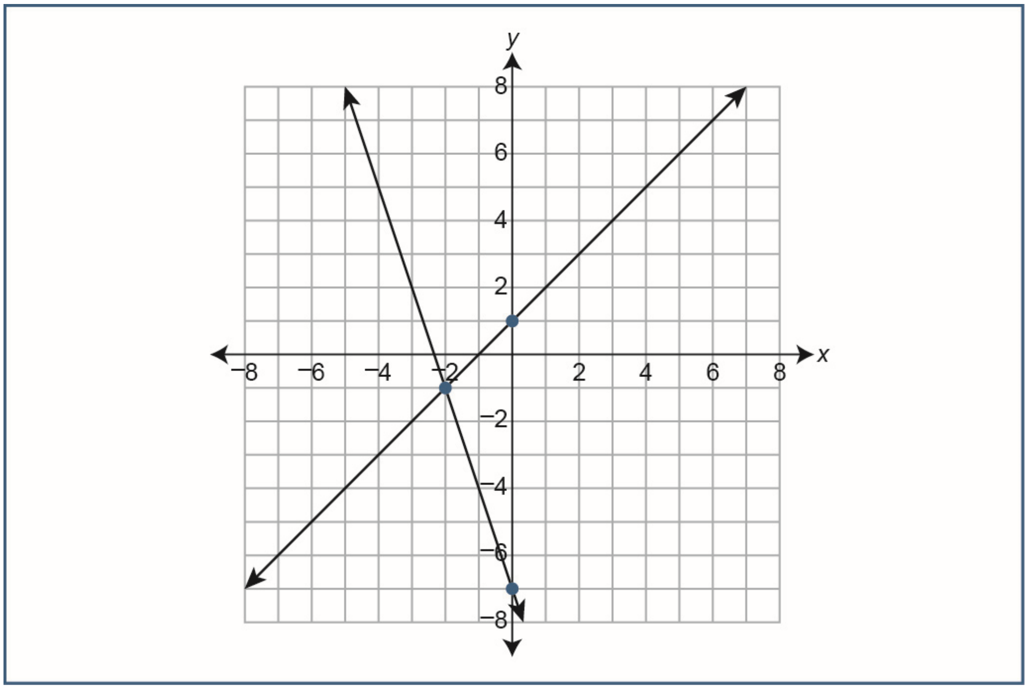
► Eligible Content

A1.1.2.2.1 Write and/or solve a system of linear equations (including problem situations) using graphing, substitution, and/or elimination.

Note: Limit systems to two linear equations.

► Instructional Supports

- Ask students to graph a system of linear equations and identify the solution. For example, ask students to determine the solution to $\begin{cases} y = -3x - 7 \\ y = x + 1 \end{cases}$ by graphing. When there is more than one equation graphed on a coordinate plane, it is called a system of equations. The solution to the system of equations is the point where the lines intersect. Therefore, students should begin by graphing the lines of the two equations to identify the point of intersection. The top equation has a *y*-intercept of -7, so students can begin by plotting a point at $(0, -7)$ on the *y*-axis. Then, the students can use the slope of -3 to locate more points on the path of the line. From the *y*-intercept, students can move 3 units down and 1 unit right (or 3 units up and 1 unit left). The bottom equation has a *y*-intercept of 1 and a slope of 1, which means the line intersects the *y*-axis at $(0, 1)$ and more points can be located by moving 1 unit up and 1 unit right (or 1 unit down and 1 unit left). A sample student graph is shown.



Next, students should identify the point of intersection from their graph. In this case, the point of intersection is at (−2, −1). Students can confirm that (−2, −1) is the solution by determining that it satisfies both equations, as shown.

$y = -3x - 7 \rightarrow -1 = -3(-2) - 7 \rightarrow -1 = 6 - 7 \rightarrow -1 = -1$ **true**

$y = x + 1 \rightarrow -1 = (-2) + 1 \rightarrow -1 = -1$ **true**

Since the point satisfies both equations, students have confirmed that the point (−2, −1) is the solution.

- Ask students to solve a system of linear equations by substitution. For example, ask students to determine the solution to $\begin{cases} y = 2x - 3 \\ 3x + 4y = -1 \end{cases}$. Students should note that y is isolated in the top equation, so this variable is a good option for substituting into the bottom equation. Since y is equal to $2x - 3$, students can substitute this for y in the bottom equation, as shown:

  $3x + 4y = -1 \rightarrow 3x + 4(2x - 3) = -1$

  Now the equation has a single variable, so students can now solve for x, as shown.

  $3x + 4(2x - 3) = -1$

  $3x + 8x - 12 = -1$ *(distributive property)*

$11x - 12 = -1$ (combining like terms)

$11x = 11$ (addition property of equality)

$x = 1$ (multiplication property of equality)

Since $x = 1$, students can substitute that value into either of the original equations to determine the value for y. Since y is isolated in the top equation, students may choose to use the top equation, as shown.

$y = 2x - 3$

$y = 2(1) - 3$ (substitution)

$y = 2 - 3$ (simplify)

$y = -1$ (simplify)

The values for x and y are $x = 1$ and $y = -1$, so the solution to this system of equations is $(1, -1)$. Students can confirm this by sketching a graph of the two lines to see that they intersect at $(1, -1)$ or by substituting both values into the original equations to ensure the point satisfies both equations.

- Ask students to solve a system of linear equations by elimination.
    - For example, ask students to solve $\begin{cases} 3x + y = 3 \\ 4x - y = 11 \end{cases}$ using the elimination method. Elimination is done by adding the two equations with the goal of eliminating one variable. For a variable to be eliminated when the equations are added, the variable being eliminated must have the same coefficient but have opposite signs in each equation. Students should begin by examining the two equations to see if any of the variables are ready to be eliminated. In this case, the y variable has the same coefficient and opposite signs. The y variable can be eliminated when the two equations are added, so the elimination method can be immediately applied, as shown.

$$\begin{array}{r} 3x + y = 3 \\ +\ \ 4x - y = 11 \\ \hline 7x + 0 = 14 \end{array}$$

Students can now solve for x.

$7x = 14$

$x = 2$

Students have now determined that $x = 2$ and can substitute the x-value into either of the original equations to solve for y, as shown.

$3(2) + y = 3$

$6 + y = 3$

$y = -3$

Therefore, the solution to the system of equations is $(2, -3)$.

- For another example, ask students to solve $\begin{cases} 2x - 4y = 14 \\ x + 3y = 2 \end{cases}$ using the elimination method.

Since neither variable has the same coefficient with opposite signs, at least one of the equations will need to be multiplied by a constant. Since the x variable in the bottom equation has a coefficient of 1 and the x variable in the top equation has a coefficient of 2, students can multiply the bottom equation by -2, as shown.

$$\begin{cases} 2x - 4y = 14 \\ (-2)(x + 3y = 2) \end{cases} \rightarrow \begin{cases} 2x - 4y = 14 \\ -2x - 6y = -4 \end{cases}$$

Now the x's have the same coefficient with opposite signs, so students can add the two equations to eliminate the x variables.

$$\begin{array}{rr} & 2x - 4y = 14 \\ + & -2x - 6y = -4 \\ \hline & 0x - 10y = 10 \end{array}$$

Now students can solve for y by multiplying both sides of the equation by $-\frac{1}{10}$.

$-10y = 10 \rightarrow y = -1$

Students can now substitute the y-value into either one of the original equations to solve for x.

$2x - 4y = 14$

$2x - 4(-1) = 14$ (substitution)

$2x + 4 = 14$ (simplify)

$2x = 10$ (addition property of equality)

$x = 5$ (multiplication property of equality)

Therefore, the solution to the system of equations is $(5, -1)$.

- Ask students to identify properties of systems of equations that have no solutions. For example, ask students to solve the following system of equations graphically and/or algebraically.

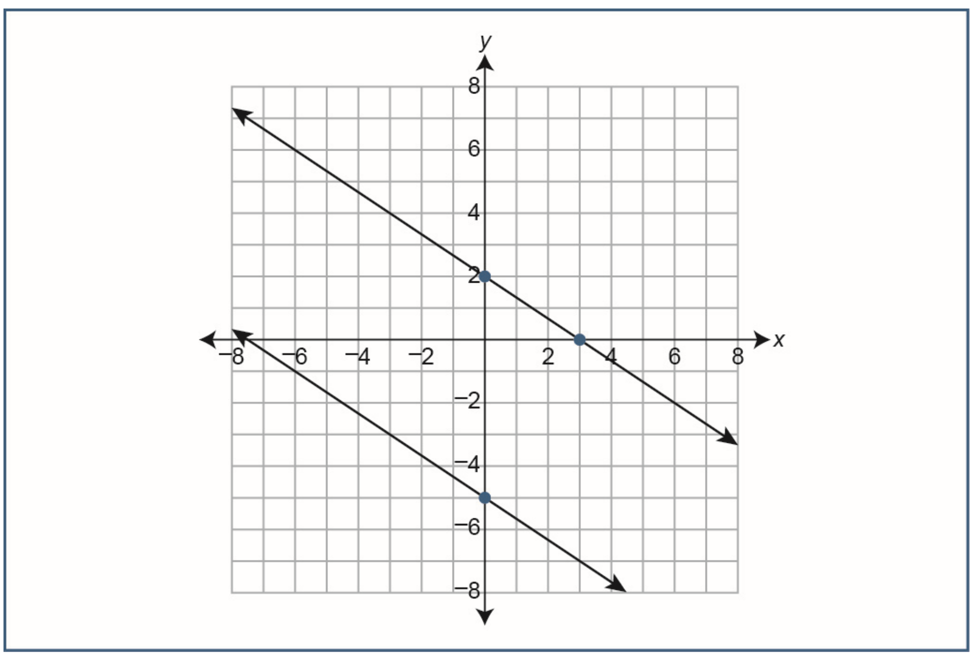
$$\begin{cases} 3y + 2x = 6 \\ y = -\dfrac{2}{3}x - 5 \end{cases}$$

Students can start by creating a graph of the two equations. The top equation, $3y + 2x = 6$, can be graphed by determining the x- and y-intercepts by substituting $x = 0$ and $y = 0$ into the equation, as shown:

x-intercept: $3(0) + 2x = 6 \rightarrow 2x = 6 \rightarrow x = 3 \rightarrow (3, 0)$

y-intercept: $3y + 2(0) = 6 \rightarrow 3y = 6 \rightarrow y = 2 \rightarrow (0, 2)$

The bottom equation is in slope-intercept form, so it can be graphed since the y-intercept is -5 and it has a slope of $-\frac{2}{3}$. This coordinate plane shows a student's possible graph.



Students should note that the graphs of the lines don't intersect because the lines are parallel to one another. Therefore, this system of equations has no solution.

Furthermore, students can solve the system of equations algebraically. Since the bottom equation is solved in terms of y, the substitution method is probably the better approach. Students can begin by substituting $-\frac{2}{3}x - 5$ into the top equation for y and then solve for y, as shown:

$3y + 2x = 6$

$3(-\frac{2}{3}x - 5) + 2x = 6$ (substitution)

$-2x - 15 + 2x = 6$ (distributive property)

$-15 = 6$ (combining like terms)

Since the equation ends in a false statement of $-15 = 6$, the system of equations has no solution. Students can conclude that a system of equations made up of two parallel lines will result in no solution. Additionally, a system of equations that results in an untrue equation when solved algebraically will also have no solution, which also means it is composed of two parallel lines. In conclusion, a system of equations that has a solution represents the intersection point of two lines, and parallel lines never intersect.

- Ask students to identify properties of systems of equations that have infinitely many solutions. For example, ask students to solve the following system of equations graphically and/or algebraically.

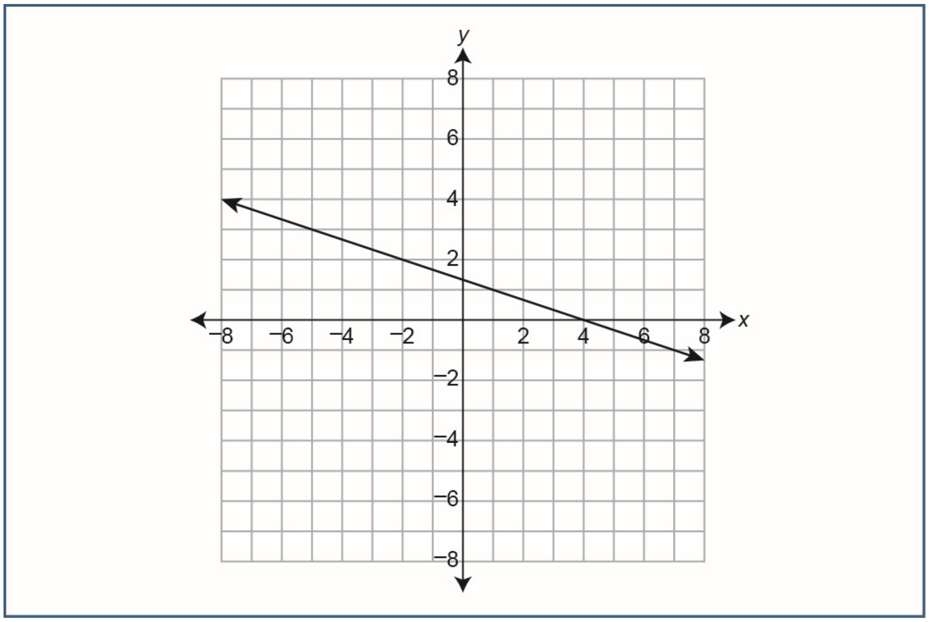
$$\begin{cases} 2x + 6y = 8 \\ -x - 3y = -4 \end{cases}$$

Students can begin by creating a graph of the two equations. An approach students can take for the creation of the graph is to put both equations into slope-intercept form, as shown.

$2x + 6y = 8 \rightarrow 6y = -2x + 8 \rightarrow y = -\frac{1}{3}x + \frac{4}{3}$

$-x - 3y = -4 \rightarrow -3y = x - 4 \rightarrow y = -\frac{1}{3}x + \frac{4}{3}$

Students should note that the two equations are the same in slope-intercept form. Therefore, the lines of the two equations coincide, meaning they are the same line when graphed, as shown.



Since the lines coincide, the students can conclude that this system of equations has infinitely many solutions.

Now students can solve the system of equations algebraically to see what the results of a system of equations with infinitely many solutions resemble. Since the variables are aligned, students may choose to use the elimination method. Multiplying the bottom equation by 2 will allow the *x* variables to cancel when the equations are added, as shown:

$$\begin{cases} 2x + 6y = 8 \\ 2(-x - 3y = -4) \end{cases} \rightarrow \begin{cases} 2x + 6y = 8 \\ -2x - 6y = -8 \end{cases}$$

Students should add the two equations and solve for the remaining variable.

$$\begin{array}{r} 2x + 6y = 8 \\ + \quad -2x - 6y = -8 \\ \hline 0x + 0y = 0 \\ 0 = 0 \end{array}$$

Since the equation results in a true statement, 0 = 0, this proves that the system of equations has infinitely many solutions. In conclusion, this system of equations represents the same line when solved algebraically, so the system of equations has infinitely many solutions.

- Ask students to write and solve a system of equations that models a real-world situation. For example, give students the following situation.

Two classes are competing in a recycling competition. One class plans to collect three-quarters of a pound of cans each day. The other class already had two pounds of cans collected and plans to collect one-quarter of a pound of cans each day. Write a system of equations that can model the situation. Students should start by assigning meaning to the independent variable (x) and the dependent variable (y). The two variables being studied are time (days) and the amount collected (pounds). Time is often the independent variable because it is a measure that progresses on its own. In this case, the amount collected is dependent on the number of days into the competition. Therefore, time (days) is the independent variable, x, and the amount collected (pounds) is the dependent variable, y. The first class plans to collect three-quarters of a pound each day, and that represents the rate (or slope). This class has no starting value (y-intercept), so the equation $y = \frac{3}{4}x$ represents the amount they plan to collect. The second class had already collected 2 pounds, which is the initial value (y-intercept). This class plans to collect one-quarter of a pound each day, which is the slope, so their situation is modeled by the equation $y = \frac{1}{4}x + 2$. The system of equations that represents this situation is $\begin{cases} y = \frac{3}{4}x \\ y = \frac{1}{4}x + 2 \end{cases}$.

- Ask students to write and solve a system of equations to model a real-world situation. For example, give students the following situation about 2 families buying tickets to the movies.

  The Martinez family paid \$12 for two adult tickets and two child tickets. The Smith family paid \$20 for three adult tickets and four child tickets. What is the price for an adult ticket and what is the price for a child ticket? The variables being explored are the cost of an adult ticket and the cost of a child ticket, so the variables a and c can be used. Two adult tickets and two child tickets cost \$12, so the first equation is $2a + 2c = 12$. Three adult tickets and four child tickets cost \$20, so the second equation is $3a + 4c = 20$. Therefore, the system of equations to model the situation is $\begin{cases} 2a + 2c = 12 \\ 3a + 4c = 20 \end{cases}$.

  Because the variables are lined up, students may determine that the elimination method is the best approach. $4c$ is a multiple of $2c$, so students can multiply the first equation by -2, as shown.

$$\begin{cases} (-2)(2a + 2c = 12) \\ 3a + 4c = 20 \end{cases} \rightarrow \begin{cases} -4a - 4c = -24 \\ 3a + 4c = 20 \end{cases}$$

  Students should add the equations, as shown:

$$\begin{array}{r} -4a - 4c = -24 \\ +\ \ 3a + 4c = 20 \\ \hline -1a + 0c = -4 \end{array}$$

Then students can solve the equation for the remaining variable, *a*.

$-1a + 0c = -4$

$-a = -4$ (simplify)

$a = 4$ (multiplication property of −1)

Now students can substitute the value of *a* into either of the original equations to solve for *c*.

$2a + 2c = 12$

$2(4) + 2c = 12$ (substitution)

$8 + 2c = 12$ (simplify)

$2c = 4$ (addition property of equality)

$c = 2$ (multiplication property of equality)

Finally, students can put this solution into the context of the problem.

Each adult ticket costs \$4, and each child ticket costs \$2.

▶ Identifying and Addressing Common Misconceptions

- Some students may not distribute a constant to every term within an equation as they attempt to solve using the elimination method. Ask students to determine the solution to $\begin{cases} 3x - 5y = 11 \\ x - y = 3 \end{cases}$ using the elimination method. Students may see that the bottom equation could be multiplied by -3 in order to eliminate the *x* terms but decide to distribute the constant to the first term only: $\begin{cases} 3x - 5y = 11 \\ -3x - y = 3 \end{cases}$. Ask students to check if $x - y = 3$ and $-3x - y = 3$ are equivalent equations by substituting different values into the equations. For example, (4, 1) is a solution to the first equation. However, it is not a solution to the second equation. Help students see that the two equations are not equivalent and therefore can't be used interchangeably. Remind students that the distributive property needs to be applied to each term in the equation when multiplying the second equation by -3.
- Some students may substitute one equation into itself instead of into the other equation when solving a system of equations by substitution. For example, ask students to solve the system $\begin{cases} 5x + 3y = 8 \\ x + y = 2 \end{cases}$. Some students may solve the second equation for *y* to get $y = 2 - x$; then, they may incorrectly substitute $2 - x$ into the second equation instead of into the first to get $x + 2 - x = 2$ and conclude that there are infinitely many solutions since $2 = 2$. Ask students to examine their work and explain where the equation $5x + 3y = 8$ is a part of the process. When students notice that they never used the other

equation, remind them that both equations must be used in the process of solving because the solution must satisfy both equations.

- Some students may think that a solution to a system of equations is any point that satisfies either of the equations. Ask students to determine the solution to $\begin{cases} y = 9 - x \\ 2x - y = -3 \end{cases}$. Students operating with this misconception may give a point from just one of the lines, such as (1, 8). Have students check their solution by graphing the system as well as the point they listed.

► Enrichment

- Ask students to solve a system of equations made up of three or more linear equations.
- Ask students to write a system of linear equations that has a given solution.
- Ask students to determine what quadrilateral is formed by a system of equations made up of four linear equations.

► Key Terms

coincide, elimination method, infinite solutions, intersecting lines, no solution, parallel lines, point of intersection, substitution method, system of linear equations

A1.1.2.2.2

▶ Module, Assessment Anchor, Anchor Descriptor

Operations and Linear Equations & Inequalities

Linear Equations

Write, solve, and/or graph systems of linear equations using various methods.

▶ Eligible Content

A1.1.2.2.2 Interpret solutions to problems in the context of the problem situation.

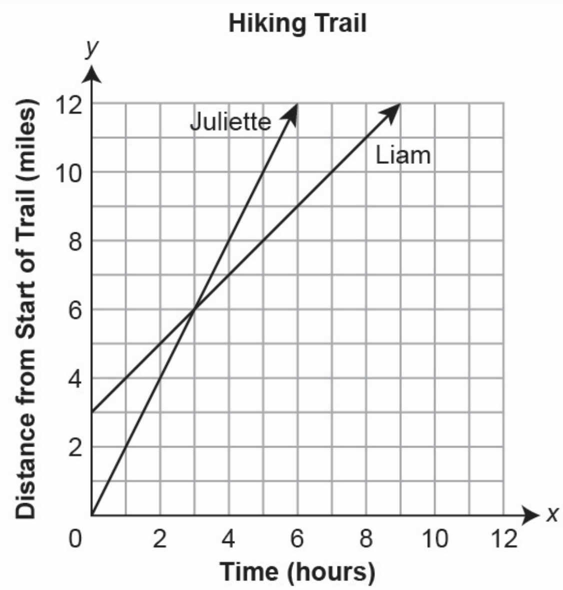
Note: Limit systems to two linear equations.

▶ Instructional Supports

- Ask students to interpret a solution provided to a system of equations within the context. Give the students the following situation.

  A class and the school's principal are doing a fundraiser that is a walking competition around the school's track. The principal walks 0.75 miles per hour and gets a 1.75-mile head start. The student representing the class walks 2.5 miles per hour. What does the solution (1, 2.5) mean within the context of this situation? The variables being explored are total miles walked and total time spent walking. It makes the most sense that the independent variable (x) be time, since the amount of time impacts how many miles are completed. Therefore, after 1 hour, both the principal and the student will have walked 2.5 miles. This can be double-checked using the information provided to see how many miles will be completed after 1 hour of walking. The principal has already walked 1.75 miles and will walk 0.75 more miles in 1 hour, which is 2.5 miles. The student walks 2.5 miles per hour, which means the student will walk 2.5 miles in 1 hour. Therefore, the interpretation is correct.

- Ask students to interpret the graph of a system of equations within the context. Ask students to interpret the system of equations shown in the following graph.



Students should begin by identifying the solution to the system of equations as (3, 6) and defining the variables as (time, distance). To interpret this within the context, both Juliette and Liam will be 6 miles from the start of the trail after hiking for 3 hours.

- Ask students to use the solution to a system of equations to answer further questions within the context. For example, give students the following situation.

A student at a snack shop bought 2 pears and 1 apple for \$2.10. Another student bought 3 apples and 1 pear for \$2.30. How much more does a pear cost than an apple? Students should define the variables where *a* represents the cost of an apple and *p* represents the cost of a pear and write the following system.

$$\begin{cases} 2p + a = 2.1 \\ 3a + p = 2.3 \end{cases}$$

Students can solve the equation by using elimination or substitution. If students choose to use substitution, they could start by solving the first equation for *a*.

$$2p + a = 2.1 \rightarrow a = 2.1 - 2p$$

Then students should substitute $2.1 - 2p$ in for a in the second equation and solve.

$$3(2.1 - 2p) + p = 2.3$$

$$6.3 - 6p + p = 2.3 \qquad \text{(distributive property)}$$

$6.3 - 5p = 2.3$	(combining like terms)
$-5p = -4$	(addition property of equality)
$p = 0.8$	(multiplication property of equality)

Then, students should substitute $p = 0.8$ into either equation and solve for a.

$2(0.8) + a = 2.1$

$1.6 + a = 2.1$

$a = 0.5$

In the context of this situation, a pear costs $0.80 and an apple costs $0.50. This means that a pear costs $0.30 more than an apple.

▶ Identifying and Addressing Common Misconceptions

- Some students may interpret the solution as being the solution to just one equation instead of both equations. For example, give students the following situation.

  Lucas and Theo each sell y tickets in x days. The system $\begin{cases} y = 9x \\ y = 6x + 15 \end{cases}$ represents Lucas's results in the top equation and Theo's results in the bottom equation. Ask students to interpret the solution (5, 45) within the context of the situation. Some students may respond incorrectly by reasoning that the solution means that Lucas will have sold 45 tickets after 5 days. Remind students that a system of equations represents the relationship between two equations. Have students describe the relationship between the two equations. They should respond that the system compares the number of tickets Lucas and Theo have each sold. Ask students if (5, 45) must apply to Lucas, Theo, or both without performing any calculations. Help students see that it must apply to both people because it is a solution to the entire system of equations, not just to a single equation. Therefore, the better description of the solution is that both people will have sold an equal number of tickets, 45, after 5 days of selling tickets.

▶ Enrichment

- Ask students to create a word problem involving a system of equations that goes with a given solution. For example, ask students to write a word problem that uses a system of equations that has a solution of 6 cookies and 8 cupcakes.

- Ask students to interpret solutions to real-world problems that are modeled by three or more linear equations.

▶ Key Terms

dependent variable, independent variable, intersection, solution, system of linear equations

A1.1.3.1.1

▶ Module, Assessment Anchor, Anchor Descriptor

Operations and Linear Equations & Inequalities

Linear Inequalities

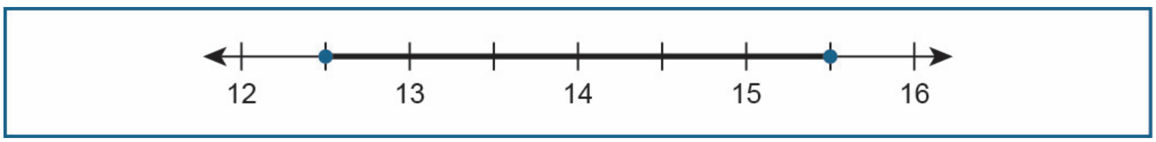
Write, solve, and/or graph linear inequalities using various methods.

▶ Eligible Content

A1.1.3.1.1 Write or solve compound inequalities and/or graph their solution sets on a number line (may include absolute value inequalities).

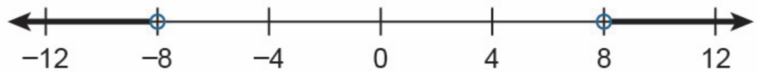
▶ Instructional Supports

- Ask students to write a compound inequality to match a solution set shown on a number line. For example, ask students to write a compound inequality whose solutions are shown on this number line.



  Because all the solutions lie between two numbers, the inequality will be of the form __ $< x <$ __ or __ $\leq x \leq$ __. Because both endpoints are closed (filled-in) circles, $\leq$ should be used instead of $<$ for both cases. The left endpoint is 12.5 and the right endpoint is 15.5. Therefore, the inequality modeling this situation is $12.5 \leq x \leq 15.5$.

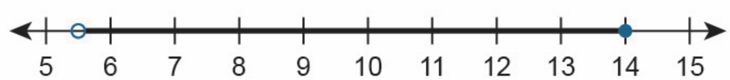
- Ask students to graph the solutions to a solved compound inequality on a number line. For example, ask students to graph the inequality $|x| > 8$. Students should recall that the absolute value of a number represents its distance from 0. In this case, the distance from 0 must be greater than 8. So, one set of solutions is any number greater than 8. Additionally, any number less than −8 also has a distance greater than 8 from 0. Students should put both sets of numbers together on a number line to represent the solutions. A potential student response is provided.



- Ask students to solve a compound inequality and graph the solution set on a number line.
  - For example, give students the inequality $6 < 2x - 5 \leq 23$. Compound inequalities of this form can be split into two inequalities with "and" in between ($6 < 2x - 5$ AND $2x - 5 \leq 23$) that are solved separately. However, it can also be solved as a double inequality at once, where steps are applied to all three sections of the inequality. Students can take this approach, as shown.

$6 < 2x - 5 \leq 23$	
$6 + 5 < 2x - 5 + 5 \leq 23 + 5$	(addition property of equality)
$11 < 2x \leq 28$	(simplify)
$(\frac{1}{2})11 < (\frac{1}{2})2x \leq (\frac{1}{2})28$	(multiplication property of equality)
$5.5 < x \leq 14$	(simplify)

Since x must be between both numbers, students will graph all numbers strictly greater than 5.5 and less than or equal to 14, as shown.



  - For another example, give students the inequality $|4x + 9| > 3$. Students should recall that absolute value represents the distance from 0, so the expression must have a distance that is greater than 3 from 0. This can be represented as $4x + 9 > 3$ OR $4x + 9 < -3$. Students can follow the process to solve a linear inequality that is shown here.

$4x + 9 > 3$	
$4x + 9 + (-9) > 3 + (-9)$	(addition property of equality)
$4x > -6$	(simplify)
$(\frac{1}{4})4x > (\frac{1}{4})(-6)$	(multiplication property of equality)
$x > -\frac{3}{2}$	(simplify)

$4x + 9 < -3$

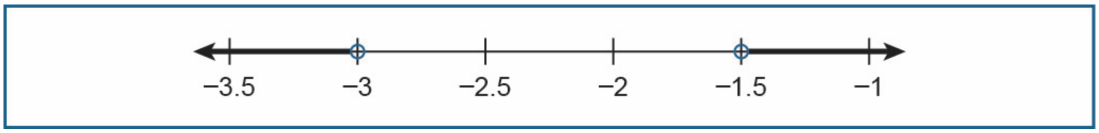
$4x + 9 + (-9) < -3 + (-9)$ (addition property of equality)

$4x < -12$ (simplify)

$(\frac{1}{4})4x < (\frac{1}{4})(-12)$ (multiplication property of equality)

$x < -3$ (simplify)

Since these inequalities are connected with an "or" statement, the solutions will satisfy one inequality or the other. Therefore, students should graph all the numbers that are greater than −1.5 or all the numbers that are less than −3, as shown.



- For another example, give students the inequality $|3x - 8| < 7$. Students should interpret this situation as the expression's distance from 0 being less than 7. That means that $3x - 8$ must be less than 7 while also being greater than −7. This is expressed as $3x - 8 < 7$ AND $3x - 8 > -7$. This can also be expressed as $-7 < 3x - 8 < 7$. Then students can solve the compound inequality by applying all operations to all 3 expressions, as shown.

$-7 < 3x - 8 < 7$

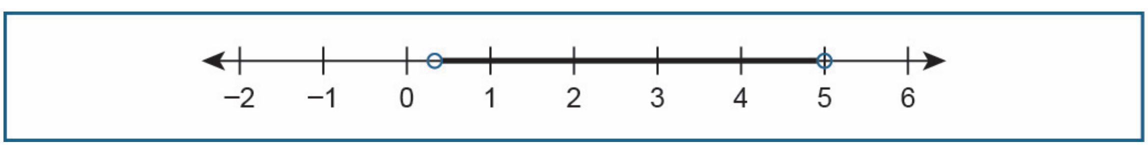
$-7 + 8 < 3x - 8 + 8 < 7 + 8$ (addition property of equality)

$1 < 3x < 15$ (simplify)

$(\frac{1}{3})1 < (\frac{1}{3})3x < (\frac{1}{3})15$ (multiplication property of equality)

$\frac{1}{3} < x < 5$ (simplify)

This inequality describes a set containing numbers that are greater than $\frac{1}{3}$ and also less than 5. Students can represent this on a number line, as shown.



- Ask students to write a compound inequality to model a real-world situation and determine the solutions. For example, ask students to write an inequality to model the following situation.

  Mr. Perez makes a list of students whose quiz scores are more than 20 points away from the class average. The class average was 75. Write an inequality to represent the scores, x, that will be on Mr. Perez's list. As students contemplate this problem, they should start by noting that the difference between x and 75 must be more than 20. The scores can be more than 20 points above or more than 20 points below the average; this implies that absolute value will be used. The difference is represented by $x - 75$, so the absolute value of the difference is $|x - 75|$. This difference needs to be greater than 20, which yields the final inequality of $|x - 75| > 20$.

  To determine the specific scores that meet the requirements, students must solve the inequality. The scores do not have an absolute value greater than 20. To determine the specific scores that meet the requirements, students need to solve $x - 75 > 20$ and $x - 75 < -20$. Adding 75 to both sides of the two inequalities yields $x > 95$ OR $x < 55$. Therefore, any score greater than 95 or less than 55 will be on Mr. Perez's list.

► Identifying and Addressing Common Misconceptions

- Some students might graph an "and" inequality as an "or" inequality. Ask students to graph $6 \leq x \leq 11$. Some students may incorrectly graph this as two rays, with one going to the left from 6 and the other going to the right from 11. Ask students to pick values from their graph and check them in the inequality. Students may pick 20 and discover that $6 \leq 20 \leq 11$ is not true. Ask students to find which numbers meet the requirement and compare the results with the graph. Help students see that they have shaded the opposite parts of the graph and that they should have shaded the numbers between 6 and 11 (including 6 and 11).

- Some students may rewrite every absolute value inequality as a double-compound inequality. Ask students to determine the solutions to $|2x + 8| > 10$. Students who mistakenly apply the reasoning for a double-compound inequality will begin by writing $-10 < 2x + 8 < 10$. Have students check different numbers in the original inequality and in the inequality they wrote. Students should discover that numbers that satisfy one inequality do not satisfy the other inequality. Therefore, their inequalities are not equivalent. Help students apply the correct reasoning about the original absolute value inequality in terms of distance from 0. Students should see that they're looking for numbers that make $2x + 8$ more than 10 units from 0. Therefore, they need $2x + 8 > 10$ OR $2x + 8 < -10$.

► Enrichment

- Ask students to solve absolute value equations using similar reasoning. For example, ask students to solve $|5x + 24| = 42$.
- Ask students to write a word problem that matches a given compound inequality.

► Key Terms

absolute value inequality, closed circle, compound inequality, double inequality, endpoint, number line, open circle, solution set

A1.1.3.1.2

▶ Module, Assessment Anchor, Anchor Descriptor

Operations and Linear Equations & Inequalities

Linear Inequalities

Write, solve, and/or graph linear inequalities using various methods.

▶ Eligible Content

A1.1.3.1.2 Identify or graph the solution set to a linear inequality on a number line.

▶ Instructional Supports

- Ask students to solve a one-variable linear inequality. For example, ask students to solve the following inequality: $8 - x > 12 + 3x$. Students should solve the inequality as shown.

$8 - x > 12 + 3x$	
$8 - x + (-3x) > 12 + 3x + (-3x)$	(addition property of equality)
$8 - 4x > 12$	(simplify)
$8 - 4x + (-8) > 12 + (-8)$	(addition property of equality)
$-4x > 4$	(simplify)
$(-\frac{1}{4})(-4x) < (-\frac{1}{4})(4)$	(multiplication property of equality)
$x < -1$	(simplify)

Note that the inequality symbol remained as a (>) until both sides were multiplied by $-\frac{1}{4}$. The inequality should switch after the multiplication is complete, in the next step. Students should try placing a few values in $x < -1$ in the original inequality to check their answer.

- Ask students to solve a one-variable linear inequality and represent the solutions on a number line. For example, ask students to determine and graph the solutions to the inequality shown.

$-3(2x + 5) \leq 22$

Students should begin by solving the inequality as shown.

$-3(2x + 5) \leq 22$

$-6x - 15 \leq 22$ (distributive property)

$-6x - 15 + 15 \leq 22 + 15$ (addition property of equality)

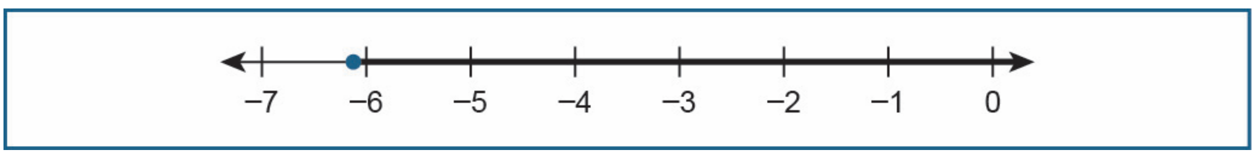
$-6x \leq 37$ (simplify)

$(-\frac{1}{6})(-6x) \leq (-\frac{1}{6})(37)$ (multiplication property of equality)

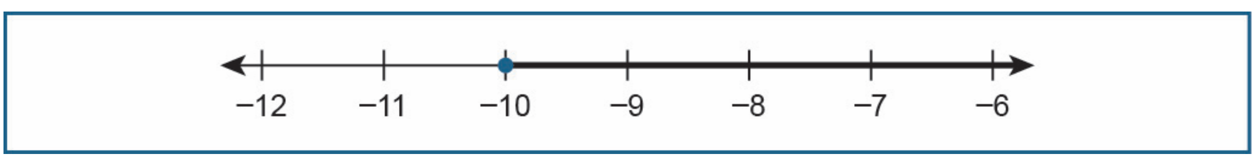
$x \geq -6\frac{1}{6}$ (simplify)

Note that the direction of the inequality does not change when the distributive property is applied in the first step, because the −3 is multiplied only on the left side of the inequality instead of on both sides. However, the direction of the inequality does change after both sides of the inequality are multiplied by $-\frac{1}{6}$.

Next, students should graph the solutions on a number line. Because the inequality symbol used here includes the meaning "or equal to," the endpoint of $-6\frac{1}{6}$ is graphed with a closed circle, which indicates numbers to the right of $-6\frac{1}{6}$ on the number line. The endpoint should be graphed to the left of −6 because $-6\frac{1}{6}$ is a little farther from 0 than −6 is. A sample student number line is shown.



- Ask students to determine an unknown value in a linear inequality, given its solution set. For example, ask students to solve for n in the inequality $14 + 3x - (5x - n) \leq 30$ if it has the following solutions.



Students can begin by identifying the solutions on the number line. Since the solutions are all to the right of −10, the solutions are either $x > -10$ or $x \geq -10$. Since the circle is closed, that means −10 is included in the solution set and the solutions are represented by $x \geq -10$. Students should then simplify $14 + 3x - (5x - n) \leq 30$ to compare the solutions, as shown.

$14 + 3x - (5x - n) \leq 30$

$14 + 3x - 5x + n \leq 30$ (distributive property)

$(14 + n) - 2x \leq 30$ (simplify and commutative property)

$(14 + n) - 2x + -(14 + n) \leq 30 + -(14 + n)$ (addition property of equality)

$-2x \leq 30 - 14 - n$ (distributive property)

$-2x \leq 16 - n$ (simplify)

$(-\frac{1}{2})(-2x) \leq (-\frac{1}{2})(16 - n)$ (multiplication property of equality)

$x \geq -\frac{1}{2}(16 - n)$ (simplify)

$x \geq -8 + \frac{n}{2}$ (distributive property)

Students should compare the solutions to $x \geq -8 + n$ to the solutions from $x \geq -10$. Then, students should set the inequality equal to -10 to solve for n, as shown.

$-8 + \frac{n}{2} = -10$

$-8 + \frac{n}{2} + 8 = -10 + 8$ (addition property of equality)

$n = -4$ (simplify)

Therefore, $n = -4$.

- Ask students to write and solve a one-variable linear inequality to model a real-world situation. For example, give students the following situation.

  Caleb sells scarves. Each scarf requires \$15 for supplies, and he sells each scarf for \$20. He spent \$200 for advertising. What number of scarves can Caleb sell to earn a profit? Students should identify the variable x as being the number of scarves. The money that Caleb earns is represented by $20x$ dollars. The expenses are represented by $200 + 15x$. He wants the money earned to be greater than the expenses, so the inequality that represents this situation is $20x \geq 200 + 15x$. Then, students should solve the inequality, as shown.

$20x \geq 200 + 15x$	
$20x + (-15x) \geq 200 + 15x + (-15x)$	(addition property of equality)
$5x \geq 200$	(simplify)
$\frac{1}{5}(5x) \geq \frac{1}{5}(200)$	(multiplication property of equality)
$x \geq 40$	(simplify)

Therefore, Caleb will earn a profit whenever he sells 40 or more scarves.

▶ Identifying and Addressing Common Misconceptions

- Some students may switch the direction of the inequality symbol when distributing a negative number. Ask students to solve the inequality $-5(-2x + 1) \leq 12$. Some students may change the inequality to $10x - 5 \geq 12$ when distributing the -5. Remind students that the direction of the inequality changes only when the same number is multiplied by **both** sides of the inequality. Ask students to look at their work and determine whether they multiplied both sides of the inequality by the same number. When they realize they didn't, direct them to see that the inequality does not change directions.
- Some students may forget to change the direction of an inequality when multiplying both sides of the inequality by a negative. Ask students to solve the inequality $-3x + 7 < 10$. Students who fail to change the direction of an inequality may respond with $x < -1$. Have students test different numbers from their solution in the original inequality, such as -5 or -10. Students should discover that their solutions don't satisfy the original inequality, which means their solutions are incorrect. Remind students that the direction of the inequality changes whenever both sides are multiplied by a negative number. Have students check their work for this step and update their solution accordingly.

▶ Enrichment

- Ask students to write an inequality that meets certain requirements and has a given solution. For example, ask students to write an inequality that has at least 4 steps in its solution process, requires the distributive property, and has a solution of $x < -4$.
- Ask students to solve the same inequality in two different ways, with as much difference in the two approaches as possible.

► Key Terms

graph, greater than, greater than or equal to, less than, less than or equal to, linear inequality, negative, number line, solution set, variable

A1.1.3.1.3

▶ Module, Assessment Anchor, Anchor Descriptor

Operations and Linear Equations & Inequalities

Linear Inequalities

Write, solve, and/or graph linear inequalities using various methods.

▶ Eligible Content

A1.1.3.1.3 Interpret solutions to problems in the context of the problem situation.

Note: Linear inequalities only.

▶ Instructional Supports

- Give students a real-world inequality that has not been solved for a variable and ask students to interpret the meaning of the variable. For example, give students the following situation.

  An online shopping market offers two payment options to members. A monthly subscription costs \$14.98 per month plus a one-time fee of \$20. A lifetime subscription costs a one-time fee of \$199. A member is trying to determine how many months of membership could be purchased that would cost less than a lifetime membership. The inequality $20 + 14.98x \leq 199$ represents the situation. Describe what x represents in the context of this situation.

  Students should begin by comparing the inequality to the context, paying close attention to the variable. The variable is multiplied by 14.98, which corresponds with the phrase "14.98 per month" from the context. Therefore, students can conclude that the variable x represents the number of months. More specifically, x represents the maximum number of months that can be purchased for an amount that is cheaper than the cost of a lifetime membership.

- Ask students to interpret the meaning of a provided solution set to an inequality that models a real-world situation. For example, give students the following situation.

  A child's ticket at a movie theater can be purchased for children who are x years old. The acceptable values of x are $5 \leq x < 12$. Describe the ages of children who can buy a child's ticket.

Students should note that x, the age of the child, must be 5 or older but younger than 12. Therefore, children that are 5, 6, 7, 8, 9, 10, or 11 years old can buy a child's ticket.

- Ask students to solve and interpret the solutions to a provided inequality that models a real-world situation. For example, give students the following situation.

Emma has \$30 to buy a water bottle. She will use a 20%-off coupon and a \$5-off coupon. The inequality $0.8x - 5 \leq 30$ represents this situation. Solve this inequality for x and interpret the solution within the context.

Students can begin by solving the inequality, as shown.

$0.8x - 5 \leq 30$

$0.8x - 5 + 5 \leq 30 + 5$ (addition property of equality)

$0.8x \leq 35$ (simplify)

$(\frac{1}{0.8})(0.8x) \leq (35)(\frac{1}{0.8})$ (multiplication property of equality)

$x \leq 43.75$ (simplify)

Students can then define the variable and interpret the solutions within the context.

Emma is looking for the original price of a water bottle that she can afford. The solutions say that x must be less than or equal to 43.75, meaning she can only afford a water bottle with an original price of \$43.75 or less.

- Ask students to interpret a situation represented in a two-variable inequality within the context. For example, give students the following situation.

Jacob owns a coffee stand that sells large cups of coffee for \$3.50 and small cups of coffee for \$2.75. He wants to earn at least \$200 in revenue. Jacob creates the inequality $3.5x + 2.75y \geq 200$ to represent the situation. Interpret the point (20, 40) within the context of the situation.

Students should begin by defining the variables. Since x has a coefficient of 3.5 and \$3.50 is the cost for a large cup of coffee, x represents the number of large cups of coffee sold. Similar reasoning shows that y represents the number of small cups of coffee sold. The point (20, 40) means that 20 large cups of coffee and 40 small cups of coffee were sold. The problem does not say whether (20, 40) is a solution, so students must determine whether this point is a solution. Students should substitute the point into the inequality and simplify: $3.5(20) + 2.75(40) \geq 200 \rightarrow 180 \geq 200$. This is a false statement, so that means the point does not satisfy the inequality. Therefore, students must conclude that selling 20 large cups of coffee and 40 small cups of coffee does not meet the revenue goal.

► Identifying and Addressing Common Misconceptions

- Some students may interpret the solution to a real-world problem modeled by an inequality as a single value instead of a range of values. Give students the following situation.

  The inequality $x < 48$ represents the heights, in inches, of children that will be allowed to enter a playground. Describe the solutions within the context.
  Some students may incorrectly conclude that a child who is 48 inches tall can enter the playground. Ask students to substitute their solution into the inequality for x and then have them read it out loud. Students should see that $48 < 48$, or "48 is less than 48," is not true. Ask students to describe what solutions do make the inequality true and then update their description. Students should conclude that any child shorter than 48 inches can enter the playground.

- Some students may incorrectly include or exclude the boundary value when interpreting an inequality. Give students the following situation.

  The inequality $18 \leq x < 30$ represents the number of raffle tickets Josie needs to sell to earn a sweatshirt. Describe the solutions within the context.
  Some students may incorrectly conclude that Josie needs to sell more than 18 and less than 30 raffle tickets. Ask students to substitute 18 into the inequality and determine whether it satisfies the inequality. Students should conclude that $18 \leq 18 < 30$ is a true statement, so this also needs to be included in their interpretation. Therefore, students should update their solution to indicate that Josie can sell 18 **or more** raffle tickets and less than 30 tickets.

► Enrichment

- Ask students to draw comparisons between two or more inequalities modeling the same real-world context.

► Key Terms

boundary value, inequality, linear inequality, solution, variable

A1.1.3.2.1

▶ Module, Assessment Anchor, Anchor Descriptor

Operations and Linear Equations & Inequalities

Linear Inequalities

Write, solve, and/or graph systems of linear inequalities using various methods.

▶ Eligible Content

A1.1.3.2.1 Write and/or solve a system of linear inequalities using graphing.

Note: Limit systems to two linear inequalities.

▶ Instructional Supports

- Ask students to graph the solutions to a system of linear inequalities. For example, ask students to graph $\begin{cases} y \le 2x + 1 \\ -x - 3y < -6 \end{cases}$. Students should begin by writing each inequality in slope-intercept form in order to graph each inequality. The top inequality is already in slope-intercept form, but the bottom inequality needs to be rearranged. Students should apply algebraic properties to isolate the y, as shown below, and should be reminded to change the direction of the inequality symbol whenever both sides of an inequality are multiplied by a negative number.

$-x - 3y < -6$

$x - x - 3y < x - 6$ (addition property of equality)

$-3y < x - 6$ (simplify)

$\left(-\frac{1}{3}\right)(-3y) < \left(-\frac{1}{3}\right)(x - 6)$ (multiplication property of equality)

$y > -\frac{1}{3}x + 2$ (simplify)

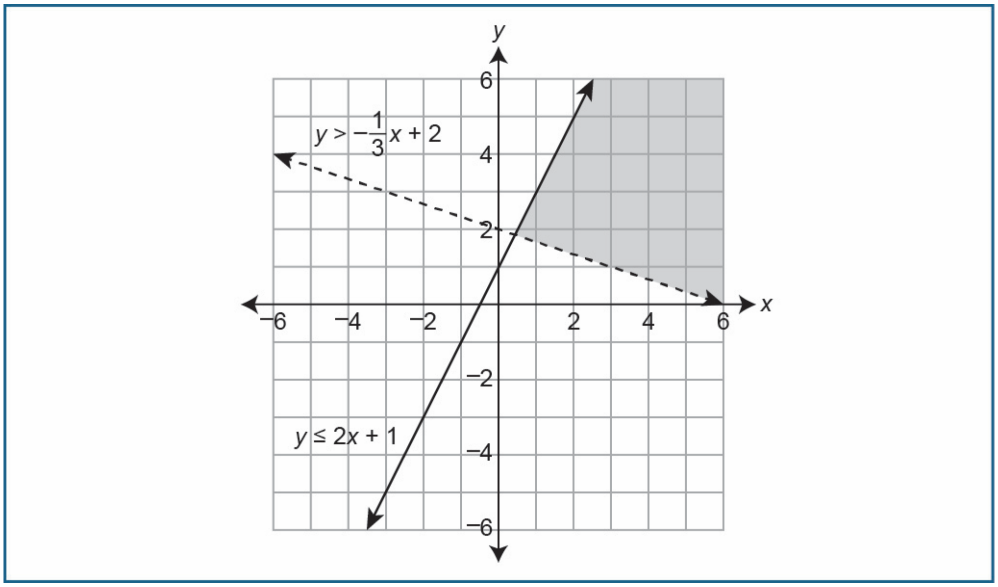
Students can begin by graphing each inequality. The top inequality is graphed with a solid line because it includes the boundary with the inequality symbol ≤. The bottom inequality is graphed with a dashed line because it does not include the boundary with the inequality symbol >. The lines are graphed using the y-intercepts and slopes. The region that needs to be shaded is determined by

$y >$ (shade above the line) or $y <$ (shade below the line). Therefore, the solid line for $y \le 2x + 1$ has the region below the line shaded, and the dashed line for $y > -\frac{1}{3}x + 2$ has the region above the line shaded. The region where the shading overlaps is the area that represents the solution to the system of inequalities. This area can be confirmed by substituting a point from the region into both inequalities to make sure that it satisfies the inequalities. For instance, students may test the point (4, 2), as shown.

$$y \le 2x + 1 \rightarrow 2 \le 2(4) + 1 \rightarrow 2 \le 9 \qquad \textbf{true}$$

$$y > -\tfrac{1}{3}x + 2 \rightarrow 2 > -\tfrac{1}{3}(4) + 2 \rightarrow 2 > \tfrac{2}{3} \qquad \textbf{true}$$

Therefore, the shaded region is confirmed. A sample student graph is provided.



- Ask students to write a system of linear inequalities to model a real-world situation. For example, give students the following situation.

  Charlie works two jobs. He works at his school's concession stand and at a fast-food restaurant. He can work no more than a combined total of 20 hours per week. He earns \$6 an hour working at the concession stand and \$8 an hour working at the fast-food restaurant. Charlie would like to earn more than \$56 per week. Write a system of inequalities to model the situation.

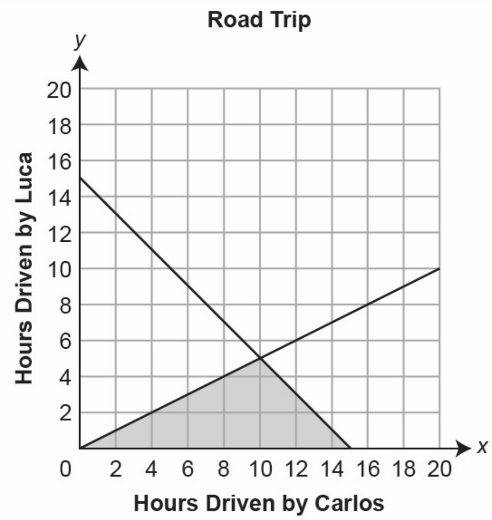
  Students should begin by defining the variables. The two things that vary are the number of hours worked at each place. The variables can be defined in either order, but students are most likely to define them in the order in which they are presented, where x is the number of hours worked at the concession stand and y is the number of hours worked at the fast-food restaurant. The total hours he works must be less than or equal to 20, so one of the inequalities can be represented as $x + y \leq 20$. He also wants to earn more than \$56 per week. The amount he earns at the concession stand is represented by $6x$ (\$6 times the number of hours worked), and the amount he earns at the fast-food restaurant is represented by $8y$ (\$8 times the number of hours worked). Therefore, $6x + 8y > 56$. The system of inequalities that models this situation is $\begin{cases} x + y \leq 20 \\ 6x + 8y > 56 \end{cases}$.

- Ask students to graph the solutions to a real-world problem that can be modeled with a system of inequalities. For example, give students the following situation.

  Carlos and Luca are creating a driving schedule for a cross-country road trip. They will share the driving between the two of them, and they estimate the drive will take no more than 15 hours. Carlos wants to drive at least twice as much as Luca drives. Create a graph to show the different numbers of hours Carlos and Luca could each drive.

  Students should begin by labeling variables, defining the number of hours Carlos drives as x and the number of hours Luca drives as y. Next, students should write a system of inequalities to model the situation. Since the total driving time will be no more than 15 hours, one of the inequalities is represented by $x + y \leq 15$. Since Carlos wants to drive at least twice as much as Luca drives, $x \geq 2y$. Therefore, the system of inequalities is $\begin{cases} x + y \leq 15 \\ x \geq 2y \end{cases}$.

  Then, students can graph the system. The inequalities can be rewritten in slope-intercept form, changing the system to $\begin{cases} y \leq -x + 15 \\ y \leq 0.5x \end{cases}$. Both lines will be solid, with shading below the line, because y is less than or equal to, as opposed to just "less than." A sample student graph is provided.



► Identifying and Addressing Common Misconceptions

- Some students may shade the wrong side of a linear inequality when graphing. For example, ask students to graph the inequality $y > 8 - 2x$. Students who shade the wrong side of a linear inequality may shade below the line. Ask students to locate a point from their shaded region and substitute it into the inequality. After the students discover that their point doesn't satisfy the inequality, ask them to try a point on the other side. Direct students to see that shading for inequalities of the form "$y >$" or "$y \geq$" is always above the line (and vice versa) because the y-values (which are along the vertical axis) must be greater than the values along the line.
- Some students may include all the shading from each inequality in a system instead of shading only the overlap. Ask students to graph the solutions to $\begin{cases} y > 3x - 5 \\ y < -\frac{1}{2}x + 3 \end{cases}$. Some students may shade three sections of the graph: the overlap, the section shaded only by the top inequality, and the section shaded only by the bottom inequality. Remind students that a solution to a system must satisfy both inequalities, not just one. It would be helpful to shade one inequality in one color and the other inequality in a contrasting color. Shading can also be used for lines that go in different directions. Some method should be chosen to make the three regions easier to see. Have students substitute values from the three different sections they have shaded to see which points satisfy both inequalities. Students should notice that only points in the overlap satisfy both inequalities.

► Enrichment

- Ask students to graph the solutions to a system of inequalities that contains three or more inequalities.
- Ask students to create a system of four inequalities whose solution is a given quadrilateral, such as a parallelogram.

► Key Terms

dashed boundary line, graph, overlap, region, shading, slope-intercept form, solid boundary line, solutions, system of linear inequalities, region

A1.1.3.2.2

► Module, Assessment Anchor, Anchor Descriptor

Operations and Linear Equations & Inequalities

Linear Inequalities

Write, solve, and/or graph systems of linear inequalities using various methods.

► Eligible Content

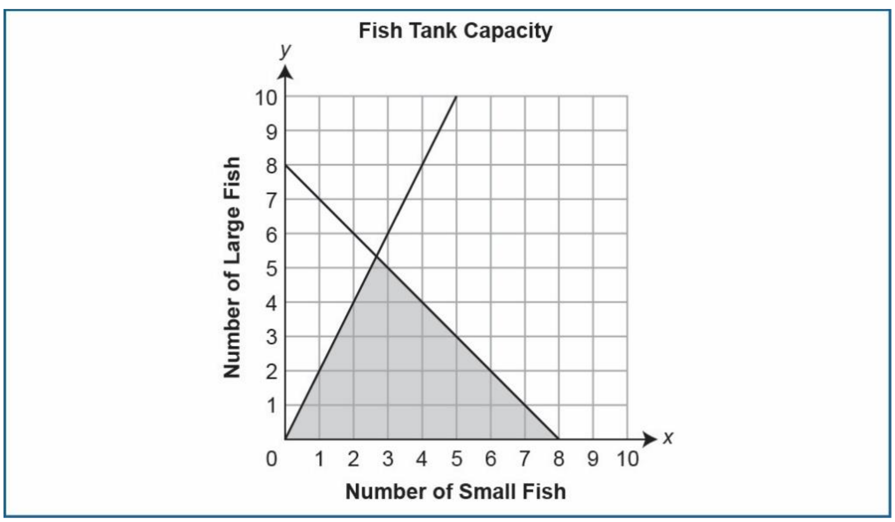
A1.1.3.2.2 Interpret solutions to problems in the context of the problem situation.

Note: Limit systems to two linear inequalities.

► Instructional Supports

- Ask students to interpret the meaning of the shaded region in a system of inequalities modeling a real-world situation. For example, ask students to describe the meaning of the shaded region in the following situation.

  A fish tank's capacity has several limitations, as shown in the graph. What does the shaded region represent?



  Students should notice that ordered pairs represent (number of small fish, number of large fish) and that the shaded region represents the pairs that meet the capacity requirements of the fish tank.

- Ask students to interpret the meaning of a point being modeled by a system of inequalities within the context of a real-world situation. For example, ask students to interpret the meaning of the point (2, 5) within the following situation.

  Lena is buying coffee and donuts for her friends. Each cup of coffee costs \$4, and each donut costs \$2. She can spend no more than \$20 and wants to buy more than 5 items. The system $\begin{cases} 4x + 2y \le 20 \\ x + y > 5 \end{cases}$ models the situation.

  Students should begin by identifying the variables that represent the cost of a cup of coffee and the cost of a donut. The *x* represents the number of cups of coffee bought and the *y* represents the number of donuts bought. The top equation represents the amounts of money where the coefficients represent the cost for each item. The point (2, 5) represents Lena buying 2 cups of coffee and 5 donuts. To complete the interpretation, students need to determine whether the point is a solution that satisfies both inequalities, as shown.

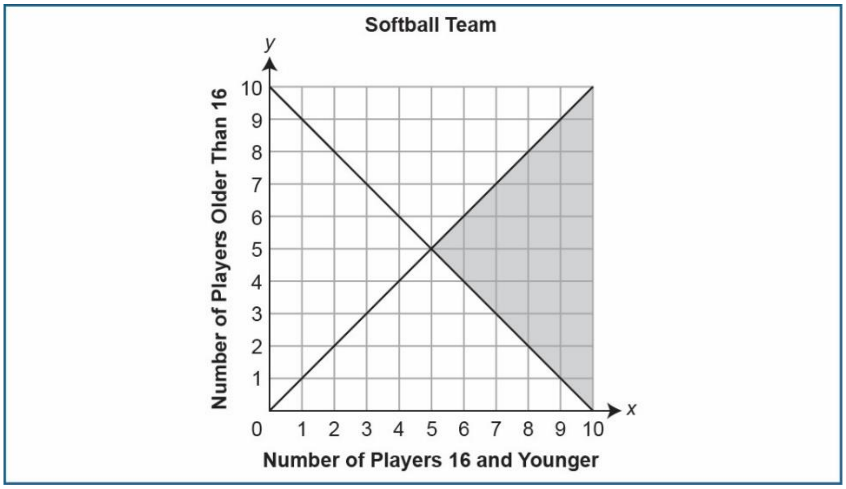
  $$\begin{cases} 4x + 2y \le 20 \\ x + y > 5 \end{cases} \rightarrow \begin{cases} 4(2) + 2(5) \le 20 \\ 2 + 5 > 5 \end{cases} \rightarrow \begin{cases} 18 \le 20 \\ 7 > 5 \end{cases}$$

  Since both inequalities are satisfied, point (2, 5) is a solution. Therefore, buying 2 cups of coffee and 5 donuts meets all the requirements.

- Ask students to use a system of inequalities modeling a real-world problem to determine a solution. For example, give students the following situation.

  Luis is forming a summer league softball team. The requirements for *x*, the number of players 16 years old and younger, and *y*, the number of players older than 16 years old, is modeled by $\begin{cases} x + y \ge 10 \\ x \ge y \end{cases}$. Describe two possible combinations for the team.

  Students can graph the system to get a better understanding of the solutions. To graph the system, students should solve both inequalities and transformed them into slope-intercept form. The top inequality can have $-x$ added to both sides of the inequality, which yields $y \ge -x + 10$. The bottom inequality can be rewritten in the opposite order as long as special attention is paid to the inequality symbol. Therefore, the system of inequalities is equivalent to $\begin{cases} y \ge -x + 10 \\ y \le x \end{cases}$. Then, students can graph the system, as shown.



Students can use the graph to identify solutions by locating points in the shaded region. Students can identify points such as (9, 3), (10, 7), or (9, 8). To describe a team, students can interpret any of these ordered pairs from the shaded region. For instance, students may conclude that the team could be made up of 9 players 16 and younger and 3 players older than 16.

▶ Identifying and Addressing Common Misconceptions

- Students may think that a solution to a system of inequalities is the only solution instead of one of many solutions. Ask students to describe the meaning of the point (90, 20) in the context of the system $\begin{cases} y \leq 3x \\ x + y > 100 \end{cases}$, where x is the number of hamburgers and y is the number of hot dogs a concession stand prepares. Some students may incorrectly conclude that the point means that the concession stand must prepare exactly 90 hamburgers and 20 hot dogs to meet the requirements. Ask students to consider the point (100, 50) by substituting 100 for x and 50 for y in both inequalities. Students should discover that the point is also a solution because it satisfies both inequalities and because it falls in the shaded region of the graph. Therefore, 100 hamburgers and 50 hot dogs is another viable solution. Remind students that inequalities and systems of inequalities often have multiple solutions. Therefore, students shouldn't describe a solution as the only solution unless they are confident there cannot be any other viable solutions.

- Students may think that a point is a solution to a system of inequalities if it satisfies only one of the inequalities. Ask students to describe the meaning of the point (20, 10) in the system $\begin{cases} x + y \geq 60 \\ x \geq 20 \end{cases}$, where x is the number of minutes spent running and y is the number of minutes spent lifting weights during a workout. Some students may incorrectly conclude that the point (20, 10) means that one option for the workout is to run for 20 minutes and lift weights for 10 minutes, since the point satisfies the bottom inequality. Remind students that solutions to a system of inequalities must satisfy both inequalities. When the students realize that (20, 10) doesn't work in the first inequality, ask them to revise their interpretation of the point. Since the point doesn't satisfy both inequalities, the combination of 20 minutes of running and 10 minutes of lifting weights does not meet the requirements and thus isn't a viable solution.

► Enrichment

- Ask students to describe solutions that are modeled by a system of three or more linear inequalities within the context of a real-world situation.
- Ask students to describe solutions that are modeled by a system composed of a linear inequality and a quadratic inequality within the context of real-world situations.

► Key Terms

solution, system of linear inequalities

A1.2.1.1.1

▶ Module, Assessment Anchor, Anchor Descriptor

Linear Functions and Data Organizations

Functions

Analyze and/or use patterns or relations.

▶ Eligible Content

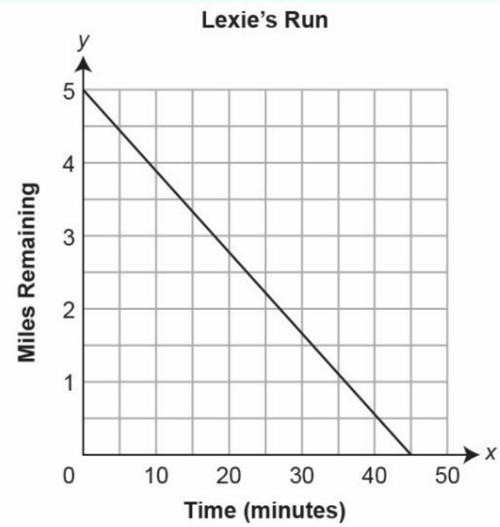
A1.2.1.1.1 Analyze a set of data for the existence of a pattern and represent the pattern algebraically and/or graphically.

▶ Instructional Supports

- Ask students to create a graph of a linear pattern given several data points from the relationship. For example, give students the following situation.

  Lexie keeps track of how many miles, y, are remaining in her run after x minutes of running at a steady pace. Two data points are given: (9, 4) and (13.5, 3.5). The pattern continues with the same rate of change.

  Students should note that, because the number of remaining miles (y) decreases at a steady rate, the relationship is linear. Students can then plot the given points and then use the slope to locate more points on the line. The slope is calculated by dividing the difference in the y-values by the difference in the x-values: $\frac{4-3.5}{9-13.5} = \frac{0.5}{-4.5} = -\frac{1}{9}$. The graph shown represents this situation.



- Ask students to write an equation that models a linear pattern, given several data points from the relationship. For example, give students the following situation.

  A school fair charges an entry fee for each person. A person can buy game tickets for an additional cost. The table shown represents the relationship between a person entering the school fair and buying game tickets.

  School Fair Game Tickets

Number of Tickets	Total Cost (dollars)
5	7.50
12	11.00
20	15.00

  The pattern continues. Write an expression to describe the pattern.
  Students can begin by determining whether the relationship is linear by checking the slopes between sets of points shown in the table. The slope from 5 to 12 tickets is $\frac{11-7.5}{12-5}=\frac{3.5}{7}=\frac{1}{2}$. The slope from 12 to 20 tickets is $\frac{15-11}{20-12}=\frac{4}{8}=\frac{1}{2}$. So, students can conclude that the relationship is linear with a slope of $\frac{1}{2}$. Students should then solve for the y-intercept. One approach is to substitute the slope (m) and a point from the table, (x, y), into the slope-intercept formula and solve for b, as shown.

  $y = mx + b$

$$7.5 = \left(\frac{1}{2}\right)(5) + b$$

$$7.5 = 2.5 + b$$

$$5 = b$$

Therefore, the equation that models the situation is $y = \frac{1}{2}x + 5$.

- Ask students to write an expression that can be used to determine the *n*th term of a pattern, given several data points from the pattern. For example, give students the following problem.

  The first three terms of a pattern are 1, 7, and 13. The pattern continues. Write an expression to represent the *n*th term.

  The pattern is increasing by 6 each time. This means the expression can be represented as $6n + \_\_$. The missing value can be determined by substituting a value for n into the expression and making it equal to the corresponding term. For instance, when $n = 2$, the term is 7. Therefore, $6(2) + \_\_ = 7 \rightarrow 12 + \_\_ = 7 \rightarrow \_\_ = -5$. Therefore, the expression $6n - 5$ represents the *n*th term of the expression.

▶ Identifying and Addressing Common Misconceptions

- Some students may determine the slope and not consider the *y*-intercept when writing the equation of a linear relationship. Ask students to write the equation of the line that represents a linear pattern that includes (4, 4), (8, 10), and (10, 13). Students who do not consider the *y*-intercept may calculate the slope and conclude that the final equation is $y = 1.5x$. Ask students to graph the equation and plot the given points to see if they coincide. Students should note that the three given points form a line that has the same slope as $y = 1.5x$, but the graphed line of the equation is in a different location. Remind students that not all linear relationships go through the origin and that the *y*-intercept must always be calculated for an equation.

- Some students may represent the first term in a pattern as $n = 0$ instead of $n = 1$. Ask students to write an expression to model the *n*th term of a pattern whose first three terms are 27, 25.25, and 23.5. Some students may incorrectly respond with $27 - 1.75n$. Ask students to substitute $n = 3$ into the expression to see if they get 23.5, since the third term is 23.5. Students should note that the term and the output are not the same. Remind students that the first term (27) represents $n = 1$, not $n = 0$.

► Enrichment

- Ask students to draw comparisons between two different linear patterns by graphing or writing an equation for each pattern. For example, ask students to describe the differences between the rates of change and the starting values of the following two relationships, as shown in the tables.

Callie

Hours	Pages Remaining
3	175
5	125
6	100

Marco

Hours	Pages Remaining
1	279
4	216
7	153

- Ask students to write an equation, given a comparison to another linear relationship. For example, ask students to write an equation of a linear relationship that has a slope that is twice as steep as a linear pattern that goes through the points (1.5, 4), (3, 7), and (6, 13).

► Key Terms

data set, equation, graph, linear, *n*th term, pattern, relationship, slope, table, *y*-intercept

A1.2.1.1.2

▶ Module, Assessment Anchor, Anchor Descriptor

Linear Functions and Data Organizations

Functions

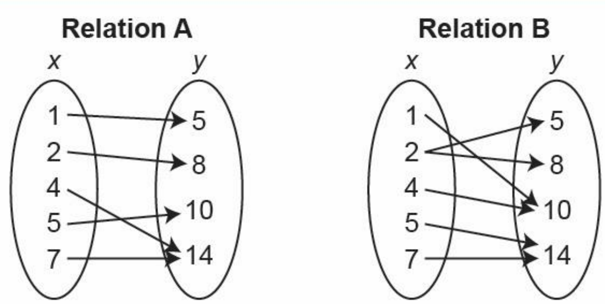
Analyze and/or use patterns or relations.

▶ Eligible Content

A1.2.1.1.2 Determine whether a relation is a function, given a set of points or a graph.

▶ Instructional Supports

- Ask students to examine relations presented in various forms to determine whether each given relation is a function; that is, each input (x-value) has exactly one output (y-value). For example, give the students the two relations represented by the mappings shown. Ask students to determine whether each relation is a function and to justify their response.



Students should observe that relation A is a function, as each input from set x maps to only one output in set y. Ask students to further analyze inputs 4 and 7 in relation A, and guide them to conclude that a function may have multiple input values sent to the same output value, as long as each input has only one output. Students should observe that relation B is not a function, as the input value 2 maps to two different outputs, 5 and 8.

Then, ask students to rewrite each relation as a set of ordered pairs. Students should rewrite relation A as $\{(1,5),(2,8),(4,14),(5,10),(7,14)\}$ and relation B as $\{(1,10),(2,5),(2,8),(4,10),(5,14),(7,14)\}$. Students should conclude that, when written as a set of ordered pairs, the relation cannot have two ordered pairs with the same *x*-value but different *y*-values to be a function.

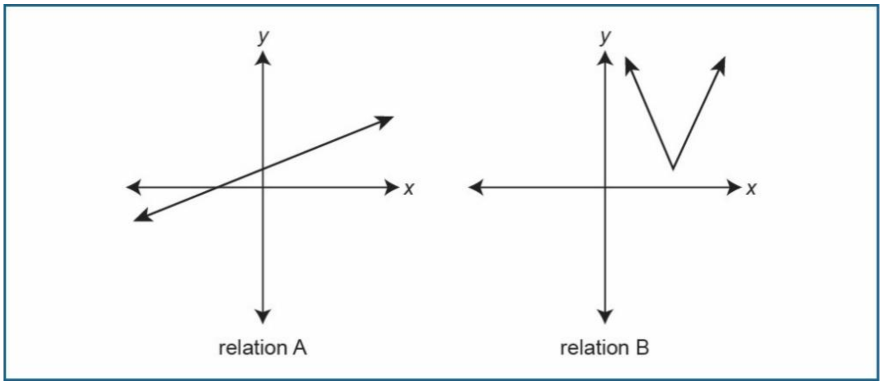
- Ask students to explore a series of relations given as sets of ordered pairs and determine whether each relation is a function. For example, ask students to determine whether each of the following relations represents a function.

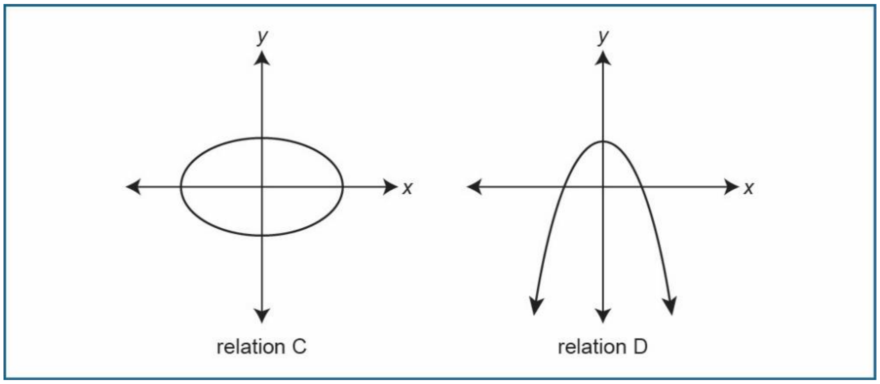
$$\mathrm{M} = \{(0,5),(2,5),(5,5),(10,5)\}$$
$$\mathrm{N} = \{(-3,2),(-1,1),(2,-4),(4,-7),(5,-9)\}$$
$$\mathrm{P} = \{(-1,2),(3,7),(-1,8),(2,11)\}$$

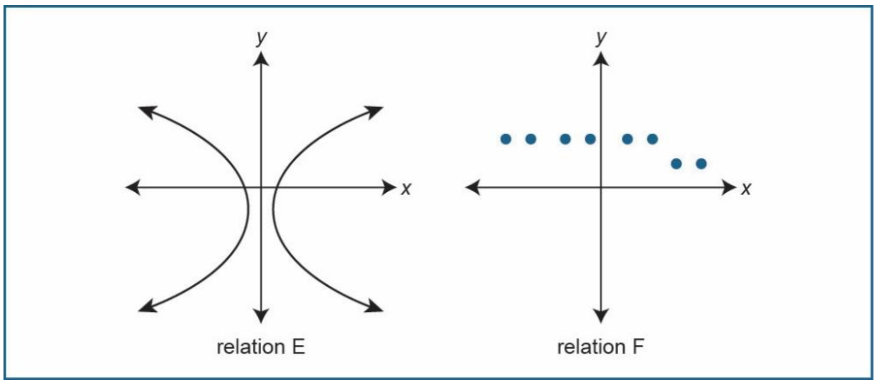
  Students should conclude that relations M and N are functions, as each input, or *x*-value, maps to exactly one output, or *y*-value. Students should observe that relation P has the input value –1 mapping to output values of both 2 and 8 and conclude that relation P is therefore not a function.

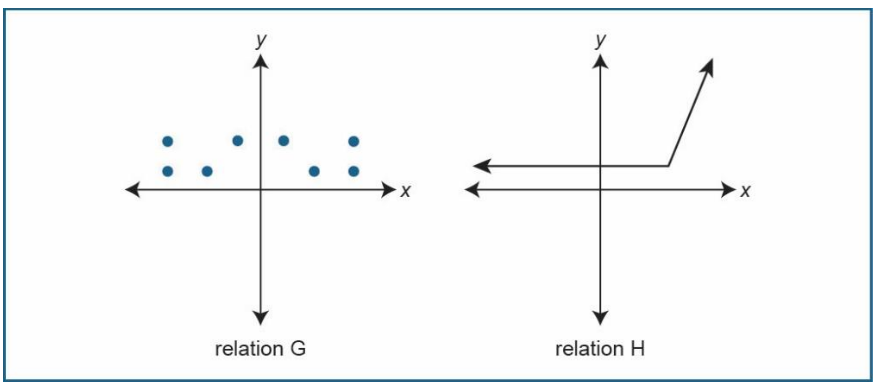
- Ask students to use coordinate pairs to create their own examples of relations that are functions. For example, ask students to define a relation that is a function with input values of {1, 2, 3, 4} and at least one output value of {0}. Students should create a variety of responses containing four points, such as $\{(1,0),(2,5),(3,10),(4,15)\}$. Ask students to compare their responses and determine whether similar sets of ordered pairs also meet the requirements, like $\{(1,0),(2,0),(3,0),(4,0)\}$ or $\{(1,0),(2,-5),(3,-10),(4,-15)\}$. Additionally, ask students to create relations that are not functions. For example, ask students to define a relation that is not a function and that has at least one input value of {–1} and one output value of {1}. Students may respond with a set of ordered pairs such as $\{(-1,1),(1,2),(-1,3),(2,3)\}$.

- Ask students to explore different graphical representations and determine whether each relation is a function. For example, give students the following graphs and have them determine which represent functions.

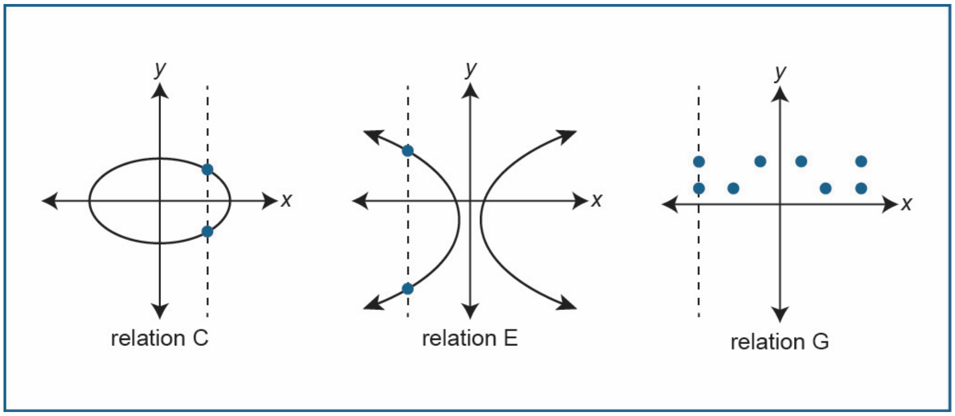








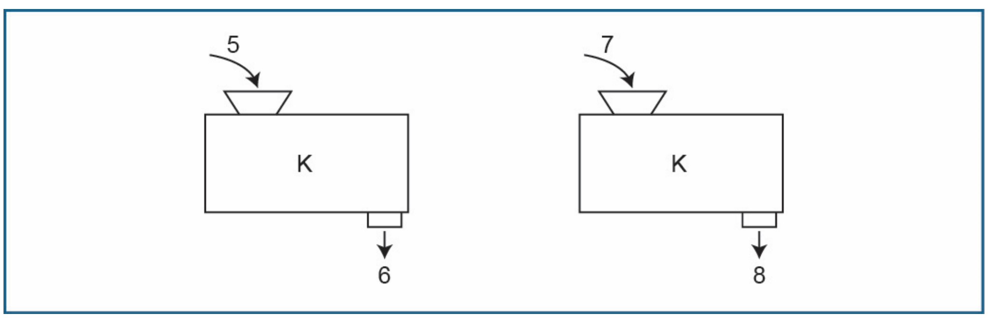
Students should observe that A, B, D, F, and H are all functions, as each input (x-value) corresponds to only one output (y-value). Students should also conclude that C, E, and G are not functions, as each contains instances where an x-value corresponds to more than one y-value, as shown.



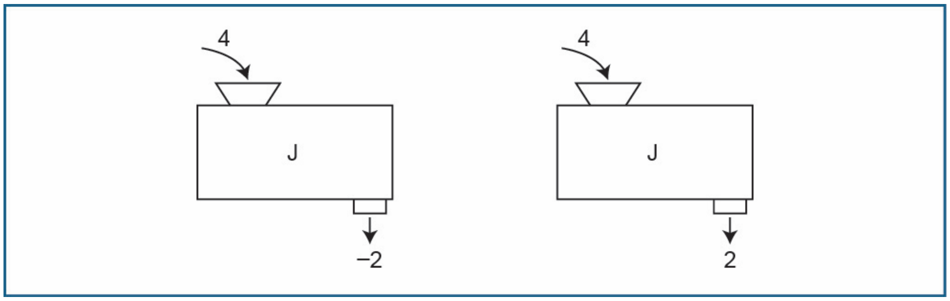
Students may abstract the use of a vertical line test to determine whether the graph of a relation represents a function. Ask students to explain why the vertical line test is conclusive. Students should explain that when a vertical line intersects a graph in more than one point, there are two coordinate pairs with the same x-value and different y-values, and therefore the relation is not a function.

- Ask students to define a function in their own words, providing both examples and nonexamples of functions. Direct students to utilize function machines, mapping diagrams, sets of ordered pairs, or graphs to define and represent their examples and nonexamples of functions. Students should respond by describing functions as relationships where each unique input value maps to one, and only one, output value. A sample student response is provided.

  A function is a relation between two groups of numbers that takes every input value to only one output value. When a relation maps one input value to multiple output values, it is no longer a function. For example, if relation K takes an input value and adds 1 to that value to get an output, the relation will be a function, as each input will have exactly one output, as shown.



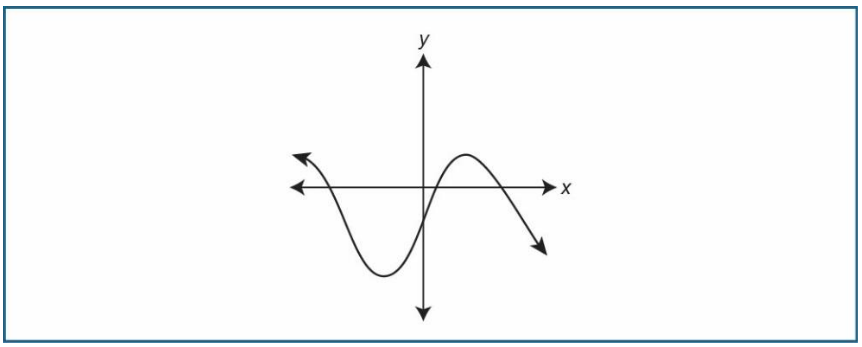
A relation J that takes an input like 4 and outputs two different options, like 2 and –2, will not be a function, as shown.



▶ Identifying and Addressing Common Misconceptions

- Some students may think that a relation with 2 or more unique input values that lead to the same output value is not a function. Ask students to determine whether the set of ordered pairs $\{(0, 4), (1, 4), (2, 4)\}$ is a function. Students who conclude that the relation is not a function since the output value of 4 corresponds to different *x*-values may not understand the difference between relations and functions. Remind students that a function requires each input to map to one and only one output but the value of the output does not need to be unique across different input values. In this case, $\{(0, \mathbf{4}), (1, \mathbf{4}), (2, \mathbf{4})\}$ is a function but the set of ordered pairs $\{(\mathbf{3}, 2), (\mathbf{3}, 5), (\mathbf{3}, 8)\}$ is not a function.

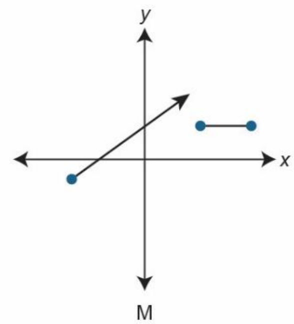
- Students may confuse the vertical line test with a horizontal line test for graphical relationships. Ask students to determine whether the graph shown represents a function.



  Students who confuse a vertical line test with a horizontal one will conclude that the graph is not a function. Remind students that a function can map multiple inputs to the same value as an output, which will allow there to be multiple values across a horizontal line in a graph.

- Some students may think $f(x)$ represents "f times x" instead of the output value from an input value of "x." Ask students to represent $f(2) = 5$ graphically. Students who draw a number line with a point at 2 and another point at 5 may have this misconception about input and output values. They may think of $f(x)$ as being the product of f and x instead of as an output value, which is graphed on the coordinate plane as a y-value. Emphasize that $f(x)$ represents function notation and not multiplication and that the graph of $f(2) = 5$ is the coordinate pair (2, 5) in a plane instead of on a number line.

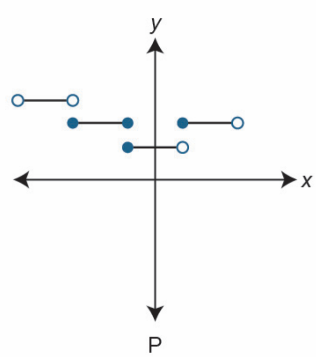
- Students may misinterpret graphical representations of relations by failing to consider the extension of a relation indicated by an arrow as part of the graph. Ask students to determine whether relation M, shown on the graph, represents a function.



Students who misrepresent graphical representations may see that the current display passes a vertical line test visually and assume it does represent a function. Ask students to extend the line with an arrowhead to demonstrate explicitly how one input will map to two outputs.

► Enrichment

- Ask students to identify when piecewise relations, given as both graphs and equations, satisfy the conditions of being a function. For example, give students relation P and relation Q as shown and ask them whether each represents a function. Then ask how the relations may be modified to allow them to satisfy the condition of being a function.



$$Q = \begin{cases} x + 3, & x \leq 0 \\ 7, & 0 \leq x \leq 5 \end{cases}$$

- Ask students to explore the relationship $x = y^2$ and determine whether y represents a function of x. Ask students to create a table of values and graph points on a standard coordinate grid. Additionally, ask students to include points in the table where $y = -4$ and $y = -9$.

- Ask students to examine the distinction between relations and functions between sets in the real world. Include contexts where the relation may or may not be a function depending on how the input and output sets are defined. For example, ask a student to determine whether the change a cashier gives back to a customer after each purchase may represent a function and justify the reasoning, or whether the number of points a basketball player scores each game over the course of a season represents a function. Additionally, ask students to create new relationships that will satisfy the requirements of a function in the real world with different contextual domain and range sets. For example, ask students to create relations that map the set of {students in class} to {classroom objects} that are functions and are not functions. Students can assert that relations that map a student to an individual desk may satisfy the requirement of a function, but a relation that maps students to their writing utensils may not satisfy the requirement, as an individual student may have multiple writing utensils.

► Key Terms

dependent variable, element, function machine, graph, independent variable, input, mapping, ordered pairs, output, relation, set, vertical line test, *x*-value, *y*-value

A1.2.1.1.3

▶ Module, Assessment Anchor, Anchor Descriptor

Linear Functions and Data Organizations

Functions

Analyze and/or use patterns or relations.

▶ Eligible Content

A1.2.1.1.3 Identify the domain or range of a relation (may be presented as ordered pairs, a graph, or a table).

▶ Instructional Supports

- Ask students to identify both the domain and the range in a relation expressed as a set of ordered pairs. For example, ask students to determine the domain and the range of the relation $\{(-2, 4), (-1, 7), (0, 2), (1, 4), (2, 12)\}$. Students should observe that the domain is the set $\{-2, -1, 0, 1, 2\}$ and the range is the set $\{2, 4, 7, 12\}$. Students should order the numbers in the set from least to greatest and not repeat any values.
- Ask students to identify both the domain and the range in a relation expressed in a table. For example, present students with the following situation.
  Paulette is playing a video game. For each level she completes in the game, she records the length of time, in seconds, it takes to complete the level and the number of points she earns. The table shows her results.

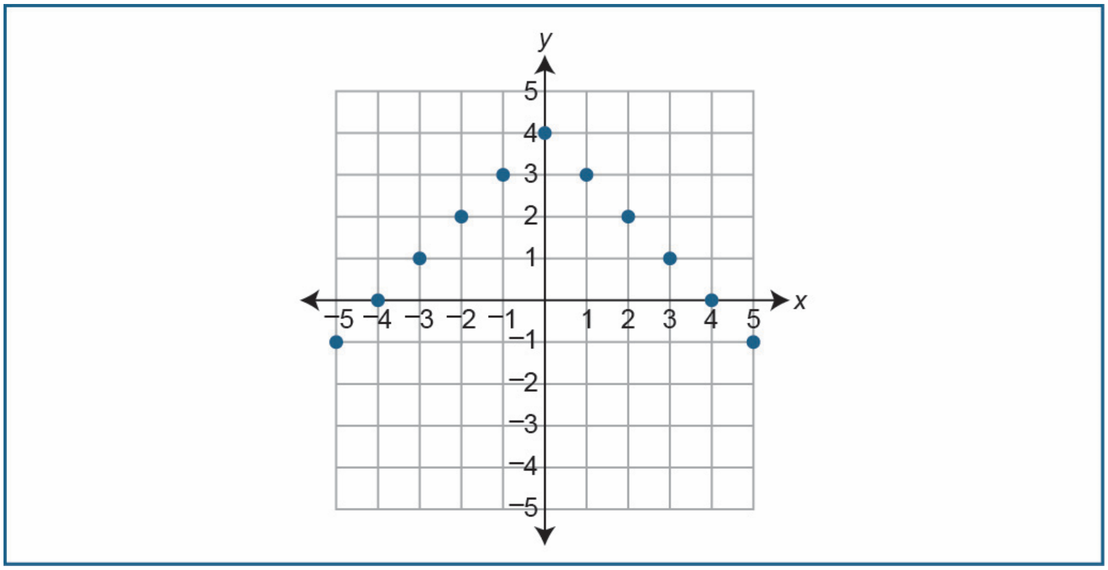
  Video Game Levels Completed

Time (seconds)	31	34	38	46	72
Points Earned	138	150	172	230	481

  Students should determine that the number of points Paulette earns is a function with respect to the length of time it takes to complete a level.

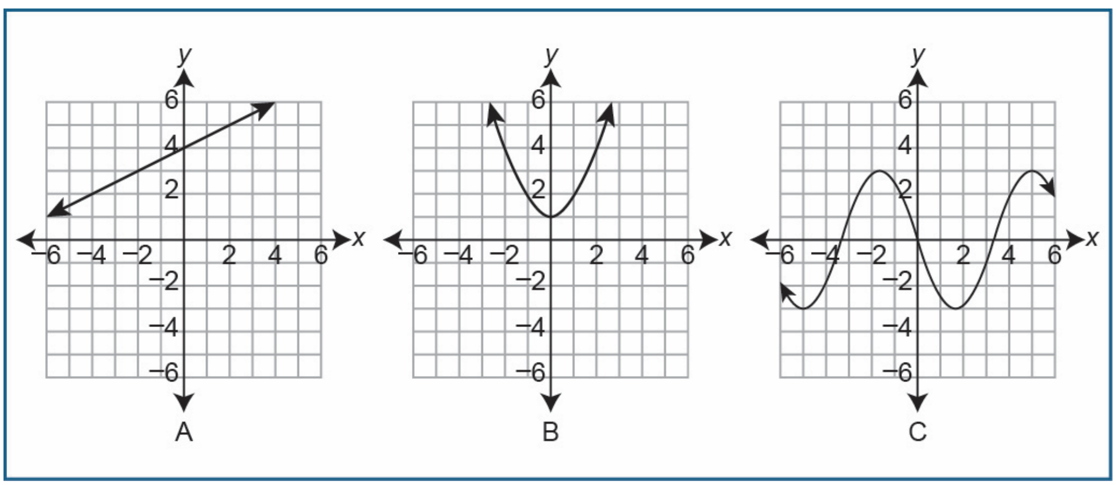
  Students should conclude that the domain of the data in the table, time, is $\{31, 34, 38, 46, 72\}$ and the range, points earned, is {138, 150, 172, 230, 481}.

- Ask students to identify both the domain and the range in the graph of a discrete relation. For example, give students the graph shown.



Students should observe that the domain of the graph is $\{-5, -4, -3, \ldots, 2, 3, 4, 5\}$ and the range of the graph is $\{-1, 0, 1, 2, 3, 4\}$. Students should not repeat any values in the range.

- Ask students to identify both the domain and the range in graphs of continuous relationships. For example, give students the graphs shown.



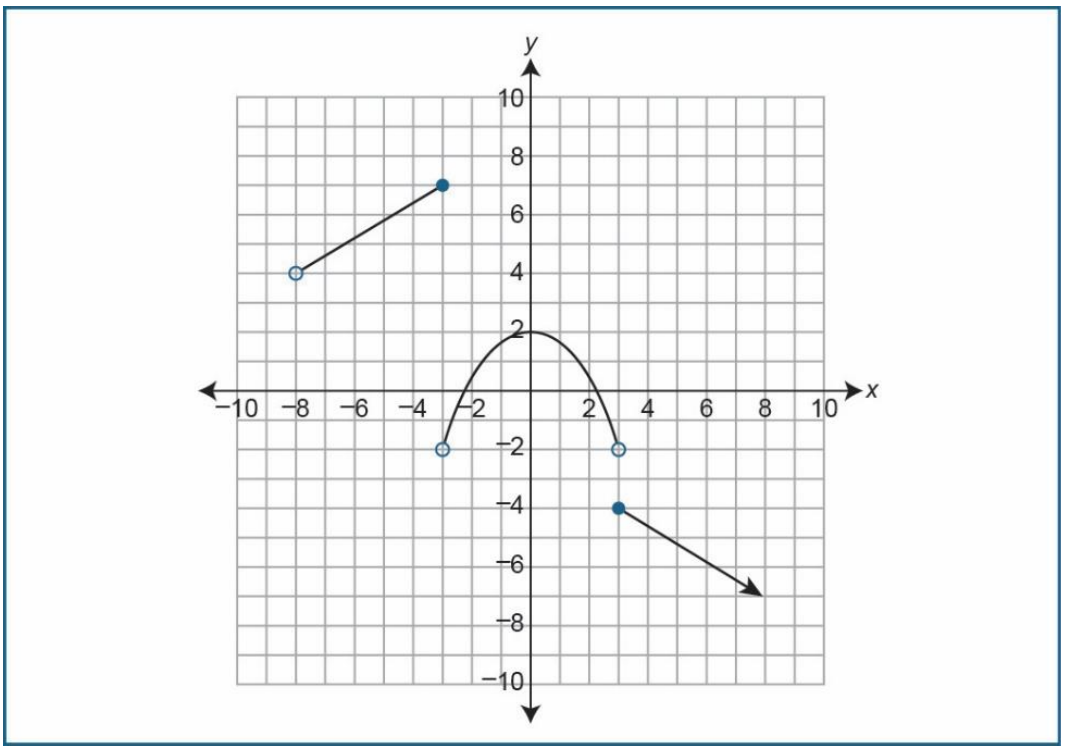
Students should investigate the end behaviors and search for possible endpoints in each graph and then identify the domain and the range using inequalities or interval notation.

- Students should observe that the domain of graph A consists of all real numbers, which can be represented as $-\infty < x < \infty$, or $(-\infty, \infty)$, and the range consists of all real numbers such that $-\infty < y < \infty$, or $(-\infty, \infty)$.

- Students should observe that the domain of graph B consists of all real numbers such that $-\infty < x < \infty$, or $(-\infty, \infty)$, and the range consists of all real numbers greater than or equal to 1 such that $1 \leq y < \infty$, or $[1, \infty)$.

- Students should observe that the domain of graph C consists of all real numbers such that $-\infty < x < \infty$, or $(-\infty, \infty)$, and the range consists of all real numbers such that $-3 \leq y \leq 3$, or $[-3, 3]$.

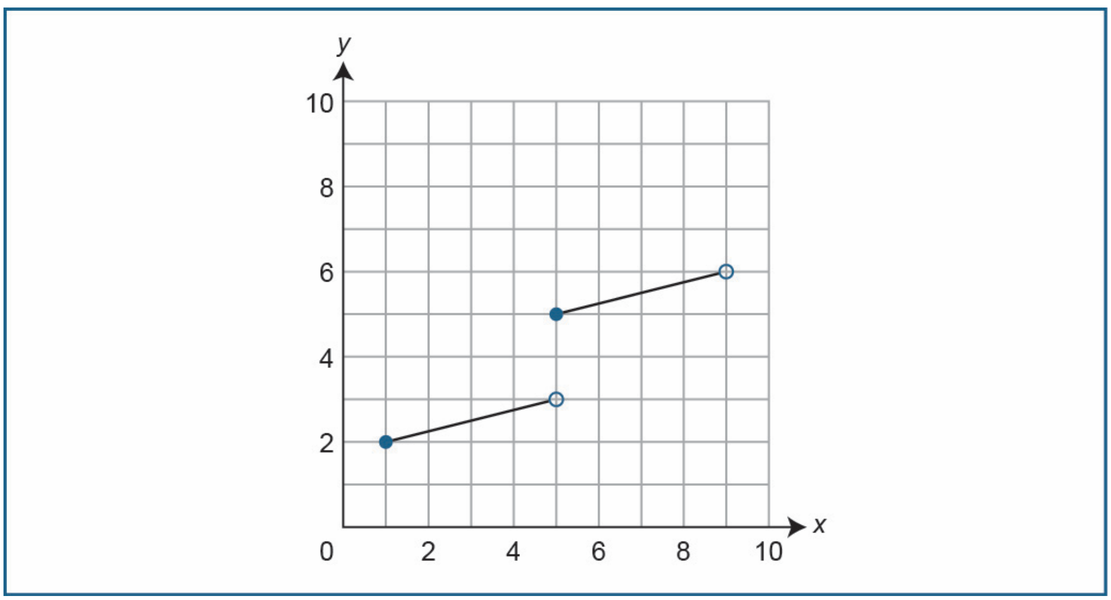
- Ask students to identify both the domain and the range in graphs of piecewise functions, and have them pay particular attention to the inclusion or exclusion of boundary points. For example, ask students to determine the domain and the range of the graph shown.



Students should recall that an open circle indicates a boundary point that is not included in the domain or the range, while a closed circle indicates a boundary point that is included in the domain and the range. Students should also understand that arrows extend a relationship indefinitely toward infinity or negative infinity. Students should then conclude that the domain of the graph consists of all real numbers greater than -8 such that $-8 < x < \infty$, or $(-8, \infty)$, as there is an explicit value for each input greater than -8, including locations where there are jumps at $x = -3$ and $x = 3$. Students should similarly conclude that the range of the graph is all real numbers less than or equal to 7 such that $-\infty < y \leq -4, -2 < y \leq 2, 4 < y \leq 7,$ or $(-\infty, -4] \cup (-2, 2] \cup (4, 7]$.

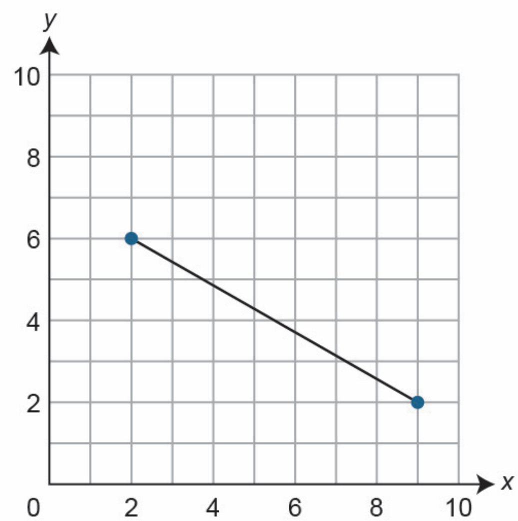
▶ Identifying and Addressing Common Misconceptions

- Students may mix up the definitions of domain and range. Ask students to identify the domain and the range of a function represented as the two points $\{(1, 5), (2, 10)\}$. Students who confuse domain and range may conclude that the domain is $\{5, 10\}$ and the range is $\{1, 2\}$. Remind students that the domain is always the set of input values, or x-values, while the range is always the resulting output values, or $f(x)$ values or y-values.
- Students may include multiple instances of repeated values when communicating the domain or the range of a relation. Ask students to identify the domain and the range of $\{(1, 2), (2, 2), (3, 2)\}$. Students who operate with this misconception about repeated values may write the range as $\{2, 2, 2\}$ instead of just the value $\{2\}$. Remind students that the domain and the range represent only the sets of possible input or output values, and it is not necessary to repeat values when listing elements of a set.
- Students may confuse open and closed circles when interpreting the boundary points of piecewise functions. Ask students to identify the range of the given graph.



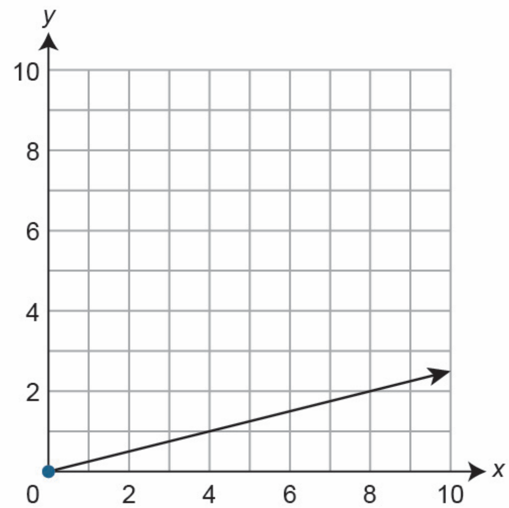
Students who confuse open or closed circles when interpreting boundary points may conclude that the range is either $[2, 3] \cup [5, 6]$ or $(2, 3] \cup (5, 6]$ or $(2, 3) \cup (5, 6)$. Remind students that closed circles on a graph indicate that a point is included in the relation, and this is communicated with a less than or equal to or greater than or equal to sign in inequality notation or with brackets [ ] in interval notation. Conversely, open circles on a graph indicate the inclusion in the relation of every possible value up to the given point but explicitly does not include the value of the point's location. This is communicated with less than or greater than signs in inequality notation or with parentheses ( ) in interval notation.

- Students may not include the entire interval of values in the domain and range of a continuous relationship. Students who operate with this misconception may list only integers within the intervals. Ask students to identify the domain of the graph shown.



Students may mistakenly conclude that the domain of the graph is the set {2, 3, 4, 5, 6, 7, 8, 9) as opposed to the set of all real values of x such that $2 \leq x \leq 9$. Remind students that a line drawn on a coordinate plane represents all possible values along the curve, and the set of those values can be expressed using inequalities. This is also true for the range.

- Students may fail to consider that the extension of a graph as shown by an arrow is part of the domain and the range of a relation. Ask students to identify the domain and the range of the graph shown.



Students may mistakenly conclude the domain of the graph is $0 \leq x \leq 10$ and the range is $0 \leq y \leq 2.5$. Remind students that an arrow on the graph of a function indicates that the function continues in the given direction without end. Ask students to draw the line continuing past the arrow. Explain that when no boundary point is given, the end behavior indicates the set of possible values extends to infinity.

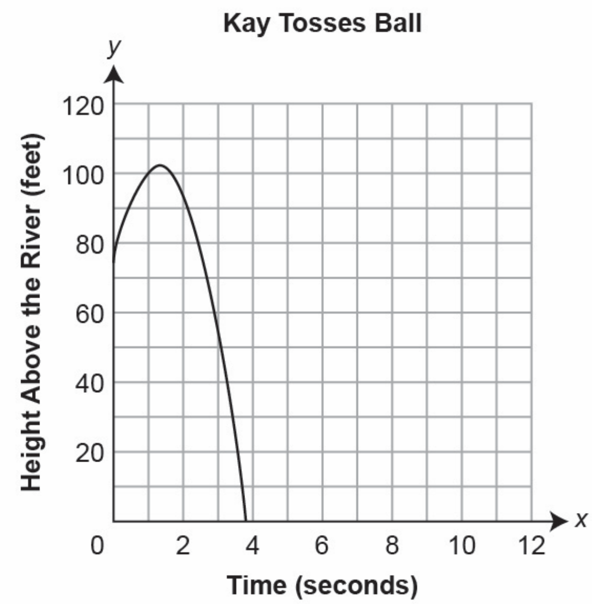
▶ Enrichment

- Ask students to identify the range of a function expressed as an equation restricted to specific input values in the domain. For example, ask students to identify the range of $f(x) = x^2 - 2x + 1$ over the domain of $\{-3, 0, 3\}$.
- Ask students to identify a reasonable domain and range in the context of a real-world situation. For example, present the following problem to students.

  Mel is studying the population growth of an invasive species of plant. Mel counts the number of invasive plants every 2 weeks and notices that they double. At the start of the study, there were 12 invasive plants. Mel has observed the growth over 10 weeks. Determine the appropriate domain and range for this function, and justify your response.
- Ask students to determine the domain and the range of common parent functions given as equations. For example, ask students to determine the domain and the range of $y = x^2$ or $y = \frac{1}{x}$ and justify their reasoning.
- Ask students to interpret the domain and the range of a given function in a real-world context

displayed in a graph. For example, present the following situation and supporting graph to students.

Kay tosses a ball upward off a bridge into a river below her. The height, in feet, of the ball, y, is modeled with the equation $y = -16x^2 + 42x + 74$, where x represents the time, in seconds, as shown in the graph.



Ask students to identify the domain and the range of this function and explain what each set represents in context. Students should conclude that the domain is the time in which the ball is in the air and it extends from exactly 0 seconds to about 3.8 seconds, while the range indicates the height of the ball above the river, which ranges from 0 feet to just over 100 feet.

► Key Terms

boundary points, closed circle, continuous, discrete, domain, element, end behavior, function, inequality, infinity, interval notation, open circle, piecewise function, range, set

A1.2.1.2.1

▶ Module, Assessment Anchor, Anchor Descriptor

Linear Functions and Data Organizations

Functions

Interpret and/or use linear functions and their equations, graphs, or tables.

▶ Eligible Content

A1.2.1.2.1 Create, interpret, and/or use the equation, graph, or table of a linear function.

▶ Instructional Supports

- Ask students to create a linear function describing a real-world situation. Students will need to determine the rate of change and the initial value of the linear function. For example, give students the following situation.
  Ren has 70 baseball cards in his collection. Throughout the summer, he plans to purchase a pack of 8 cards each week to add to his collection. Determine the linear function representing the number of cards, $f(x)$, Ren will have in his collection after different numbers of weeks, x.
  Students should determine that the initial number of cards in Ren's collection, 70, is the y-intercept. Students should then determine that the 8 cards Ren will add to his collection each week represent the rate of change. Students can then write the linear function in slope-intercept form as $f(x) = 8x + 70$.

- Ask students to create an equation, graph, or table to model a real-world linear function presented in context. For example, give students the following situation.
  Lucia is renting a pressure washer from a hardware store. The store charges a flat fee of \$35 plus \$9 per day for the rental.

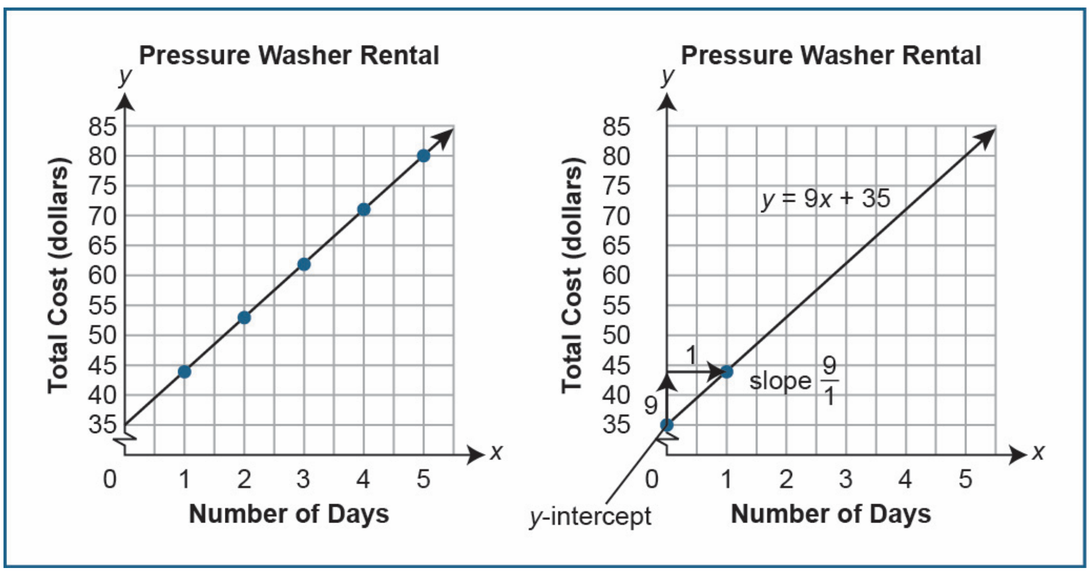
  - Ask students to write an equation that represents the total cost, y, as a function of the number of days, x, the pressure washer is rented. Students should identify the flat fee of \$35 as the initial value and the \$9 per day rental fee as the rate of change. Students should then write the equation $y = 9x + 35$ to model the total cost of the rental for x days.

- Ask students to create a table of values to represent the total cost of the rental for different numbers of days. Students may create and use an equation to populate a table, or they may generate table values through reasoning and successive counting.

Pressure Washer Rental

Number of Days, *x*	1	2	3	4	5
Total Cost (dollars), *y*	44	53	62	71	80

- Ask students to graph a line to model the relationship between the length of the rental, *x*, and the total cost, *y*. Students may use one of two methods to create their graph. They may create a table of values to plot as ordered pairs and connect them to form a line on a coordinate grid, or they may identify the initial value and the rate of change in the relationship and use those key elements to sketch a graph.

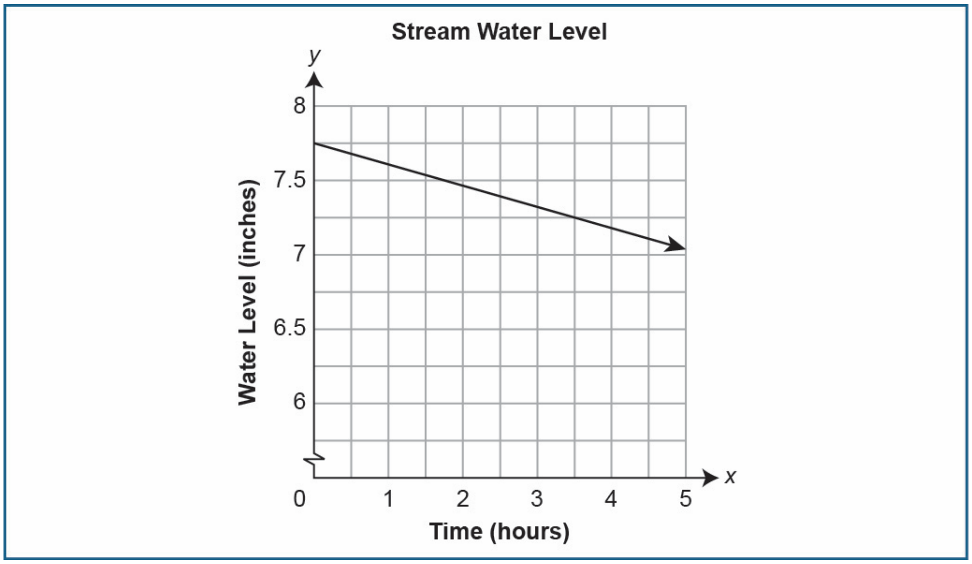


When sketching their graphs, students should label appropriate units for both their *x*- and *y*-axes and create an appropriate scale to represent their function values.

- Note: Students may prefer to interpret linear functions through successive reasoning or by finding a rate of change and an initial value. Encourage students, no matter the method they use, to analyze how values in the relationship change and which remain constant.

- Ask students to interpret the rate of change and the initial value of a linear function, with respect to its context, when given in an equation, table, or graph. For example, give students the following situation.

Presley has created a graph that represents the water level of a local stream over the course of a sunny afternoon, where x is the time, in hours, since noon, and y is the water level in inches.



Ask students to describe in their own words what the point (0, 7.75) means in relation to the context. Students should conclude that the y-intercept or initial value in a graph represents the water level when Presley began recording, which started at 7.75 inches.

- Give students the following situation.

  The total cost of hiring a caterer for an event depends on the total number of guests, as shown in the table.

Catering Services

Number of Guests	15	20	25	30	35
Total Cost (dollars)	270	335	400	465	530

Lori determines that the rate of change for this relationship is 13. What does this mean in the context of the problem?

Students should observe in the table that as the number of guests increases by 5, the total cost increases by $65, which leads to a rate of change of $\frac{\$65}{5} = \frac{\$13}{1}$, or $13 per guest. Students should conclude that the rate of change represents the cost of the caterer per guest at the event.

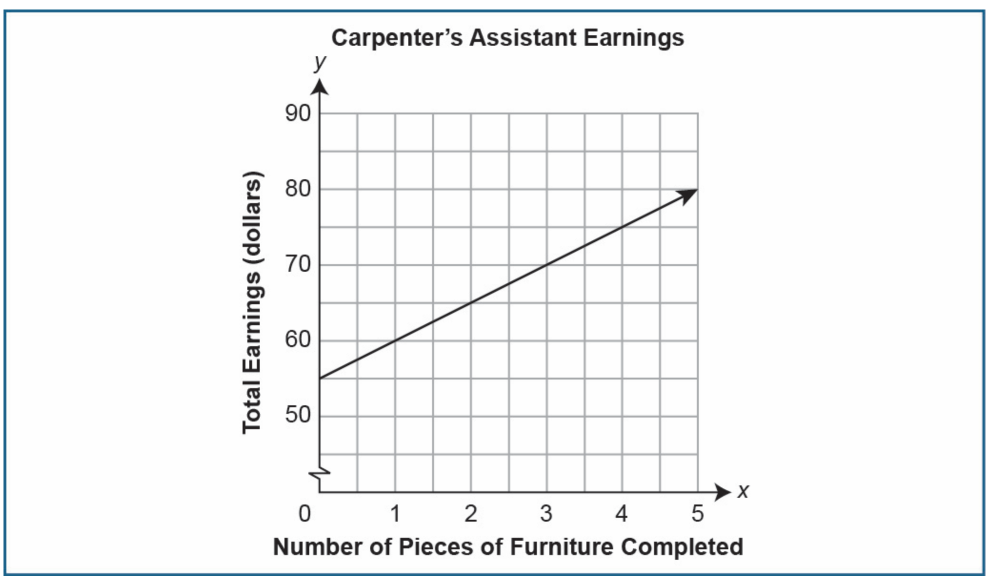
- Give students the following situation.

  A student is studying the number of butterflies in a wildflower field during the summer months. The student observes that the number of butterflies, b, in the field each week can be approximated with the linear function $b = 175 + 23w$, where *w* is the number of weeks since the study began. Interpret both the initial value and the rate of change of the provided linear function and explain each interpretation in your own words.

  Students should identify 175 as the initial value and conclude that it represents the number of butterflies present in the field when the study began, while 23 represents the rate of change and models the approximate increase in the number of butterflies counted in the field each week.

- Ask students to use the equation, table, or graph of a linear function to solve problems by identifying function values. For example, give students the following situation.

  A carpenter’s assistant earns a base rate each day she works, plus a bonus for every piece of furniture she completes that day. A graph of her daily pay is shown.

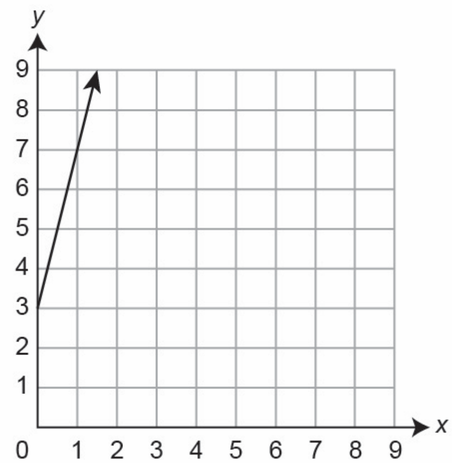


  Based on this information, how much will the carpenter’s assistant earn in a day in which she completes 4 pieces of furniture?

  Students should follow the *x*-axis on the graph to a value of 4 and connect it to the vertical height of the line at that point to identify a function value of 75 and therefore conclude that the carpenter’s assistant will earn \$75 for that day.

▶ Identifying and Addressing Common Misconceptions

- Students may confuse the rate of change and the initial value of a linear function presented in an equation. Ask students to identify the rate of change and the initial value in the function $f(x) = \frac{1}{2}x - 4$. Students who confuse the rate of change and the initial value of a linear function may conclude that the rate of change is -4 and the initial value is $\frac{1}{2}$. Remind students that in an equation of the form $y = mx + b$ or $f(x) = mx + b$, m is the rate of change because it is the value that is multiplied by each input value (x) and b is the y-intercept because it is the starting value.
- Students may confuse the order terms are given in an equation for the rate of change or the initial value instead of determining how the terms relate to the input variable. Ask students to identify the rate of change and the initial value in the equation $14 + 3x = y$. Students who confuse the order terms are given in an equation may assume that the rate of change is 14 and the initial value is 3, because 14 appears first in the equation and 3 appears later. Remind students that in a linear function, the rate of change is the value that is multiplied by the input variable, x, no matter the order that it appears in the equation, while the initial value or y-intercept is the output value when the input value is 0.
- Students may neglect to include the sign of the initial value or y-intercept when interpreting a linear function to create a graph or table. Ask students to create the graph of a linear function $g(x) = 4x - 3$. Students who interpret linear equations incorrectly may focus on the absolute value of b and place the starting point at (0, 3).



Remind students who make this mistake to always include the sign or operator of both the rate of change and the initial value when interpreting or plotting a linear function and that in this function the y-intercept should be located at $(0, -3)$.

- Students may use the inverse of the rate of change when creating the graph of a linear function, confusing "rise over run" with "run over rise." Ask students to create a graph for a linear function with a rate of change of $\frac{5}{3}$ and an initial value of 2. Students who mistakenly use the inverse of the rate of change may plot a point at (0, 2) and then plot a second point at (5, 5) by moving 3 units up and 5 units right from the location of (0, 2). Remind students that the rate of change represents the change in y-values or output values divided by the change in x-values or input values, which is often described as "rise over run." In this example, the rate of change is $\frac{5}{3}$, and a point can be plotted on the line by starting at any point on the line and moving 5 units up and 3 units right.

▶ Enrichment

- Ask students to create their own real-world examples of linear and nonlinear functions. After the students have provided descriptions of their examples, ask them to create a table, graph, and equation to model the linear functions they produced. Then, ask students to identify the rate of change and the initial value of their function and describe any reasonable restrictions on the domain and the range of their function.
- Ask students to create an equation, table, or graph that models a linear function given a distinct relationship with an associated linear function. For example, ask students to create a table for a linear

function $g(x)$ that has a rate of change that is twice that of the function $f(x)$ shown in the table but has the same initial value.

x	3	4	5	6	7
$f(x)$	9	11	13	15	17

Ask students to create another table for a function $h(x)$ that has the same rate of change as $g(x)$ but an initial value of -5.

- Ask students to solve a real-world problem that involves comparing linear functions presented in different ways. For example, ask students to consider the cost for using two moving companies, Get a Move On and Heavy Lifters. The cost of using Get a Move On is a one-time fee of $60 plus $4.50 per mile traveled. The table shows the total costs of using Heavy Lifters for different distances, in miles.

Heavy Lifters

Distance (miles)	10	20	30	40	50
Total Cost (dollars)	80	140	200	260	320

Ask students to determine which company would be less expensive to use for different distances, in miles. Then ask students to determine when using each company is the better deal and if there is a distance for which the services of the two companies will cost the same amount.

► Key Terms

constant, initial value, linear, nonlinear, rise over run, slope (rate of change), slope-intercept form, x-intercept, y-intercept

A1.2.1.2.2

▶ Module, Assessment Anchor, Anchor Descriptor

Linear Functions and Data Organizations

Functions

Interpret and/or us linear functions and their equations, graphs, or tables.

▶ Eligible Content

A1.2.1.2.2 Translate from one representation of a linear function to another (i.e., graph, table, and equation).

▶ Instructional Supports

- Ask students to identify the slope (rate of change) and the y-intercept (initial value) of a linear function represented in an equation, graph, or table.

  For example, ask students to identify the slope and the y-intercept in each of the following linear functions:

  $$f(x) = \frac{1}{3}x - 17,\ g(x) = 2 - 4x,\ h(x) = 2(x + 5)$$

  Students should rewrite each equation in slope-intercept form, or $y = mx + b$ form, to quickly identify the slope, m, and the y-intercept, b.

  $$f(x) = \frac{1}{3}x - 17,\ g(x) = -4x + 2,\ h(x) = 2x + 10$$

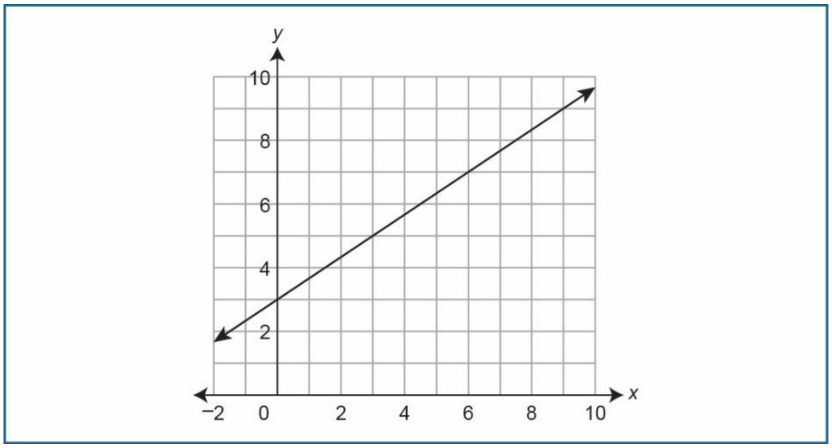
  Students should understand that these are equivalent forms of the initial equations and can therefore directly deduce the following slopes and y-intercepts.

  $f(x) = \frac{1}{3}x - 17$ $\quad m = \frac{1}{3}$ and $b = -17$

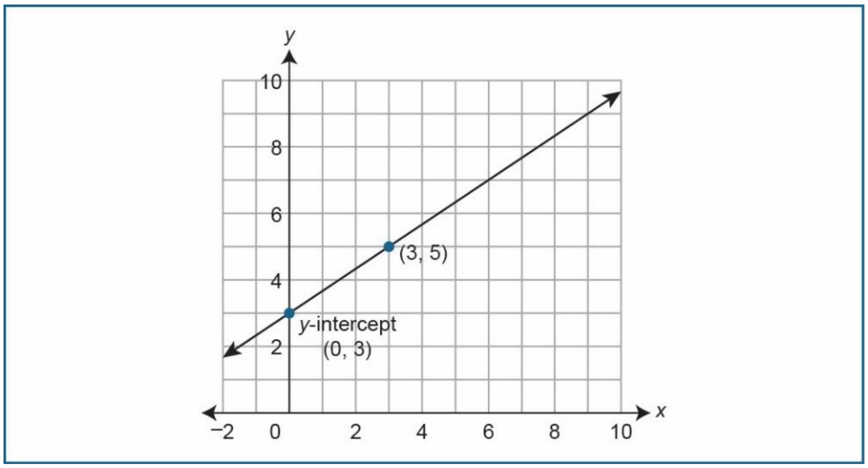
  $g(x) = -4x + 2$ $\quad m = -4$ and $b = 2$

  $h(x) = 2x + 10$ $\quad m = 2$ and $b = 10$

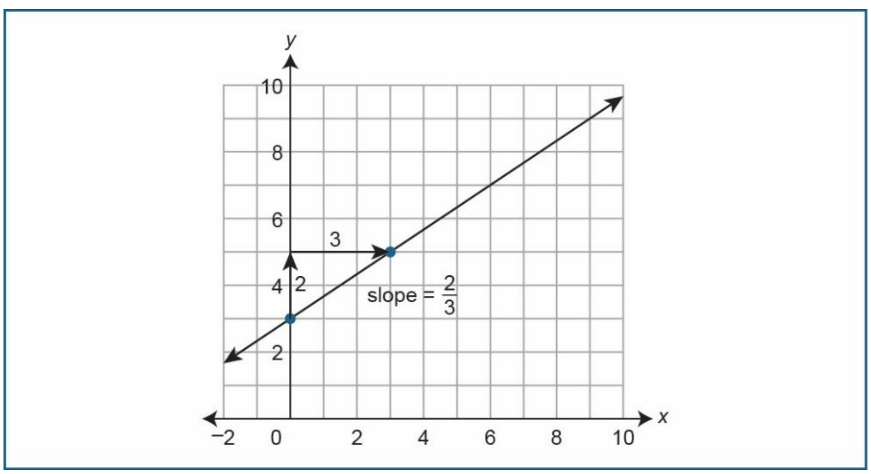
  Ask students to identify the slope and the y-intercept in the following graph so they can write an equation of the line.



Students should identify the *y*-intercept as 3.



Students should then count the rise over run from the *y*-intercept to the next lattice point to determine the slope, as shown.



Students should interpret the y-intercept as $b = 3$ and the slope as $m = \frac{2}{3}$ and write the equation $y = \frac{2}{3}x + 3$.

For another example, ask students to identify the slope and the y-intercept in a table and use them to write the equation of a linear function representing the situation that follows.

A pool cleaning service charges a set fee plus an hourly rate. The table shows the total cost for different numbers of hours needed to clean a pool.

Pool Cleaning Services

Time (hours)	2	3	4	5
Total Cost (dollars)	120	145	170	195

Students should first determine the slope by using the slope formula $m = \frac{y_2 - y_1}{x_2 - x_1}$. Students can choose any two ordered pairs from the table to determine the slope. Students can choose (2, 120) and (3, 145) to find $m = \frac{145 - 120}{3-2} = \frac{25}{1} = 25$ dollars per hour. Students can then solve for the y-intercept by substituting any ordered pair from the table into the slope-intercept form of a linear equation and solving for the variable b.

$y = mx + b$	slope-intercept form
$y = 25x + b$	Substitute $m = 25$ for the slope.
$120 = 25(2) + b$	Substitute the point (2, 120) for x and y.
$70 = b$	Solve the equation for b.
$y = 25x + 70$	equation of the linear function

Students may alternatively use repeated reasoning to extend the table backward, which will determine the initial value by subtracting the rate of change for each successive input, as shown.

Pool Cleaning Services

Time (hours)	**0**	**1**	2	3	4	5
Total Cost (dollars)	**70**	**95**	120	145	170	195

In doing so, students can determine that the initial value of the linear function is $b = 70$ dollars.

- Ask students to create a table of values and a graph of a linear function represented as an equation. For example, ask students to create a table or graph to model the following linear function in this situation.

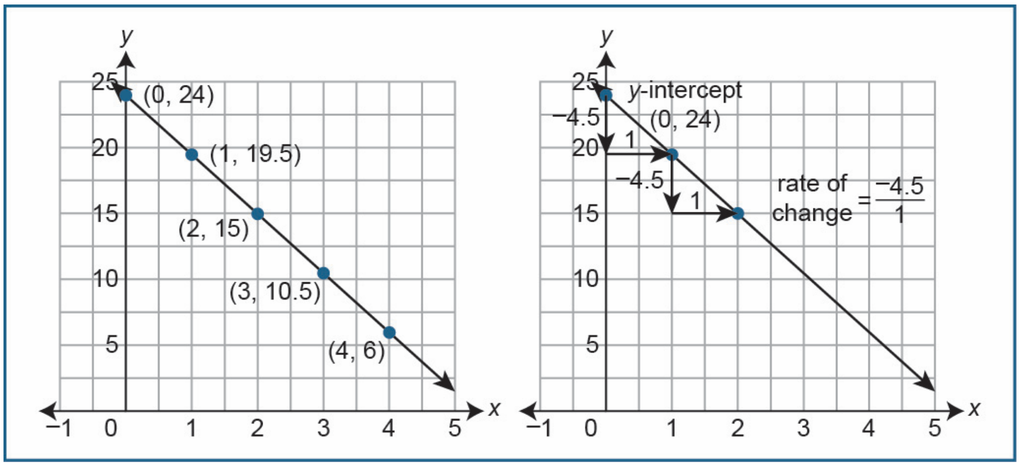
The number of cookies in the breakroom during working hours is modeled by the equation $y = -4.5x + 24$, where x is the number of hours since the cookies were placed in the breakroom and y is the total number of cookies remaining.

To create a table, students should determine ordered pairs by substituting possible x-values into the given equation and solving for the resulting y-value. Students can construct the table by beginning with $x = 0$ and adding increases in increments of 1, as shown.

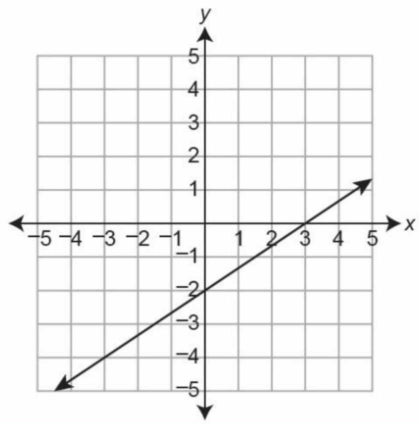
Cookies in Breakroom

Time (hours)	0	1	2	3	4
Number of Cookies	24	19.5	15	10.5	6

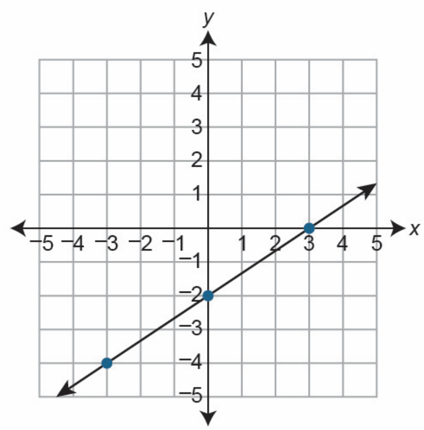
To create a graph, students may create a table of values, plot the collection of points, and connect those points with a straight line. Students may also identify and use the slope and y-intercept given in the equation.



- Ask students to create a table of values for the linear function represented on a graph. For example, provide students with the following graph of a linear equation and ask them to create a table of values to model the relationship.



Students should first identify lattice points along the graph.



Students should then identify the ordered pair for each point and enter them into a table relating x and y.

x	−3	0	3
y	−4	−2	0

▶ Identifying and Addressing Common Misconceptions

- Students may incorrectly use the formula for determining the slope from two points. Ask students to determine the slope of the linear function given in the following table.

x	2	4
y	5	1

Students who don't use the formula for determining a slope correctly may incorrectly interpret the slope formula as $m = \frac{x_2 - x_1}{y_2 - y_1} = \frac{4-2}{1-5} = \frac{2}{-4} = -\frac{1}{2}$ or $m = \frac{x_2 - y_2}{x_1 - y_1} = \frac{4-1}{2-5} = \frac{3}{-3} = -1$ or $m = \frac{y_2 - y_1}{x_1 - x_2} = \frac{5-1}{4-2} = \frac{4}{2} = 2$. Remind students that the slope formula is always the difference in y-values divided by the difference in x-values, which provides a slope of $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-5}{4-2} = \frac{-4}{2} = -2$ in this example.

- Students may misinterpret the first value provided in a table as the initial value or the y-intercept. Ask students to write an equation to model the linear function shown in the table.

x	3	4	5	6
y	2	4	6	8

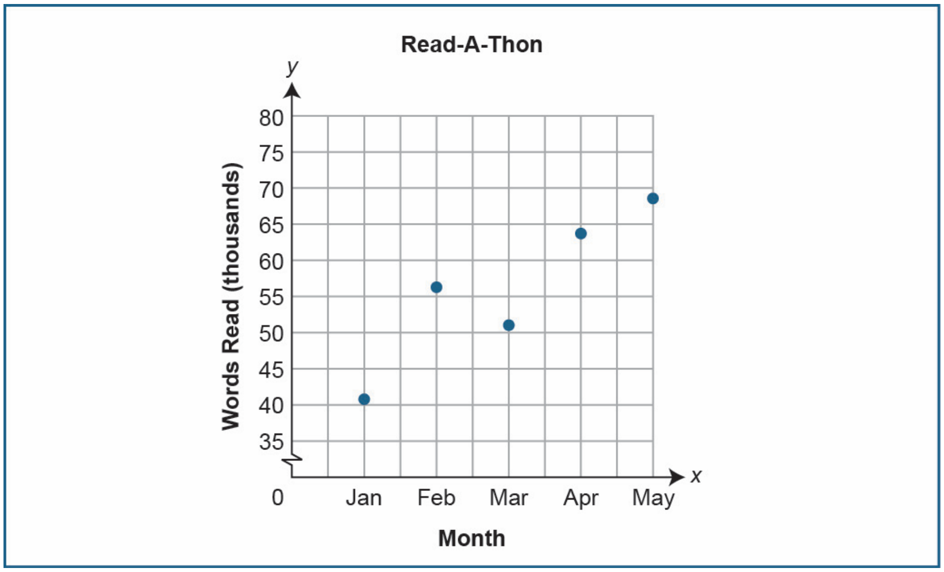
Students who misinterpret the first value provided in a table may interpret the initial value or y-intercept of the function as $b = 2$, which leads to an equation of $y = 2x + 2$. Remind students that the y-intercept or initial value is the function value when the input is $x = 0$. When that function value is not given explicitly in a table, students need to use successive reasoning or algebra to determine the missing value instead. In this example, students may use the rate of change of 2 to count back and create table values to zero to conclude that $b = -4$.

x	**0**	**1**	**2**	3	4	5	6
y	**−4**	**−2**	**0**	2	4	6	8

► Enrichment

- Ask students to draw a line of best fit to model real-world data and then write an equation that represents that line of best fit. For example, give students the following situation.

  Martina's school is hosting a charity read-a-thon during the entire second semester. The number of words read by students in the school each month is displayed in the graph.



- Ask students to first draw a line of best fit to model the linear relationship. Then, ask students to write an equation for their line of best fit. Ask students to continue their exploration by predicting the number of words read in June (if the second semester includes the month of June) based on the collected data.
- Ask students to make predictions about a data set when given a table of values that represents a linear function. For example, ask students to use the following table describing the amount of water in a pond to predict in what year the pond will dry up if it continues to lose water at the same rate.

Pond's Water Level

Year	2010	2012	2014	2016	2018
Gallons (thousands)	480	458	436	414	392

- Ask students to solve word problems comparing linear functions represented in different ways. For example, provide students with the following context: Agatha is ordering pizza for an event she is hosting at the library. Pele's Pizza provides the following table to customers to show the cost for a given number of large pizzas plus a delivery fee.

Pele's Pizza

Number of Pizzas	5	10	15	20	25
Total Cost (dollars)	78	138	198	258	318

The total cost for n pizzas at Cheesie's Pizza is $C(n) = \$32 + \$11.25n$. Ask students to determine which pizza vendor Agatha should select for varying numbers of pizzas. Then, ask students to determine when, if ever, the cost for pizzas at the two companies will equal the same amount.

► Key Terms

constant, lattice point, linear, nonlinear, slope (rate of change), slope-intercept form, x-intercept, y-intercept (initial value)

A1.2.2.1.1

▶ Module, Assessment Anchor, Anchor Descriptor

Linear Functions and Data Organizations

Coordinate Geometry

Describe, compute, and/or use the rate of change (slope) of a line.

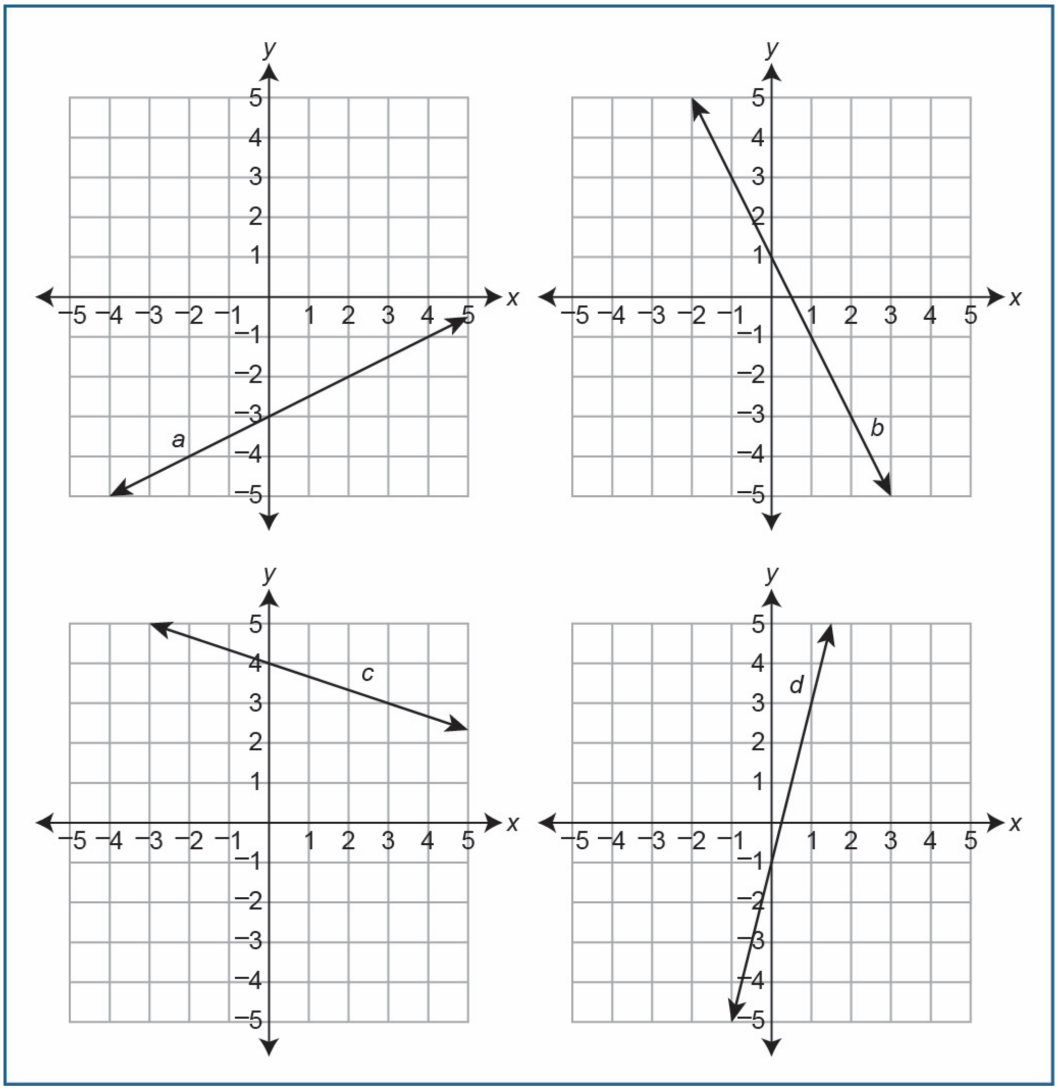
▶ Eligible Content

A1.2.2.1.1 Identify, describe, and/or use constant rates of change.

▶ Instructional Supports

- Ask students to compute a constant rate of change from a real-world context. For example, ask students to determine the average annual growth rate of a child whose height increased from 42 to 50 inches between the ages of 8 and 12. Students should interpret the height as the function value, or output, in this relationship and the age as the input. The annual average growth rate can be determined by the change in the number of inches divided by the change in the number of years. Students should divide the difference, or change, in output values, $50 - 42$, by the difference, or change, in age, $12 - 8$, to determine that the average growth rate is $m = \frac{50 - 42}{12 - 8} = \frac{8 \text{ inches}}{4 \text{ years}} = 2$ inches per year.
- Ask students to determine a rate of change in context. For example, give students the following situation.
  Wings and Things sold 3.2 tons of chicken wings in their first year of business. They sold 7.2 tons of chicken wings 5 years later. What is the average annual rate of change for the chicken wings sold during that time?
  The rate of change is the change in the output values divided by the change in the input values, so in this problem, students should subtract 3.2 tons of chicken wings from 7.2 tons of chicken wings to determine an increase of 4 tons of chicken wings over 5 years. This is an annual increase, or rate of change, of 0.8 tons of chicken wings sold. Using the slope formula results in $m = \frac{7.2 \text{ tons} - 3.2 \text{ tons}}{5 \text{ years}} = \frac{4 \text{ tons}}{5 \text{ years}} = 0.8$ tons per year.

- Ask students to describe the steepness and direction of the rate of change, or slope, of a line given its value. For example, ask students to describe the slope of $y = -\frac{2}{5}x + 14$ in their own words. Students should recognize that the slope is decreasing because the rate of change, $m = -\frac{2}{5}$, is negative.
- Ask students to determine whether the rate of change of a line shown in a graph is positive or negative and characterize it by the steepness. For example, ask students to match the following collection of linear graphs in the standard (x, y) coordinate plane with the appropriate rate of change and then place the slopes in order according to the steepness of the lines.



$\boldsymbol{m = -2,} \quad \boldsymbol{m = -\frac{1}{3},} \quad \boldsymbol{m = \frac{1}{2},} \quad \boldsymbol{m = 4}$

Students should recognize that the sign of the slope indicates whether the line is increasing (positive) or decreasing (negative) as the x-value increases. Students should determine that rates of change to lines a and d are positive and line a has a slope of $\frac{1}{2}$ and line d has a slope of 4. The rates of change for lines b and c are negative and line b has a slope of -2 and line d has a slope of $-\frac{1}{3}$. Students should recall that the greater the absolute value of the rate of change, the steeper, or more vertical, the line is. Students should determine the absolute value of each slope and list the slopes in order according to steepness as $4, -2, \frac{1}{2}$, and $-\frac{1}{3}$.

- Ask students to compute the rate of change between two points on a line. For example, ask students to determine the rate of change for the line that passes through the ordered pairs (1, 4) and (4, −2). Students may recall and use the slope formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ to calculate that this line has a rate of change of $m = \frac{-2-4}{4-1} = \frac{-6}{3} = -2$.
- Ask students to describe the rate of change of horizontal and vertical lines. For example, ask students to identify the rate of change in each of the following tables.

Horizontal Line *c*

x	−6	−2	1	4
y	−3	−3	−3	−3

Vertical Line *d*

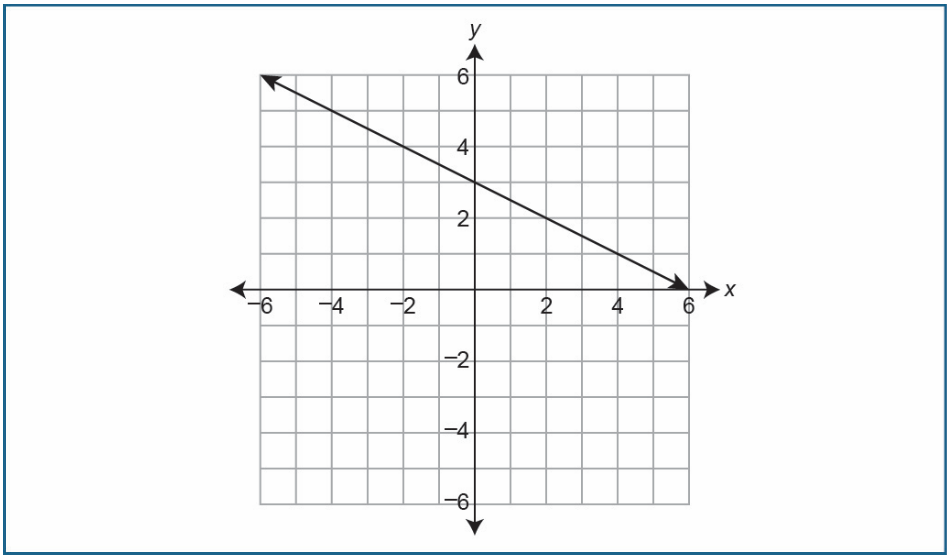
x	4	4	4	4
y	−3	0	2	5

Students should recall that a horizontal line has no change in y-value, so the rate of change of line c is 0. Students should also recall that the slope of a vertical line is undefined, so the rate of change of line d is undefined. If students need support in remembering these rules, provide a scaffold for them by reminding them that the rate of change of a line is defined by rise over run. In the case of a horizontal line, there is no “rise,” and any fraction with a numerator of 0, such as $\frac{0}{2}$, will be 0. Similarly, in the case of a vertical line, there is no “run,” or movement in the x-direction. In this case, the denominator in the rate of change fraction will be 0, and any value divided by 0, such as $\frac{2}{0}$, is undefined.

► Identifying and Addressing Common Misconceptions

- Students may misinterpret the idea of “rise over run” when working with an integer slope. Ask students to describe the slope of a line with a rate of change of $m = 4$. Students who misinterpret rise over run may describe the rate of change as moving up 1 unit and over 4 units to the right. Remind students that any integer can be interpreted as a fraction with a denominator of 1, so a rate of change of 4 can be thought of as $\frac{4}{1}$, with a rise of 4 and a run of 1. Or, students may consider only a rise of 4 and graph a vertical line. Another common misconception is that students think a slope of -3 means down 3 and left 1. Students who mistakenly believe this may put the negative sign on both the numerator and the denominator.

- Students may interpret a rate of change as its inverse by confusing "rise over run" with "run over rise." Ask students to describe how to plot a line with a slope of $m = \frac{2}{7}$ with an initial value at (0, 1). Students who misinterpret rate of change may describe moving 2 units in the *x*-direction and then 7 units up to get a second point of (2, 8). Redirect students to utilize "rise over run" to instead determine a second point of (7, 3), which helps create a line with a flatter slope that is less steep.
- Students may mix up the order of *x*- and *y*-coordinate values when applying the formula for rate of change. Ask students to substitute the coordinate pairs (3, 5) and (6, 1) into the formula for slope and share their response without solving. Students who mix up the order of coordinate values may present a formula of $m = \frac{1-5}{3-6}$ or $m = \frac{5-1}{6-3}$. Remind students that while it does not matter which point is considered the first or second point in the formula, the order of the coordinates needs to be consistent when the formula is used. Students should then arrive at a slope formula of either $m = \frac{1-5}{6-3} = \frac{-4}{3}$ or $m = \frac{5-1}{3-6} = \frac{4}{-3}$, both of which are equivalent. Students may determine the difference of the *x*-coordinates over the difference of the *y*-coordinates, which will determine the reciprocal of the slope.
- Students may mistakenly characterize horizontal lines as having an undefined slope and vertical lines as having a slope of zero. Ask students to draw two lines: first, a line with a slope of 0 that passes through the point (3, 3) and a second line with an undefined slope that passes through the point $(-1, 2)$. Students who mistakenly confuse the slopes of lines may draw a vertical line at $x = 3$ and a horizontal line at $y = 2$. Remind students that a slope with a numerator, or "rise," of 0 will be flat, so a rate of change of 0 corresponds with a horizontal line, while a slope with 0 "run" will never move in the *x*-direction, meaning it is directly vertical. Remind students that dividing by zero is not possible, which is why the presence of a zero in the denominator leads to an undefined slope. Students should correct their lines by drawing a horizontal line at $y = 3$ to represent a line with slope of 0 that passes through the point (3, 3) and a vertical line at $x = -1$ to represent a line with an undefined slope that passes through the point $(-1, 2)$.
- Students may describe and write slopes only as positive, which means they've misinterpreted lines with decreasing rates of change as increasing. Ask students to calculate the slope of the following line.



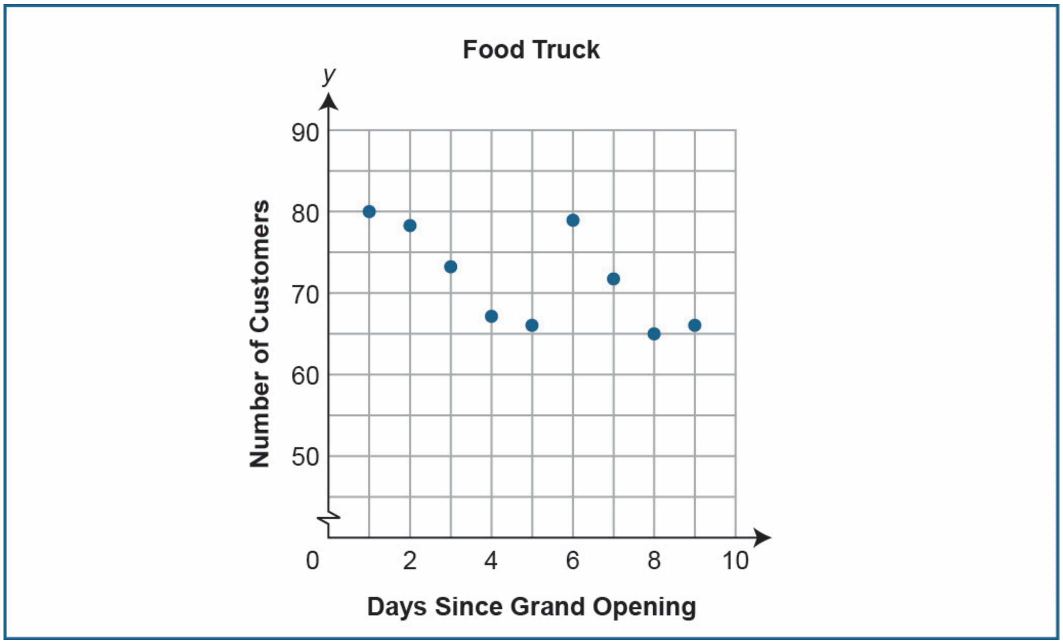
Students who misinterpret slopes in this manner may believe that the line has a rise of 1 and a run of 2 and therefore a rate of change of $m = \frac{1}{2}$. Remind students that when y-values decrease as x-values increase, the rate of change is decreasing, or negative. Students should connect this idea with “rise over run” and note that the line has a drop of 1, or a “rise” of -1, before identifying a run of 2 and concluding that the slope is $m = -\frac{1}{2}$.

► Enrichment

- Ask students to compare real-world rates of change in context. For example, ask students to compare the design of two linear water slides one of which has a vertical drop of 45 feet in height over a horizontal distance of 90 feet and another that drops 50 feet in height over a horizontal distance of 97 feet and describe which slide is steeper.
- Ask students to identify and describe slope in geometric applications. For example, ask students to determine the slope of a ski slope that drops 25 feet for every horizontal 80 feet. Then, instruct students that in real-world applications, slope is often communicated as a percent grade, which interprets the rate of change directly as a percentage, and ask students to determine the percent grade of the ski slope. Finally, ask students to use a ruler to draw a precise model of the slope. Continue by asking students to compare different representations of slope and to create a table of values that connects various rates of change represented as a fraction and a percentage.

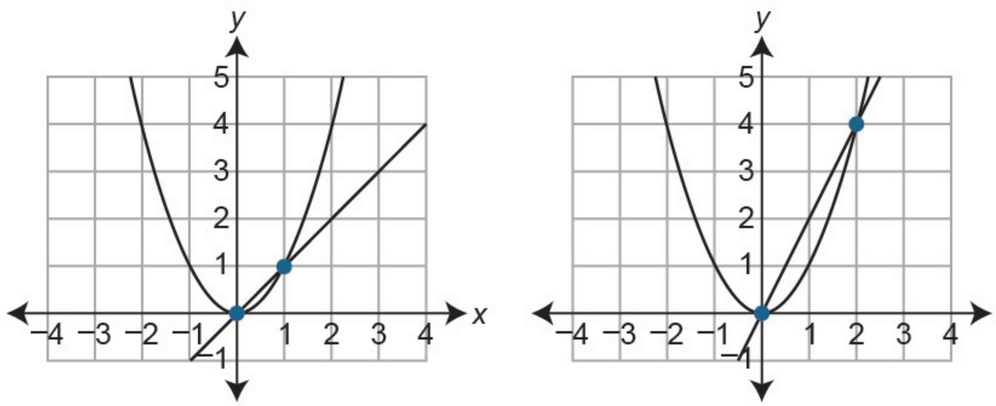
- Ask students to modify a rate of change to write a new equation or draw a new graph of a line. For example, ask students to write an equation that has the same y-intercept of $y = \frac{1}{3}x - 7$ but a rate of change that is half as steep.
- Ask students to draw a line of best fit to model real-world data and then write an equation that represents that line of best fit. For example, give students the following situation.

  A local restaurant opened a food truck and tracked the number of customers the truck served each day after its grand opening.



  Ask students to first draw a line of best fit to model the linear relationship. Then, ask students to use their line of best fit to identify a linear rate of change for the number of customers served each successive day. Ask students to continue their exploration by predicting the number of customers on the 20th day if the trend continues based on the collected data.

- Ask students to determine and compare the constant rate of change for different sets of points across nonlinear functions. For example, ask students to determine the constant rate of change between different pairs of points along the curve of the function $y = x^2$ and to compare these rates of change. Ask students to determine the constant rate of change between (0, 0) and (1, 1). Then, ask students to determine the constant rate of change between (0, 0) and (2, 4) and ask them to create a line between those points. Ask students to compare the rates of change for each set of points and draw conclusions about the constant rates of change.



► Key Terms

constant rate of change, ordered pair, slope, steepness

A1.2.2.1.2

▶ Module, Assessment Anchor, Anchor Descriptor

Linear Functions and Data Organizations

Coordinate Geometry

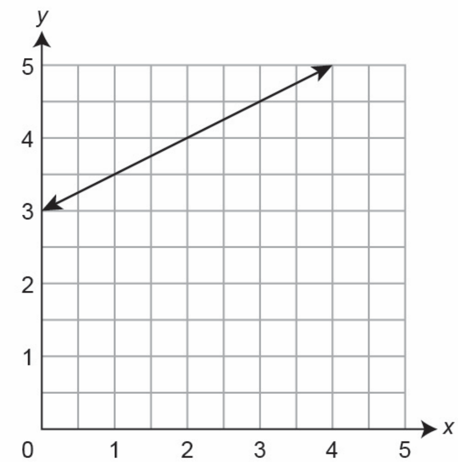
Describe, compute, and/or use the rate of change (slope) of a line.

▶ Eligible Content

A1.2.2.1.2 Apply the concept of linear rate of change (slope) to solve problems.

▶ Instructional Supports

- Ask students to use the rate of change in context to predict a future value. For example, give the students the following situation.
  A car costs \$27,500 when new. The car loses its value, on average, at a rate of \$2,300 per year. What is the expected value of the car after 4 years?
  Students should multiply the rate of change, \$2,300, by the duration of the depreciation, 4 years, to determine an expected total loss of \$9,200. They should then subtract that value from the initial cost of \$27,500 to determine an expected value of $\$27{,}500 - \$9{,}200 = \$18{,}300$.
- Ask students to use the rate of change to determine the total change in y-value given the change in x-value. For example, give students the following situation.
  A beehive produces 1.8 pounds of honey each day. How much honey should the hive be expected to produce between the 4th day of the month and the 11th day of the month?
  Students should identify the rate of change in this context as 1.8 pounds of honey each day and the change in x-values, or input values, to be $11 - 4 = 7$ days. Students can then set up the equation $1.8 \text{ lb per day} = \frac{\text{change in amount of honey}}{7 \text{ days}}$ and then multiply the rate of change of 1.8 pounds per day by 7 days to yield an expected value of 12.6 pounds of honey produced.
- Ask students to identify the rate of change of a line and use it to predict a future value given the graph of the line. For example, provide students with the following graph. Students should first determine the rate of change of the line. Then, ask students to use the rate of change to determine the y-value of the line when $x = 20$.



Students can use different methods to determine that the rate of change is $\frac{1}{2}$. They can then use several methods to determine the y-value when $x = 20$.

- One method would be to recognize that the initial value of this line is at $y = 3$ when $x = 0$. Understanding that the rate of change multiplied by the change in x gives the total increase or decrease in y from a given point will allow students to determine the change in y to be $\frac{1}{2} \bullet 20 = 10$ units. If the y-value starts at 3 and increases by 10 units, then $y = 13$ at $x = 20$.

- Students may also use the rate of change formula to determine the change from a point such as (4, 5). By using $m = \frac{1}{2}$, our first coordinate as (4, 5), and our second coordinate as (20, y), we can write an equation:

$$m = \frac{y_2 - y_1}{x_2 - y_1} \Rightarrow \frac{1}{2} = \frac{y - 5}{20 - 4}$$

This results in a linear equation that can be solved for y.

Step 1: $\frac{1}{2} = \frac{y-5}{16}$

Step 2: $8 = y - 5$

Solution: $13 = y$

- Ask students to identify the rate of change from two points and use it to predict a future value in the relationship. For example, ask students to determine the rate of change of the line that connects the points $(-3, -2)$ and (1, 10). Then, ask students to use the rate of change to determine the y-value when x is 7.
  - Students should use the slope formula to determine $m = \frac{y_2 - y_1}{x_2 - y_1} = \frac{10 - (-2)}{1 - (-3)} = \frac{12}{4} = 3$.

Once students have the rate of change, they may reason that the x value increases by 6 units from 1 to 7, so the function value, or y-value, will need to increase by the rate of change, 3, multiplied by 6 units, which is 18 units. Since the y-value was 10 at $x = 1$, the resulting y-value at $x = 7$ is $10 + 18 = 28$.

Students may also create an equation to determine the missing y-value by entering known values of $m = 3$, point (1, 10), and point (7, y) into the rate of change formula:

$$m = \frac{y_2 - y_1}{x_2 - y_1} \Rightarrow 3 = \frac{y - 10}{7 - 1}$$

This results in an equation that can be solved for y.

Step 1: $3 = \dfrac{y - 10}{6}$

Step 2: $18 = y - 10$

Solution: $28 = y$

► Identifying and Addressing Common Misconceptions

- Students may interpret the rate of change as a singular value and not as a rate. Give students the following situation.
  A carpenter makes 8 tables per day and has 12 tables already made. How many tables will he have after working for 3 days?
  Students who misinterpret rate of change may directly add 12 and 8 to get a result of 20 tables, or students may add 12 and 8 and multiply 20 by 3. Remind students that the rate of change provides the change across a single unit in the input value, which in this problem would be one day, so the rate of change needs to be multiplied by the total change in input values to determine the total change. Students should then multiply 8 tables per day by 3 days to determine a difference of 24 tables to add to the initial 12 tables, leading to a result of 36 tables.
- Students may enter coordinate values in the wrong order when applying the rate of change formula. Ask students to determine the rate of change between (3, 4) and (7, 2). Students who misunderstand the rate of change formula may either place the x-values in the numerator of the formula $m = \frac{x_2 - x_1}{y_2 - y_1}$ or change the order of coordinates in the formula $m = \frac{y_2 - y_1}{x_1 - x_2}$, leading to possibly incorrect rates of change of $m = \frac{7-3}{2-4} = \frac{4}{-2} = -2$ or $m = \frac{2-4}{3-7} = \frac{-2}{-4} = \frac{1}{2}$. Remind students that the rate of change is

always taken as the change in y-values, or function values, divided by the change in x-values, or input values. Additionally, remind students that while the two points can be used in any order and result in the same rate of change when using the formula, the coordinates within each point need to be consistently entered first, or second. In this problem, students should determine the rate of change as $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-4}{7-3} = \frac{-2}{4} = -\frac{1}{2}$.

- Students may confuse the rate of change with the initial value, or y-intercept, when interpreting a linear relationship in context. Give students the following situation.

  A flock of migrating Arctic terns have flown 1,200 miles on their current trip. As they travel, they fly at a rate of 450 miles per day. Use this information to determine the total distance the terns might travel over an additional week's time.

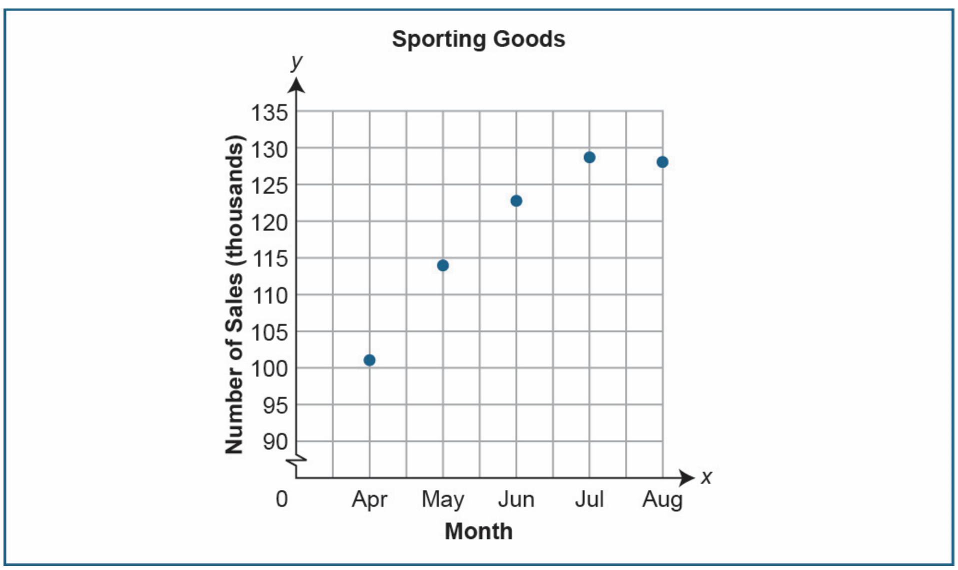
  Students who confuse the rate of change may multiply 1,200 miles by 7 days to anticipate a distance traveled of 8,400 miles, or they may additionally add that sum to 450 miles to anticipate a distance traveled of 8,850 miles. Remind students that the rate of change in a context represents a value that changes over time or some other input, and the initial value is a constant. Students should then identify the rate of change in this problem as 450 miles, which they can multiply by 7 days to get a total increase in distance of 3,150 miles. They can then add this distance to 1,200 to determine a total distance traveled of 4,350 miles.

► Enrichment

- Ask students to compare and use rates of change provided in different units. For example, give students the following situation.

  Ken and Larisa are trying to compare their recent hiking trips. Ken measured his recent hike as 17 miles long, which he completed in 5.5 hours. Larisa measured her hike as 22 kilometers long, which she completed in 3.8 hours. Who hiked at a faster pace? If both Ken and Larisa plan on hiking for 5 hours the following day at the same speeds, how much farther will the faster hiker travel?
- Ask students to create a story that represents a context with a linear rate of change given specific values. For example, ask students to write a story that centers around a rate of change of 8 and contains a data point of $(4, -1)$. Students will need to be creative in reasoning with negative values. Provide a scaffold by suggesting geometric or financial contexts, where negative values like below sea level or debt are common. Once students create their stories, ask them to share them with each other, and ask other students to determine a future data point.

- Ask students to solve a real-world problem where the rate of change shifts at a certain function value. For example, give students the following situation.
  At 7:00 a.m., the temperature in Greenville is 78 degrees. For the next 8 hours, the temperature increases at $1\frac{1}{4}$ degrees per hour. Then, the temperature decreases by $1\frac{1}{2}$ degrees per hour for the following 6 hours. What was the temperature at 7:00 p.m.?
  Ask students to sketch a graph of the temperature throughout the day and to name the rate of change(s). Further, ask students to determine the average rate of change from 7:00 a.m. to 7:00 p.m., and have them compare what this value gives in relation to the original problem and which provides a more accurate story; be sure the students explain their answers.
- Ask students to explain why the visual slope in a graph may be misleading. For example, provide students with graphs that show particularly steep or flat slopes, particularly graphs that display real-world contexts but use nonstandard scales. Ask students to compare data representing the same context or similar scenarios that have been placed on graphs with different scales, and have the students draw conclusions about visual slope.
- Ask students to use a rate of change in a linear model or a linear equation to predict values in a data set and then compare those values to actual data that deviate from the predictions. Have students draw conclusions from the differences and similarities. For example, give students the following situation.
  A sporting goods store predicted in March that sales would increase throughout the year at a rate of $13,000 per month from their current monthly total of $89,000. They then tracked their data over those months on the graph shown.



How do the store's predicted sales compare with their actual sales? What would you predict the sales to be in September? Justify your answer.

▶ Key Terms

expected value, slope (rate of change)

A1.2.2.1.3

▶ Module, Assessment Anchor, Anchor Descriptor

Linear Functions and Data Organizations

Coordinate Geometry

Describe, compute, and/or use the rate of change (slope) of a line.

▶ Eligible Content

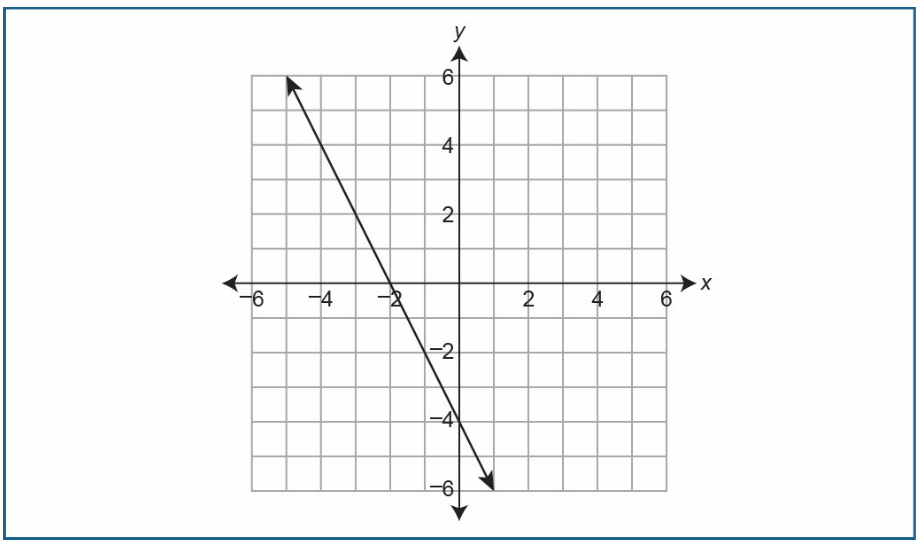
A1.2.2.1.3 Write or identify a linear equation when given

- the graph of the line,
- two points on the line, or
- the slope and a point on the line.

Note: Linear equation may be in point-slope, standard, and/or slope-intercept form.

▶ Instructional Supports

- Ask students to write an equation when given the graph of a line. For example, provide students with the following graph and ask them to write an equation for the given line.



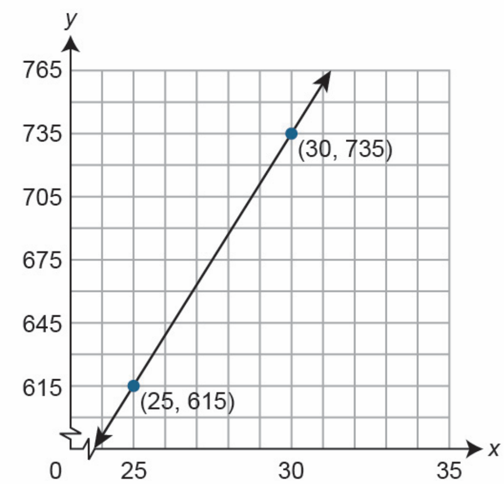
Students may approach this problem by directly identifying the slope and *y*-intercept and then entering those values into slope-intercept form: $y = mx + b$. Students should identify the *y*-intercept as the point where the line crosses the *y*-axis (0, −4). Then, to identify the slope, or rate of change, students should select a second point on the line, such as (1, −6), and trace the rise over run between the *y*-intercept and the selected point. With a rate of change identified as $m = -2$ and a *y*-intercept identified as $b = -4$, students should substitute these values into slope-intercept form as $y = -2x - 4$.

- Alternatively, students may derive the equation in a different form.
  - Students may use a single identified point, such as (1, −6), and a derived slope from counting on the graph, $m = -2$, to write an equation in point-slope form: $y - y_1 = m(x - x_1)$. Students using this strategy should write the linear equation as $y - (-6) = -2(x - 1)$, or $y + 6 = -2(x - 1)$.
  - Students may identify two points on the line graph, such as (0, −4) and (1, −6), and then determine the slope by using the slope formula, which results in $m = -2$. Then, they may use this slope with one of their selected points to write an equation in point-slope form, such as $y - (-4) = -2(x - 0)$, or $y + 4 = -2x$.
  - Students can take an equation written in slope-intercept form, such as $y = -2x - 4$, and transform that equation into standard form (*Ax* + *By* = *C*): $2x + y = -4$.

- Ask students to write a linear equation given a slope and a point. For example, ask students to write a linear equation that passes through the point (5, 4) and has a slope of $m = \frac{1}{3}$. The point-slope form of a linear equation, $y - y_1 = m(x - x_1)$, provides one method for students to create an equation in this context. Students should substitute the point (5, 4) into the general equation for (x_1, y_1) and $\frac{1}{3}$ in for *m* to write the linear equation of $y - 4 = \frac{1}{3}(x - 5)$.
- Ask students to write a linear equation given two points on a line. For example, ask students to write an equation to model the line that passes through the points (−3, 5) and (−1, −1). Students should first identify the rate of change, or slope, by using the slope formula $m = \frac{y_2 - y_1}{x_2 - x_1}$. Students should determine that the rate of change is $m = \frac{-1 - 5}{-1 - (-3)} = \frac{-6}{2} = -3$. Students may then select one of the two given points to substitute, along with the rate of change, into the general equation for point-slope

form of a line, $y - y_1 = m(x - x_1)$, leading to possible equivalent linear equations of $y - 5 = -3(x + 3)$ or $y + 1 = -3(x + 1)$.

- Ask students to identify the equation for a line on a graph where the y-intercept is not provided. For example, ask students to write an equation in point-slope form for the following line.



Students should identify two points on the graph, use those points to determine the rate of change, and then write an equation in point-slope form. The line passes through the points $(25, 615)$ and $(30, 735)$. Students should first identify the rate of change, or slope, by using the slope formula $m = \frac{y_2 - y_1}{x_2 - x_1}$. Students should determine that the rate of change is $m = \frac{735 - 615}{30 - 25} = \frac{120}{5} = 24$. Students may then select one of the two points to substitute, along with the rate of change, into the general equation for point-slope form of a line, $y - y_1 = m(x - x_1)$, leading to possible equivalent linear equations of $y - 615 = 24(x - 25)$.

▶ Identifying and Addressing Common Misconceptions

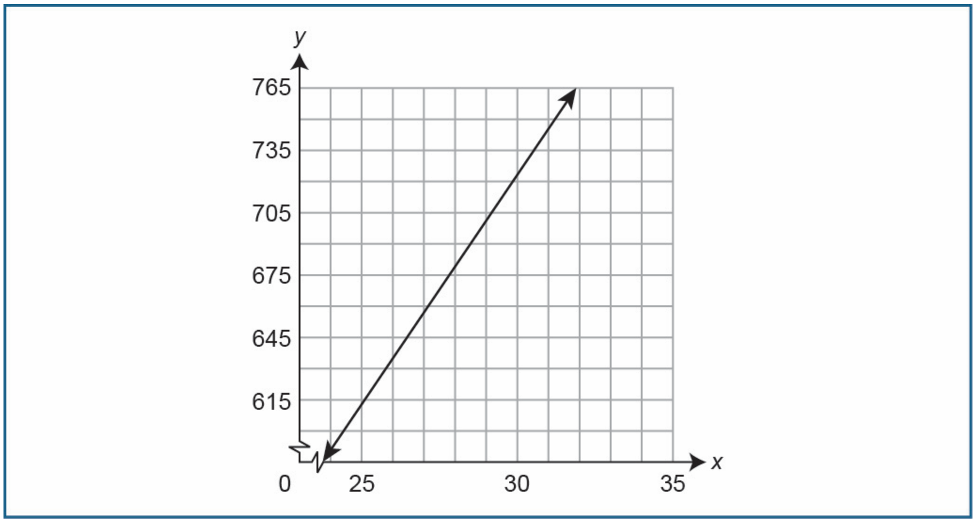
- Students may not recognize equivalent forms of linear equations. Ask students to determine whether any or all of the following linear equations represent the same line: $-2x + y = 1$, $y - 3 = 2(x - 1)$, and $y = 2x + 1$. Students who don't recognize equivalent forms of linear equations may interpret each of these equations as representing separate lines because they contain different values and signs and differences in the order of terms. Ask students who are operating with this misconception to create a table of values for each equation by using the same x-values and then compare the results. Once this process has been completed, remind students that equations may have many different equivalent forms, since performing the same algebraic operations to both sides of an equation does not modify the relationship between variables.

- Students may substitute the wrong coordinate or forget to include the sign when entering a coordinate pair into the point-slope form of a linear equation. Ask students to use the point-slope formula to write a linear equation with a slope of $m = 2$ and a point of $(-1, -5)$. Students who incorrectly substitute coordinates or forget to include the sign when entering a coordinate pair may create a linear equation of $y + 1 = 2(x + 5)$, having swapped the locations of the *x*- and *y*-coordinates in the equation. They may also incorrectly create the equation $y - 5 = 2(x - 1)$ by assuming that the negatives in the given point are satisfied by the subtraction in point-slope form. Remind students of the general form $y - y_1 = m(x - x_1)$ and guide students to create a habit of labeling each coordinate in the given point as x_1 and y_1. Additionally, remind students to include parentheses and maintain signs and operators before the variable when substituting a value in for a variable. In this problem students should correctly substitute $y - (-5) = 2\big(x - (-1)\big)$, which may be rewritten as $y + 5 = 2(x + 1)$.

- Students may forget to use the distributive property when rewriting linear equations in point-slope form when determining equivalent linear equations. Ask students to determine whether the following linear equations are equivalent: $y - 5 = 4(x + 2)$ and $y = 4x + 7$. Students who forget to use the distributive property may believe that these linear equations are equivalent by rewriting $y - 5 = 4(x + 2)$ as $y - 5 = 4x + 2$ without distributing the rate of change, 4, to both terms. The equation $y - 5 = 4x + 2$ would then be determined as equivalent to $y = 4x + 7$ by adding 5 to both sides of the equation. Remind students that the term outside the parentheses must be multiplied by every term inside the parentheses when using the distributive property. By using this property, students should rewrite $y - 5 = 4(x + 2)$ as $y - 5 = 4x + 8$, which is equivalent to $y = 4x + 13$ and is clearly not equivalent to $y = 4x + 7$.

► Enrichment

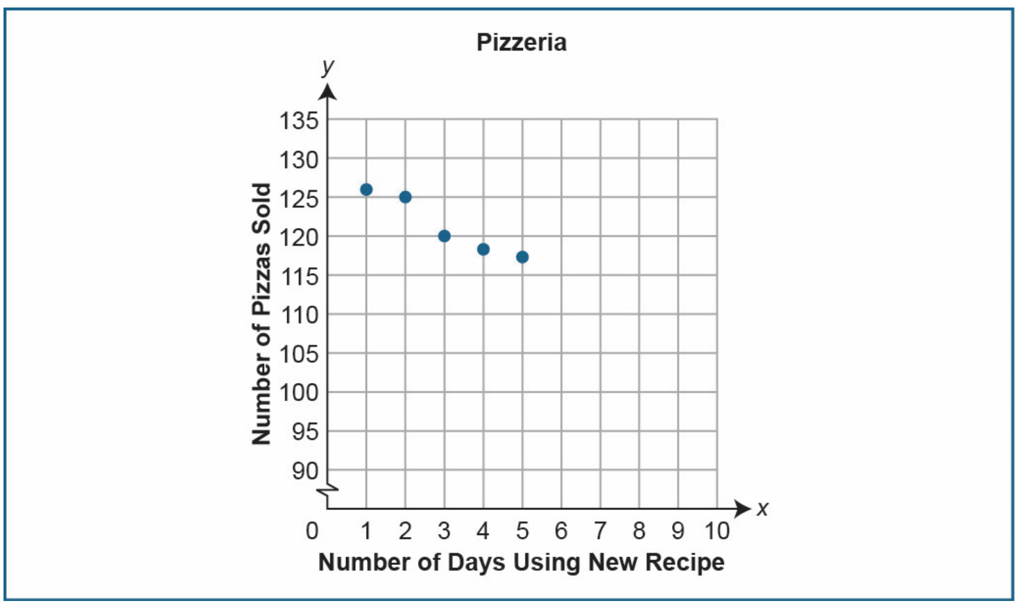
- Ask students to identify the equation for a line on a graph where the *y*-intercept is not provided. For example, ask students to approximate and write an equation in point-slope form for the following line.



Students should identify two points on the graph, use those points to determine the rate of change, and then write an equation. The graph shows lattice points on the line, but the students could have rounded when lattice points are not clearly defined.

- Ask students to generate a line of best fit to model a given data set and then write an equation for their line. For example, provide students with the following situation.

  The owners of a pizzeria are using a new recipe for their pizza sauce. They tracked the number of pizzas sold each day since implementing the new recipe on the graph shown.



  Ask students to draw a line of best fit that models the data and then write an equation for that line. Additionally, ask students to use their equation to predict how many pizzas will be sold on day 10.

- Ask students to use a linear equation in context to anticipate when a function value will occur. For example, give students with the following situation.
  Isabelle is saving money from her weekend landscaping job to buy a car. She has $4,500 saved and earns $650 each weekend that she works. If the car she wants to buy costs $18,800, how many weekends will she need to work to save enough money?
- Ask students to rewrite linear equations given changing constraints. For example, ask students to first determine the linear equation, line A, that passes through the points (2, 3) and (5, 9). Then, ask students to write a second linear equation, line B, that has the same slope but passes through the point (−1, 7). Encourage students to continue their investigation by writing a third linear equation, line C, that is half as steep as line B but intersects line B when $x = 3$.

▶ Key Terms

initial value (y-intercept), lattice point, point-slope form, rate of change (slope), slope-intercept form, standard form

A1.2.2.1.4

▶ Module, Assessment Anchor, Anchor Descriptor

Linear Functions and Data Organizations

Coordinate Geometry

Describe, compute, and/or use the rate of change (slope) of a line.

▶ Eligible Content

A1.2.2.1.4 Determine the slope and/or y-intercept represented by a linear equation or graph.

▶ Instructional Supports

- Ask students to determine the slope and y-intercept from varying equations in slope-intercept form. For example, ask students to identify the slope and y-intercept from each of the following equations.

  $y = 2x + 4$ $\qquad$ $y = -5x + 1$ $\qquad$ $y = \frac{2}{3}x - 7$ $\qquad$ $y = -x + \frac{1}{2}$

  Students should recall the general equation for slope-intercept form as $y = mx + b$, where m is the rate of change, or slope, and b is the initial value, or y-intercept. By using this knowledge, students can identify the slope and y-intercept of each equation, as shown. They should note that these are not limited to integer values and that the positive or negative sign stays with each value. Remind students that whenever there is no value written in front of a variable in an equation, there is an implied coefficient of 1.

$y = 2x + 4$	$y = -5x + 1$	$y = \frac{2}{3}x - 7$	$y = -x + \frac{1}{2}$
$m = 2$	$m = -5$	$m = \frac{2}{3}$	$m = -1$
$b = 4$	$b = 1$	$b = -7$	$b = \frac{1}{2}$

- Ask students to determine the slope and y-intercept from an equation of a line in point-slope form. For example, ask students to identify the slope and y-intercept of the following equation.

  $$y + 5 = \frac{2}{3}(x - 3)$$

  Students should recall the general equation for slope-intercept form as $y = mx + b$ and rewrite the given linear equation as an equivalent linear equation in slope-intercept form by using properties of algebra and isolating the variable *y*. Students should first distribute and then solve for *y*, as shown.

$$y + 5 = \frac{2}{3}(x - 3)$$

$$y + 5 = \frac{2}{3} \bullet x - \frac{2}{3} \bullet 3$$

$$y + 5 = \frac{2}{3}x - 2$$

$$y + 5 + (-5) = \frac{2}{3}x - 2 + (-5)$$

$$y = \frac{2}{3}x - 7$$

Once students determine the equivalent slope-intercept form of the given equation, $y = \frac{2}{3}x - 7$, they should note that the slope is $m = \frac{2}{3}$ and the *y*-intercept is $b = -7$.

- Ask students to determine the slope and *y*-intercept of an equation of a line in standard form. For example, ask students to identify the slope and *y*-intercept of the following equation.

$$6x - 2y = 13$$

Students should recall the general equation for slope-intercept form as $y = mx + b$ and rewrite the given linear equation in slope-intercept form by using properties of algebra and isolating the variable *y*, as shown.

$$6x - 2y = 13$$

$$6x + (-6x) - 2y = 13 - 6x$$

$$-2y = 13 - 6x$$

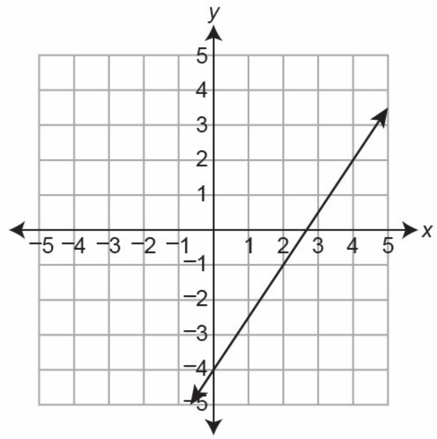
$$\left(-\frac{1}{2}\right)(-2y) = (13 - 6x)\left(-\frac{1}{2}\right)$$

$$y = -\frac{13}{2} + 3x$$

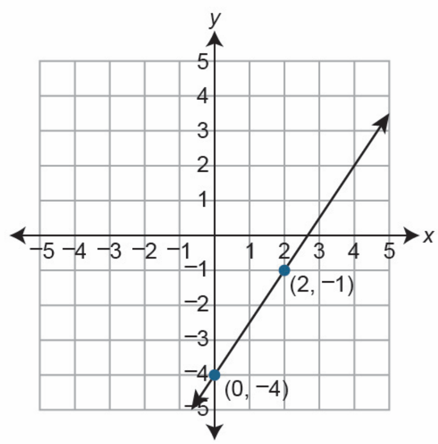
$$y = 3x - 6.5$$

Once students have determined the equivalent slope-intercept form of the given equation, $y = 3x - 6.5$, they should note that the slope is $m = 3$ and the *y*-intercept is $b = -6.5$.

- Ask students to identify the slope and *y*-intercept of a line from a graph. For example, ask students to identify the slope and *y*-intercept of the line shown.



Students may first identify the y-intercept, which is the point where the line intersects the y-axis, which in this case is the point $(0, -4)$. To identify the slope, students may identify another lattice point on the line and use rise over run to determine the slope or use these two points in the slope formula.

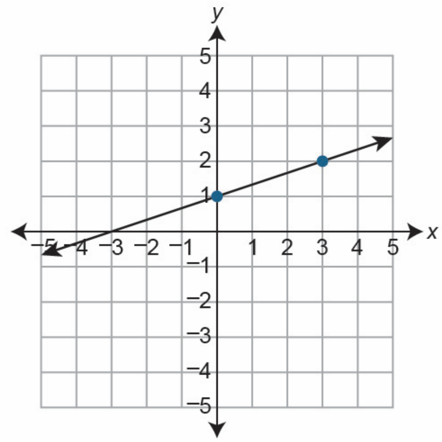


- Students using rise over run may begin at the y-intercept and count a rise of 3 units in the positive direction and a run of 2 units in the positive direction to arrive at a slope of $m = \frac{3}{2}$.
- Students using the slope formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ may substitute in the coordinates for the points $(0, -4)$ and $(2, -1)$ to conclude $m = \frac{-1-(-4)}{2-0} = \frac{3}{2}$.

▶ Identifying and Addressing Common Misconceptions

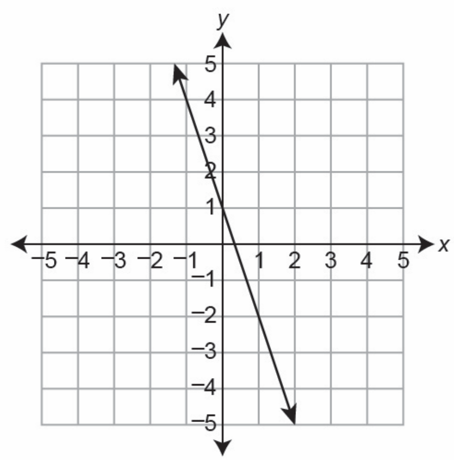
- Students may mix up the values of the slope and y-intercept in the general equation for slope-intercept form. Ask students to identify the slope and y-intercept in the equation $y = 3x - 2$. Students who mix up the values of the slope and y-intercept may conclude that the y-intercept is 3 and the slope is -2. Remind students to consider the sign of any operator leading to the initial value or the slope. Remind students of the general equation for slope-intercept form, $y = mx + b$, and connect m to the slope and b to the initial value or y-intercept. This leads to a slope of $m = 3$ and a y-intercept of $b = -2$ in the given linear equation.

- Students may misinterpret a slope of 0 or an undefined slope. Ask students to identify the slopes in the equations $y = -7$ and $x = 12$.
- Students may not include a negative sign with the y-intercept when the value is subtracted in slope-intercept form. Ask students to identify the slope and y-intercept in the equation $y = \frac{1}{2}x - 5$. Students who do not include a negative sign with the y-intercept may conclude that the initial value is $b = 5$. Remind students to consider the sign of any operator leading to the initial value or the rate of change. Since $y = \frac{1}{2}x - 5$ is equivalent to $y = \frac{1}{2}x + (-5)$ in the context of the general slope-intercept equation, $y = mx + b$, students should then conclude that the y-intercept is $b = -5$.
- Students may get confused by a line with a slope of $m = 1$, where the coefficient to the x term in slope-intercept form is not written. Ask students to identify the slope in the equation $y = x + 11$. Students who get confused by a line with a slope of $m = 1$ may interpret the slope as 0, because there is no value in the location where m would be in the general form. Remind students that whenever there is no value written in front of a variable in an equation, there is an implied coefficient of 1, and that $y = x + 11$ and $y = 1x + 11$ are equivalent, which should lead students to interpret the slope as $m = 1$.
    - Remind students about interpreting an integer slope of $m = 3$, when the denominator is not visible and there might not appear to be a rise over run.
- Students may identify the slope as its inverse: run over rise. Ask students to identify the slope of the following linear relationship.



Students who incorrectly identify the slope as its inverse may first notice that the second point is 3 units to the "right" and 1 unit "up" from the first point and incorrectly identify a rate of change of $m = \frac{3}{1}$. Remind students that the rate of change is the change in the *y*-values, or output values, divided by the change in the *x*-values, or input values, and can be interpreted through the visual idea of counting "rise over run." Students should then interpret the rate of change as having a "rise" moving up 1 unit from the initial value of (0, 1) and a "run" moving 3 units to the right, leading to a slope of $m = \frac{rise}{run} = \frac{1}{3}$.

- Students may interpret the slopes of all graphs to be positive. Ask students to identify the slope of the following graph.



Students who mistakenly interpret the slopes of all graphs to be positive may identify the *y*-intercept as (0, 1), count a "rise" of 3 units and a "run" of 1 unit to the left, and conclude that the rate of change is $m = 3$. Remind students to pay attention to the direction they are moving as they count "rise" and "run." As the numbers along each axis demonstrate, movements to the right or up result in positive values, while movements to the left or down result in negative values. Additionally, students should note whether the line is increasing or decreasing as x-values increase. Students should identify the correct rate of change as $m = -3$ in this example, with $\frac{-3}{1}$ or $\frac{3}{-1}$. Positive rates of change lead to increasing y-values as x increases and slant uphill. Negative slopes, on the other hand, lead to decreasing y-values as x increases and slant downhill.

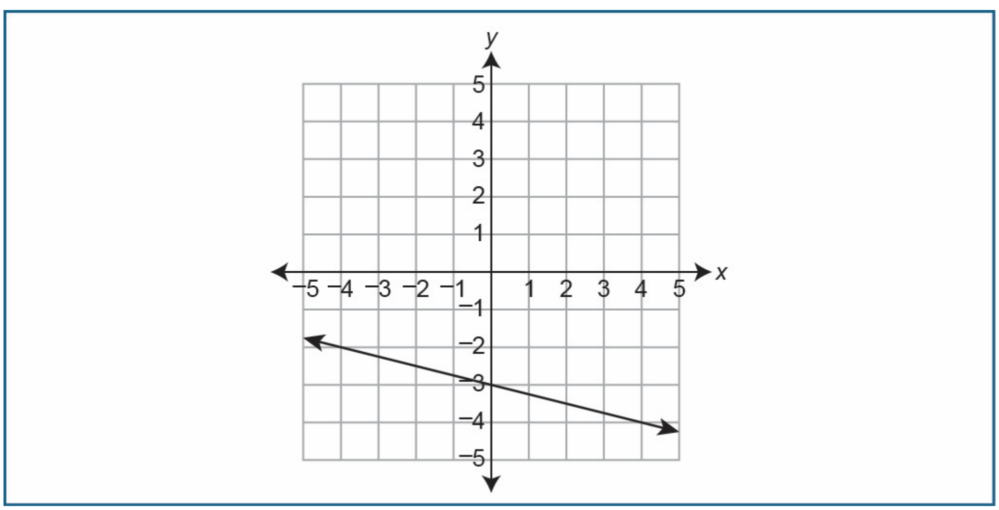
▶ Enrichment

- Ask students to interpret the rate of change and initial value from a real-world context and use those values to write a linear equation to represent the situation. For example, give students the following situation.

  A toy shop is having a promotion where every child who enters the store will receive a dinosaur toy until the shop runs out of them. The shop has 283 dinosaur toys at the start of the day. Write an equation to represent the relationship between the number of dinosaur toys the shop has remaining and the number of children who visit the store.

  Have students identify the rate of change and the initial value in this relationship. Additionally, ask students to solve problems with the equation they create, such as finding how many dinosaur toys will remain after 23 children visit the store and how many children can participate in the promotion before the store runs out of the toys.

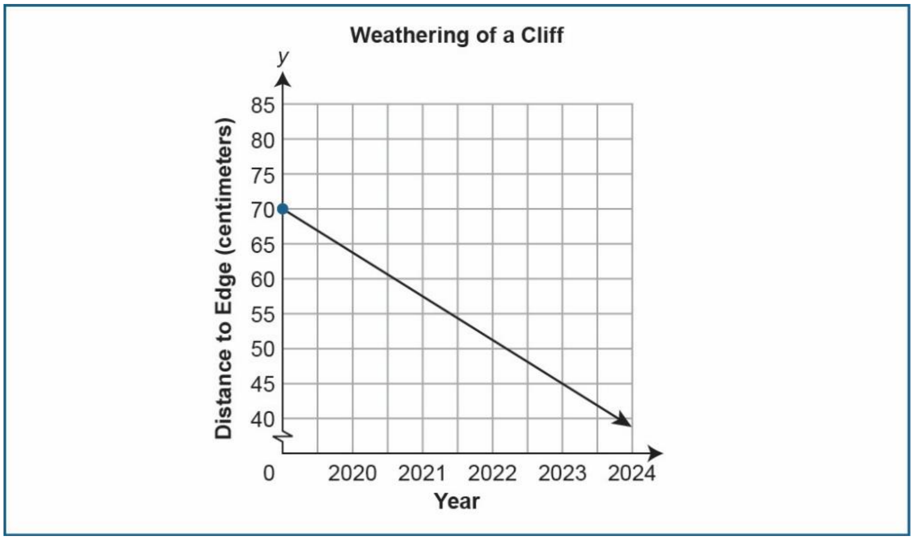
- Ask students to create related linear equations based on an understanding of slope and the y-intercept. For example, ask students to identify the equation of the line in the following graph.



Then, ask students to write a new equation that has the same slope but a y-intercept of 4. Additionally, ask students to write an equation that has the same y-intercept as the initial line but a slope that is double the magnitude and in the positive direction. Then, ask students to determine the y-value of this final line when x is equal to 10.

- Ask students to interpret the rate of change and initial value of a graph that represents a real-world scenario in context. For example, give students with the following situation

  An environmental science class has been measuring the weathering of a cliff for several years. They measure this by placing a metal locating stake in the ground at a specific point and measuring the perpendicular distance to the cliff face. The data they have collected are modeled in the graph.



  Answer these questions based on the graph: What was the distance from the locating stake to the edge of the cliff when the class began studying? If the cliff continues to weather at the same rate, what will be the distance to the cliff's edge in 2027?

► Key Terms

graph, linear equation, negative, point-slope form, positive, rise over run, slope (rate of change), slope-intercept form, standard form, y-intercept (initial value)

A1.2.2.2.1

▶ Module, Assessment Anchor, Anchor Descriptor

Linear Functions and Data Organizations

Coordinate Geometry

Analyze and/or interpret data on a scatter plot.

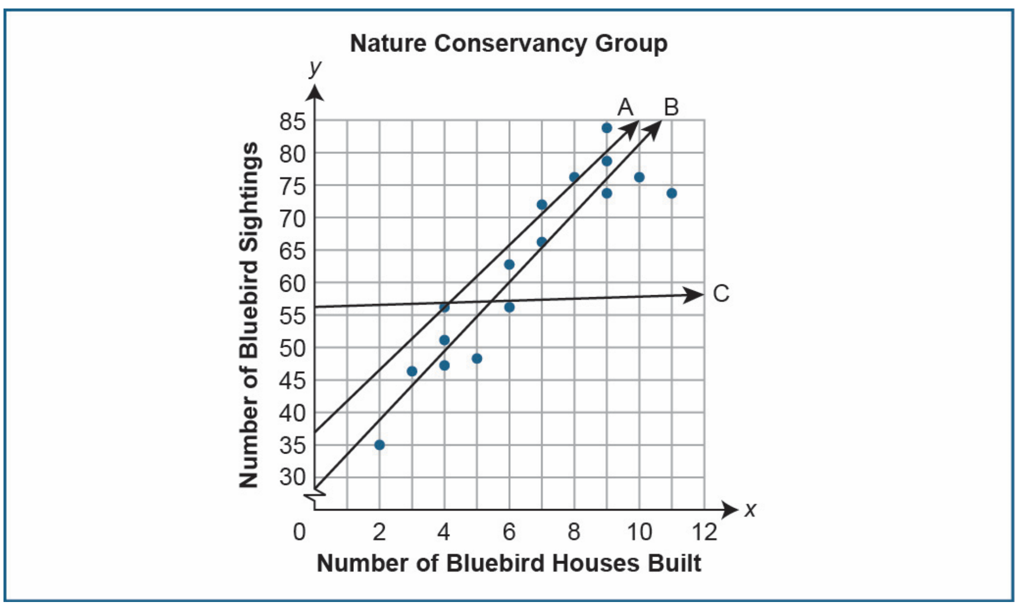
▶ Eligible Content

A1.2.2.2.1 Draw, identify, find, and/or write an equation for a line of best fit for a scatter plot.

▶ Instructional Supports

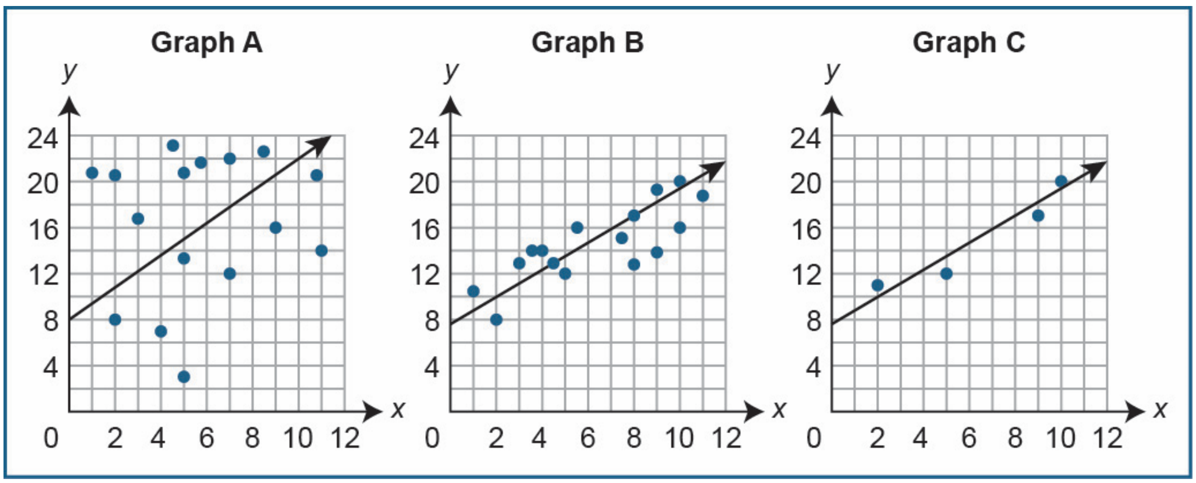
- Ask students to determine the BEST line of best fit from several options for the same data set. For example, give students the following situation.

  A nature conservancy group has built and placed bluebird houses in parks in a targeted region. They used cameras to track the number of bluebird sightings at each park and collected the data in the scatter plot shown. Which line BEST fits the data set? Justify your answer.



  Students should identify line B as being the line of best fit for this data set through visual inspection by noting that roughly half of the data points lie above the line and half of the data points lie below the line and that the data clusters in a narrow band around the line. Line C is not an appropriate line of best fit because even though roughly half of the data lies above and below the line, the data spread farther away from the line for lower and higher input values. Line A is not an appropriate line of best fit because most of the data points lie below the line and few lie above the line.

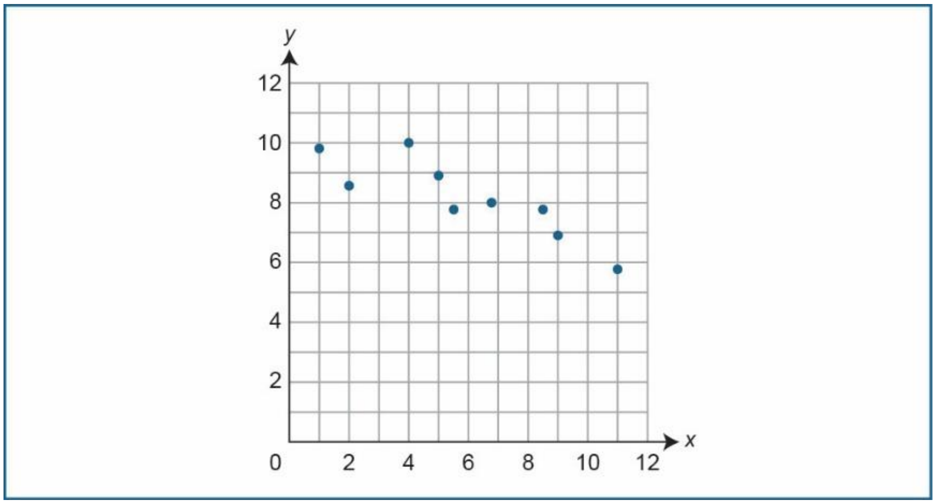
- Ask students to interpret or compare the strength of different lines of best fit. For example, ask students to compare the following data sets with the following lines drawn that have different dispersions of data.



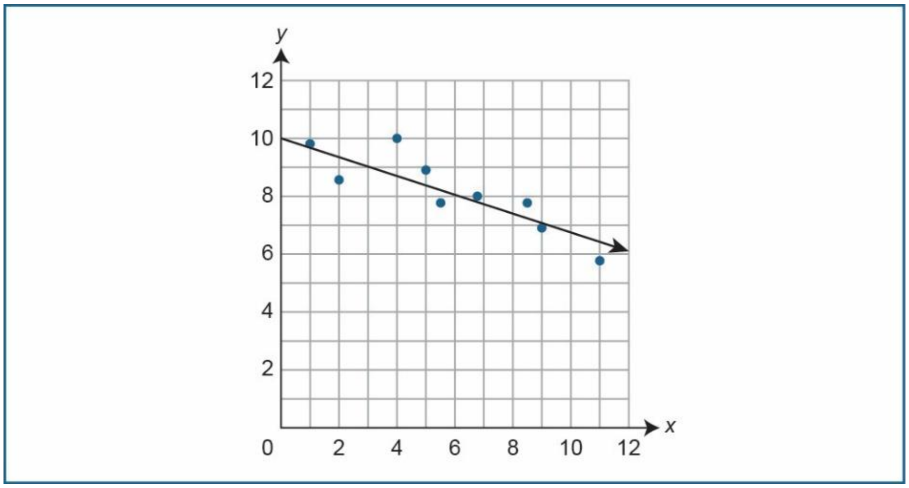
  - Students should justify that data set B has a stronger correlation for its line than data set A has because the data points in set A are scattered farther away from the line, with very few actual data points drawing near the line. Students should identify that there still appears to be a correlation for the points in set A, but it is weaker than the correlation in data set B.

  - Students should justify that data set B has a stronger correlation for its line than data set C has because, while the data in data set C are clustered very close to the line, there are very few actual data points; this provides less certainty in the data than is provided by the data in set B, which has a similar clustering but with many more data points.

- Ask students to draw a line of best fit when given a scatter plot. For example, provide students with a scatter plot showing a clear correlation, such as the one shown.

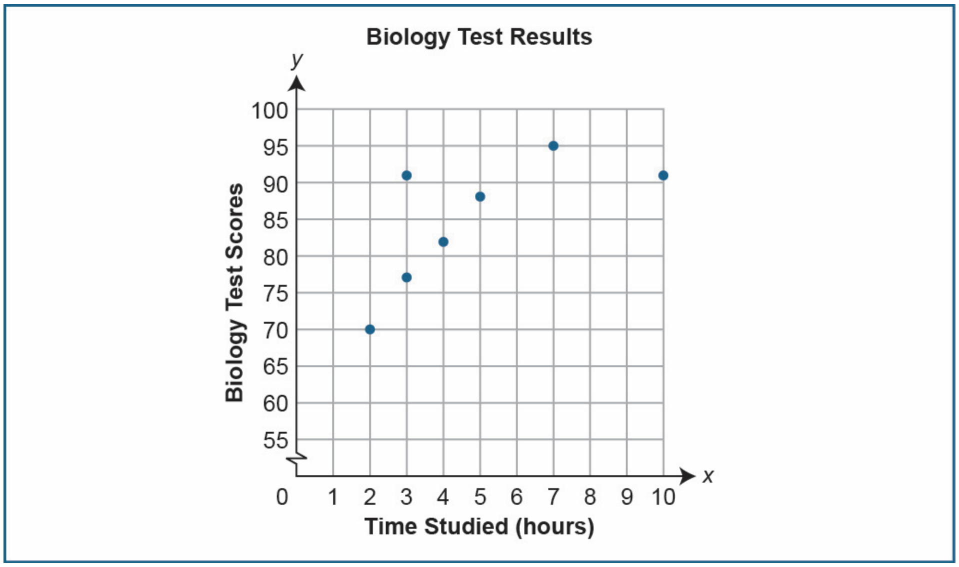


Ask students to draw a line of best fit to model the trend they see in the data. At this level, students should use visual estimation to draw a line that has roughly an equal number of points above and below the line and minimizes the average distance from the line to each point, capturing the general trend of the data.

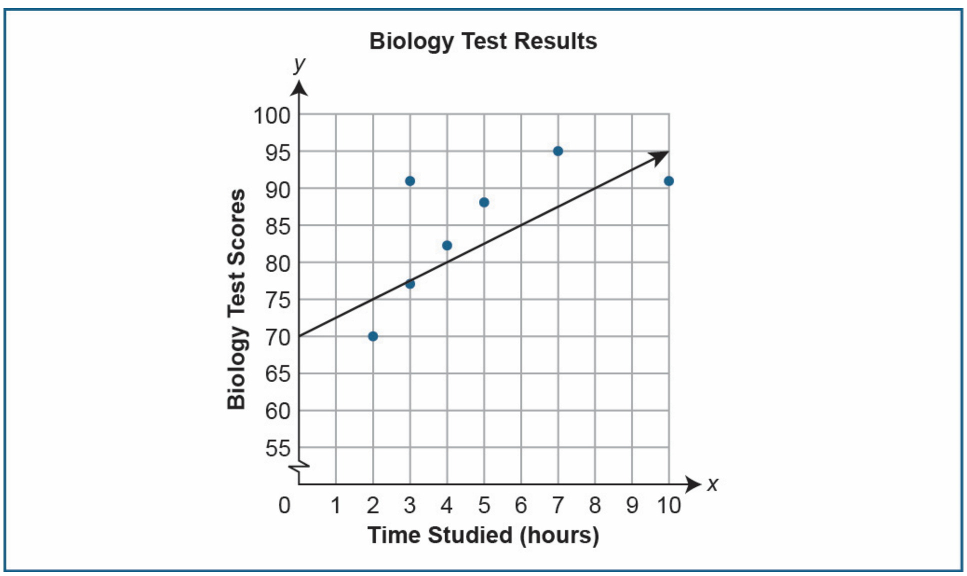


For additional consideration, ask students to compare the lines they drew with the lines drawn by their classmates by counting the points above and below each line and drawing vertical distances from each point to the lines of best fit. Students should conclude that lines with a more even distribution of points above and below the line and tighter clustering of data around the line are stronger fits for the data set.

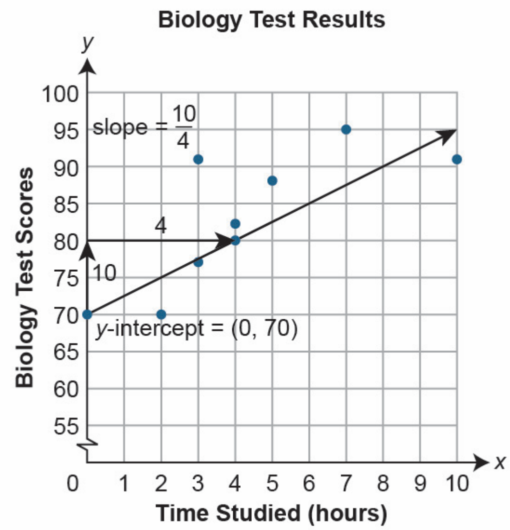
- Ask students to write an equation that describes the line of best fit for a data set and then draw the line of best fit derived from their equation. For example, give students a scatter plot and ask them to determine an equation for the line of best fit for the data set that relates the number of hours students in a class studied for a biology test, x, with the score that each student received, y, and then have them draw the line of best fit they have derived from their equation. The scatter plot below represents this data set: {(2, 70), (3, 77), (3, 91), (4, 82), (5, 88), (7, 95), (10, 91)}.



The following graph shows a line where about half the points are above and below the line.



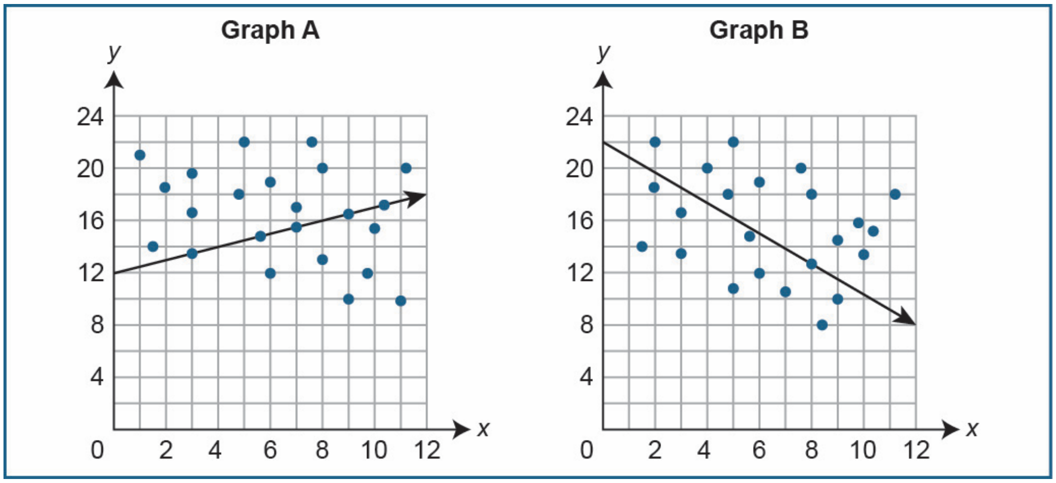
Once students have drawn a line that they believe best fits this data set, they can write an equation by identifying the y-intercept and using points on the line to approximate the slope.



Students can use any two data points to determine the slope of the line. Remind students to first locate points that pass through (or close to) lattice points. For example, a student could use (0, 70) and (3, 80) in the slope formula $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{80 - 70}{3 - 0} = \frac{10}{3}$. Then students may select one of the two selected points to substitute, along with the slope, into the slope-intercept form $y = mx + b$ to determine the y-intercept. For example, a student who selects (0, 70) along with $m = \frac{10}{3}$ would substitute them into the equation as $70 = \left(\frac{10}{3}\right)(0) + b$. Then, students should solve for b, resulting in $b = 70$. Substituting these values for slope-intercept form yields an equation for the line of best fit drawn on the graph: $y = \frac{10}{3}x + 70$. Ask students to compare their equations with this particular equation and discuss the possible differences and why there are differences.

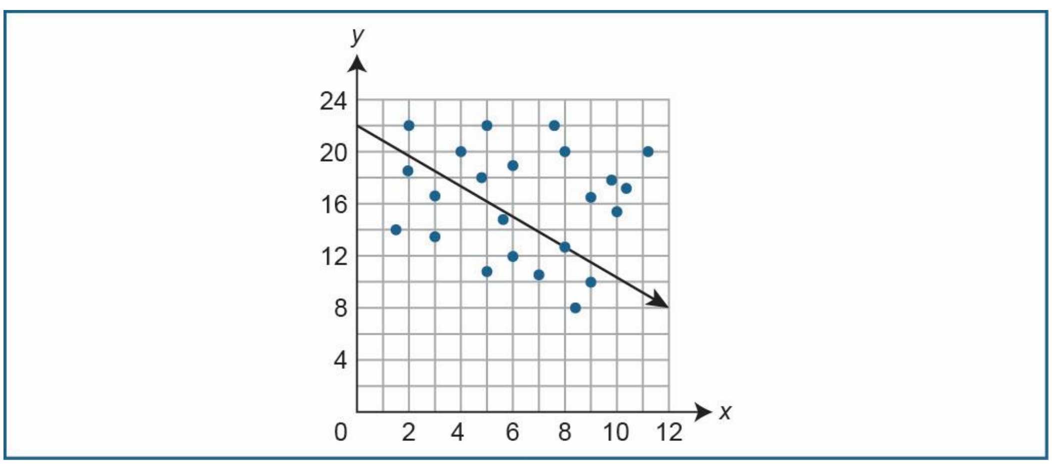
▶ Identifying and Addressing Common Misconceptions

- Students may believe that a line of best fit is a line that touches, or comes close to, the most points. Ask students to determine which graph has a stronger correlation for the following graphed lines.



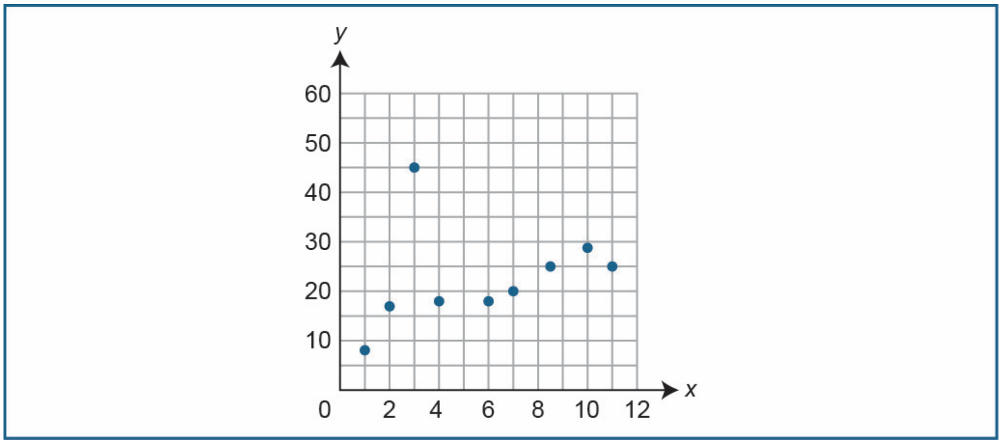
Students who have this misconception about lines of best fit may believe line A is a line of best fit because it touches more data points. Remind students that a line of best fit models the general trend of a data set, having approximately the same number of points above and below the line and minimizing the average distances from each data point to the line. Students should thus correctly interpret line B as having the line of best fit for its data set.

- Students may neglect to use a negative sign when writing the slope in an equation for a line of best fit with a negative correlation. Ask students to write an equation for the line drawn on the graph.

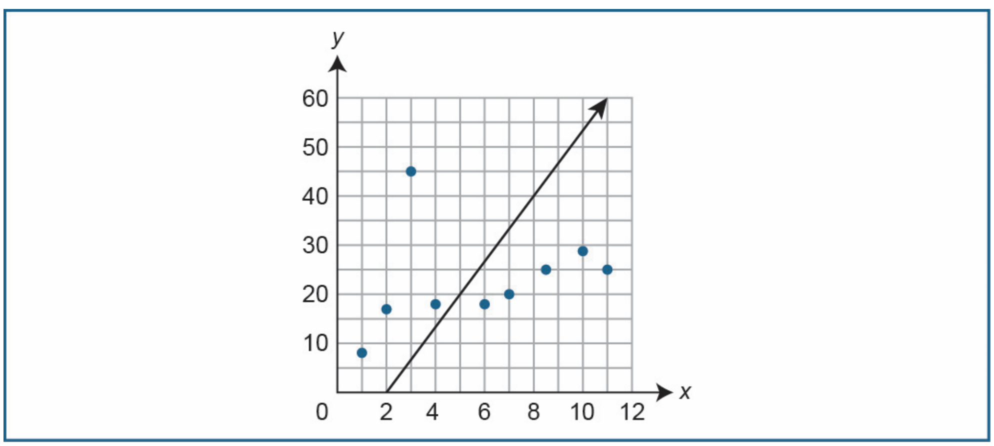


Students who have this misconception about negative signs and slopes may write an equation such as $y = \frac{5}{4}x + 22$ by seeing a "rise" of 5 and a "run" of 4 when connecting two points on the line of best fit. Remind students to pay attention to the direction in which they are moving as they count "rise" and "run." As the numbers clustered along each axis demonstrate, movements to the right or up result in positive values, while movements to the left or down result in negative values. Additionally, students should note whether the line is increasing or decreasing as *x*-values increase. Positive rates of change lead to increasing y-values as x increases and slant uphill. Negative slopes, on the other hand, lead to decreasing y-values as x increases and slant downhill. An appropriate approximation of the line drawn in this example would be $y = -\frac{5}{4}x + 22$.

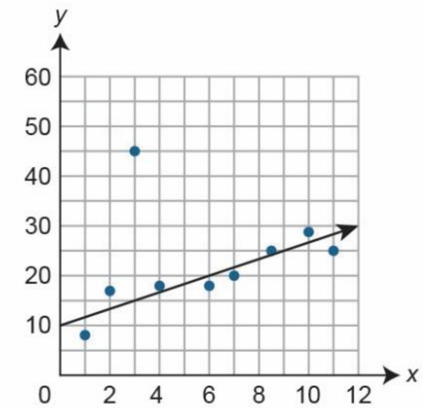
- Students may rely too much on outliers when determining a line of best fit. Ask students to create a line of best fit for a data set with a clear outlier, such as the one shown below.



Students who have this misconception about outliers and lines of best fit may overemphasize the effect of the outlier and shift the entire trend toward it.

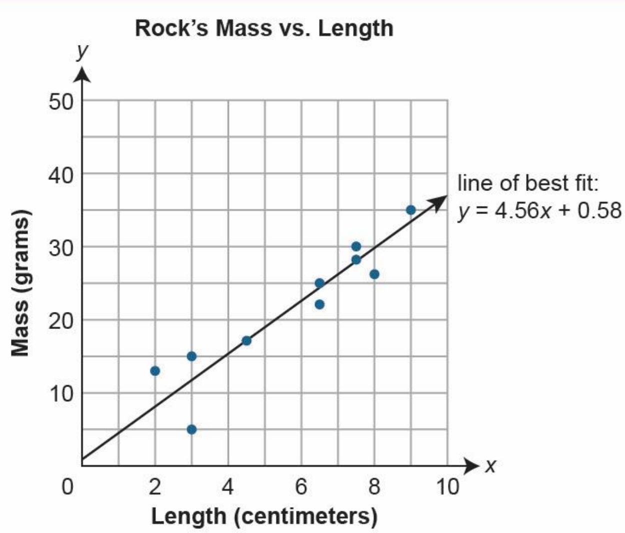


Remind students that a data set will often have data points that can be considered outliers. Outliers should be inspected with care to determine whether they are the result of a measuring error or should be considered as part of the data set. When outliers are not considered, they do not carry any extra weight, and a stronger line of best fit will follow the common trend designated by the majority of the data, as shown below.



► Enrichment

- Ask students to collect real-world data and create a scatter plot and line of best fit using the data. For example, ask each student to bring in a rock (setting some parameters of size). Then, as a class, measure the length of each rock at its longest point in centimeters and the mass of that rock in grams. Collect the data and place them in a table and ask students to plot the data in a scatter plot. Once the scatter plot is created, ask students to draw a line of best fit and then determine an equation for their line of best fit.
    - Ask students to use the line of best fit to make predictions about a data set. For example, ask students to use the line of best fit drawn in the following scatter plot showing the trend between the length and mass of rocks collected by a class to predict the mass of a rock that is 15 centimeters long at its longest point.



- Ask students to use technology such as a graphing calculator to identify and determine the correlation coefficient, *r*, of several distinct data plots and ask students to interpret how the coefficient, *r*, relates to the strength and trend of the line of best fit. For example, provide students with examples of scatter plots that have low or high values of *r* and positive or negative values for *r* and ask students to share conclusions and make predictions about the correlation coefficient in the scatter plots.
- Ask students to draw conclusions about a scatter plot, given information about the line of best fit. For example, give students the following situation.

  The students in an afterschool film club created a scatter plot connecting the length of a movie in minutes with the average rating of that movie on a scale from 1 to 10. After plotting, students drew a line of best fit that has a slope of 0 and a *y*-intercept of 6. What does this information tell us about the data set? Students should conclude that a slope of 0 indicates that there is not a clear positive or negative correlation between the two variables, and the length of a movie may not have any effect on the average rating.

► Key Terms

clustering, correlation (strong, weak), data set, line of best fit, negative, outlier, positive, rise over run, scatter plot, slope, slope-intercept form, trend, *y*-intercept

A1.2.3.1.1

▶ Module, Assessment Anchor, Anchor Descriptor

Linear Functions and Data Organizations

Data Analysis

Use measures of dispersion to describe a set of data.

▶ Eligible Content

A1.2.3.1.1 Calculate and/or interpret the range, quartiles, and interquartile range of data.

▶ Instructional Supports

- Ask students to calculate the range of a data set. For example, give students the following list of daily high temperatures for a local town in the past ten days and ask them to calculate the range.

  $$\{84°, 82°, 77°, 77°, 78°, 85°, 75°, 75°, 78°, 80°\}$$

  Students should rewrite the data set from least to greatest:

  $$\{75°, 75°, 77°, 77°, 78°, 78°, 80°, 82°, 84°, 85°\}$$

  Students should identify the greatest value in the data set, 85°, and the least value in the data set, 75°. Students should then determine the difference between the two values to determine the range of $85° - 75° = 10°$.

- Ask students to determine the quartiles (Q) and interquartile range (IQR) of a data set. For example, give students the time, in seconds, some teachers took to solve a puzzle and then ask students to calculate Q1 (quartile 1, also called the lower quartile), Q2 (quartile 2, also called the median), and Q3 (quartile 3, also called the upper quartile) of the data set along with the interquartile range (IQR). The data set is shown in the table.

Puzzle Solving

Teacher	1	2	3	4	5	6	7	8	9	10
Time (seconds)	35	41	22	18	65	47	93	28	29	53

Students should begin by rewriting the data from least to greatest, as shown.

$$\{18, 22, 28, 29, 35, 41, 47, 53, 65, 93\}$$

- Once the data are sorted, students should first determine the median value, or Q2. Since this data set has an even number of data points, 10, students can calculate the median by taking the average of the two middle values, 35 and 41, leading to a median value of $\frac{35+41}{2} = 38$ seconds.
- After students have determined Q2, they can calculate Q1 by determining the median value of the first half of the data set, or the values below Q2:

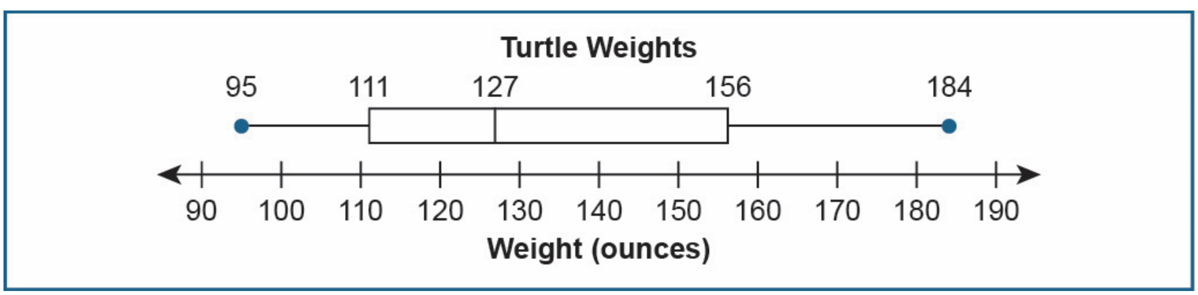
  $$\{18, 22, 28, 29, 35\}$$

  Since the number of the values is odd, the median is the middle value when the values are ordered from least to greatest, which results in a Q1 of 28 seconds.
- Students can then proceed to calculate Q3 by determining the median of the second half of the data set, or the values above Q2:

  $$\{41, 47, 53, 65, 93\}$$

  Since the number of the values is odd, the median is the middle value when the values are ordered from least to greatest, which results in a Q3 of 53 seconds.
- Finally, students can calculate the interquartile range by determining the difference between Q1 and Q3, which in this problem results in an IQR of $Q3 - Q1 = 53 - 28 = 25$ seconds.

• Ask students to read and interpret a box-and-whisker plot. For example, ask students to identify the range, interquartile range (IQR), and median in the following box-and-whisker plot, which summarizes the weights of turtles living near a local pond.



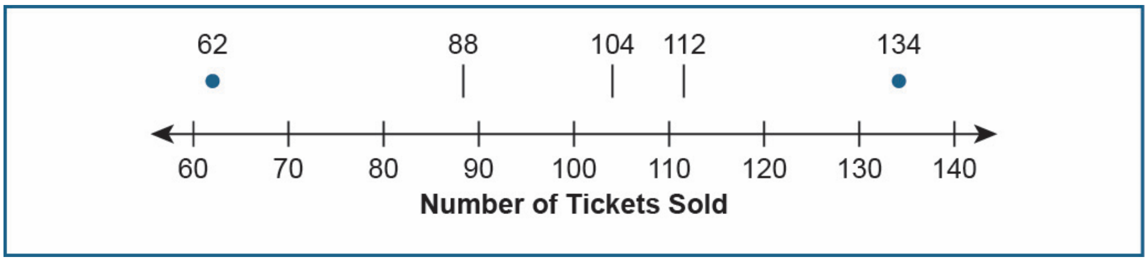
- Students should interpret the range to be the difference between the maximum (max) value of the right whisker, 184, and the minimum (min) value of the left whisker, 95. This leads to a range of $184 - 95 = 89$ ounces.
- Students should interpret the IQR as the difference between the value at the far-right edge of the box (Q3), 156, and the value at the far-left edge of the box (Q1), 111. This leads to an IQR of $156 - 111 = 45$ ounces.
- Students should interpret the median as the value where the line splits the box into two quartiles, which in this data set is 127 ounces.

- Ask students to create a box-and-whisker plot using the range and quartiles of a data set. For example, ask students to create a box-and-whisker plot for the number of tickets sold for each home volleyball game at a local school. Provide students with a data set and the five-number summary:

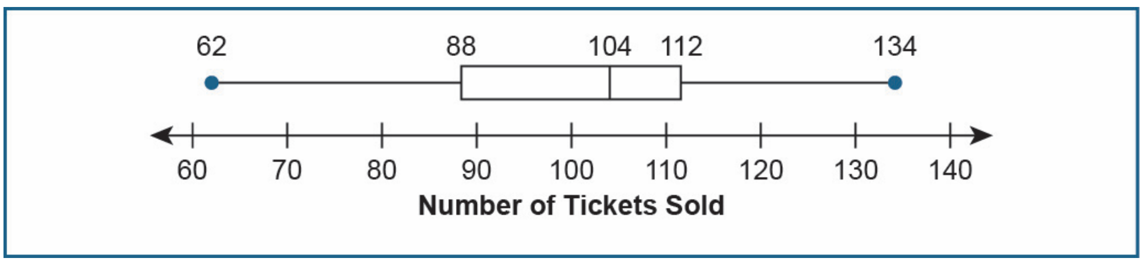
Number of tickets sold each game: $\{134, 67, 88, 93, 105, 62, 103, 110, 112, 117\}$

Five-number summary: min = 62, Q1 = 88, median = 104, Q3 = 112, max = 134

- Students should create a box-and-whisker plot (box plot) to represent the data by drawing endpoints at the min and max values and then marking vertical lines at Q1, the median, and Q3.



- Students should then connect the minimum to Q1 and connect the maximum to Q3 as "whiskers" while creating a divided "box" with Q1, the median, and Q3.



- Ask students to interpret the range, quartiles, and interquartile range (IQR) in the context of a real-world data set. For example, give students with the following situation.
  A local bowling league recorded the average score for each player in the league. The Q1 value of the data set is 185 and the Q3 value is 221. Determine whether each of the following statements is a valid conclusion based on the data and justify your answer.
  - Statement 1: 25% of the bowlers had an average score greater than 221.
    - Students can justify this as a valid statement through the definition of quartiles, which organize a data set into 4 regions, each of which contains 25% of the data.
  - Statement 2: 50% of the bowlers had an average score between 185 and 221.
    - Students can justify this as a valid statement through the definition of IQR, which defines the spread of the middle 50%, or the two interior quartiles, of a data set.
  - Statement 3: The median value of the data set must be 203.
    - Students should understand that the median value falls between Q1 and Q3 but is not necessarily the mean value of the two; therefore, this statement is false.
  - Statement 4: The least possible value of the data set is 185.
    - Since 185 is Q1, 25% of the data must fall below this data point. It would only be true that the least possible value in the data set is 185 if every value less than Q1 were equal to 185. Since there is no guarantee that this is true, the statement is false.

  Ask students to directly compare the variability (lack of consistency) in two data sets using range and interquartile range (IQR). For example, give students the following situation.

  A farmer has two chicken coops. She counts the number of eggs she collects from each coop over 8 weeks, as shown.

  Coop 1: 40, 47, 45, 32, 41, 39, 37, 45

  Coop 2: 28, 32, 34, 35, 36, 33, 21, 33

  1) Calculate the range and interquartile range for each coop over this period.

  2) Based on your calculations, which chicken coop has more variability? Explain your reasoning.

  Coop 1: 32, 37, 39, 40, 41, 45, 45, 47

  Coop 2: 21, 28, 32, 33, 33, 34, 35, 36

- Students should calculate the range of each coop by subtracting the minimum value from the maximum value.

$$\textit{Coop 1 range}: 47 - 32 = 15 \textit{ eggs}$$

$$\textit{Coop 2 range}: 36 - 21 = 15 \textit{ eggs}$$

- Students should calculate the IQR of each coop by first determining the median, Q1, and Q3 and then subtracting Q1 from Q3. Since there is an even number of data points in each data set, each median is determined by calculating the mean of the two middle values.

Coop 1 data:

$$\textit{Median}/Q2 = \frac{40+41}{2} = 40.5$$

$$Q1 = \frac{37+39}{2} = 38$$

$$Q3 = \frac{45+45}{2} = 45$$

$$IQR = 45 - 38 = 7 \textit{ eggs}$$

Coop 2 data:

$$\textit{Median}/Q2 = \frac{33+33}{2} = 33$$

$$Q1 = \frac{28+32}{2} = 30$$

$$Q3 = \frac{34+35}{2} = 34.5$$

$IQR = 34.5 - 30 = 4.5$ *eggs (value used to describe IQR but not realistic in real world)*

- Students should conclude that both data sets have the same range, but the data from Coop 1 have a greater variance seen in the greater IQR and therefore have a more unpredictable output.

▶ Identifying and Addressing Common Misconceptions

- Students may mix up interquartile range (IQR) with the total range for a data set. Provide students with a data set—including values for the median, Q1, and Q3—and then ask them to determine the range, as demonstrated in the following example.

The length of all the short films in a film festival, in minutes, are gathered in the following list:

$$\{4, 5, 7, 8, 9, 9, 10, 10, 11, 14\}$$

The median of this data set is 9 minutes, the Q1 value is 7 minutes, and the Q3 value is 10 minutes. What is the range of this data set?
Students who confuse interquartile range with the total range for a data set may subtract Q1 from Q3 to incorrectly determine a range of $10 - 7 = 3$ minutes. Remind students that the range of a data set is the difference between the absolute maximum value and the absolute minimum value, while the interquartile range (IQR) is the difference between Q3 and Q1. Students should determine the actual range of the data set to be $14 - 4 = 10$ minutes.

- Students may believe that quartiles or the median must be actual data values and incorrectly determine quartile or median values when there is an even number of data points being examined. Ask students to determine Q1, Q2, and Q3 of the data set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Students who have this misconception about quartiles and the median may select the median, or Q2, to be the value 4 or 5 and may similarly believe Q1 to be 2 or 3 and Q3 to be 6 or 7. Remind students that when they are determining the median, or middle value, of a collection of data that has an even number of entries, they need to take an average of the two middle values, which means adding the two values together and dividing by 2. In this example, students should determine $\text{Q2} = \frac{4+5}{2} = 4.5$, $\text{Q1} = \frac{2+3}{2} = 2.5$, and $\text{Q3} = \frac{6+7}{2} = 6.5$.
- Students may determine a median or a quartile value without first ordering the data from least to greatest. Ask students to determine the median of the following list of values:

$$\{9, 5, 3, 2, 7\}$$

Students who fail to order data correctly may believe that because there are 5 data points, the median will be the middle point listed, which is 3. Remind students that when they are determining any median or quartile value they must first order the data from least to greatest. Students should reorder this data set as $\{2, 3, \mathbf{5}, 7, 9\}$ and determine a median of 5.

► Enrichment

- Ask students to collect their own real-world data and then describe the range, interquartile range

(IQR), and quartile values. Additionally, ask students to tell the story of the data they collected by using variability in their data set. For example, ask students to use a stopwatch to determine the amount of time it takes them to get to school in the morning, starting when they walk through their front door and ending when they walk through the front door of the school. After students have collected the data for the class, ask them to determine the range, quartile values, and IQR. Then, ask them to summarize their data and draw conclusions based on the summary statistics they have determined.

- Ask students to decide which measure of variability may be more useful in a specified comparison. For example, ask students to compare the data set of song lengths in the most recent albums of two different artists by focusing on the range, quartiles, and interquartile range (IQR). Ask students to determine which artist has more consistency in song length and write an explanation justifying that claim.
- Ask students to use a given rule for identifying outliers to reinterpret a data set and then draw conclusions from the process. The outlier rule using the IQR to identify potential outliers to determine whether the value is "greater than 1.5 times the IQR and added to Q3" and "less than 1.5 times the IQR and subtracted from Q1." For example, provide students with the following data set that tracks the daily step count for employees in an office.

  $$\{6{,}200, 5{,}800, 6{,}400, 7{,}100, 24{,}500, 6{,}900, 7{,}000, 6{,}200, 28{,}000, 5{,}500, 5{,}300, 7{,}300\}$$

  Ask students to first determine the range, quartiles, and interquartile range (IQR) of this data set. Then, explain to students that one method for determining statistical outliers is through identifying points that are greater than 1.5 times the IQR and are then added to Q3 or less than 1.5 times the IQR and are then subtracted from Q1. Ask students to identify and remove any outliers from the data set and then use the new data set to determine an updated range, set of quartile values, and IQR. Have students compare the initial summary statistics with their new set and draw conclusions about the similarities and differences. Finally, have students explore contexts in which they may want to include the outlier values in their data sets or when they may want to exclude them based on potentially different aims for collecting and understanding the data.
- Ask students to predict missing data values given a partial data set and summary statistics. For example, give students with the following situation.
  A scientist reviewing a recent study spilled coffee on her notes. She can see that the range of values is 42 and the IQR is 18. She can also see 4 initial data points but not in any order: {103, 119, 97, 98}.

She recalls gathering at least 8 total data points. While we may never see what her actual data were, create a full list of hypothetical additional data points that would fit the information she already has.

▶ Key Terms

box-and-whisker plot (box plot), dispersion, interquartile range (IQR), maximum value, median, minimum value, outlier, quartile, range, variability

A1.2.3.2.1

► Module, Assessment Anchor, Anchor Descriptor

Linear Functions and Data Organizations

Data Analysis

Use data displays in problem-solving settings and/or to make predictions.

► Eligible Content

A1.2.3.2.1 Estimate or calculate to make predictions based on a circle, line, bar graph, measure of central tendency, or other representation.

► Instructional Supports

- Ask students to determine the mean, median, and mode of a data set. For example, ask students to determine the mean, median, and mode of the length of time ten students in a class were able to swing a Hula-Hoop before the hoop dropped, as shown in the table.

Hula-Hoop Contest

Student	1	2	3	4	5	6	7	8	9	10
Time (seconds)	3	8	24	4	68	31	17	4	12	19

Students should determine the mean by calculating the sum of the data entries for time and dividing the value by the total number of entries, 10. (Note: The students do NOT need to order the data in order to calculate the mean.)

$$\text{mean} = \frac{3 + 8 + 24 + 4 + 68 + 31 + 17 + 4 + 12 + 19}{10} = 19$$

Students should conclude that the mean, or average, amount of time students in the class were able to swing the Hula-Hoop was 19 seconds.

Students should determine the median by ordering the data from least to greatest and then determining the middle value if the number of data points is odd or the average of the two values that are closest to the middle if the number of data points is even. The data set is written from least to greatest, as shown.

$$\{3, 4, 4, 8, 12, 17, 19, 24, 31, 68\}$$

Since the number of data points, 10, is even, the median is the average of the two data points that are closest to the middle.

$$\{3, 4, 4, 8, 12, \mathbf{17}, 19, 24, 31, 68\}$$

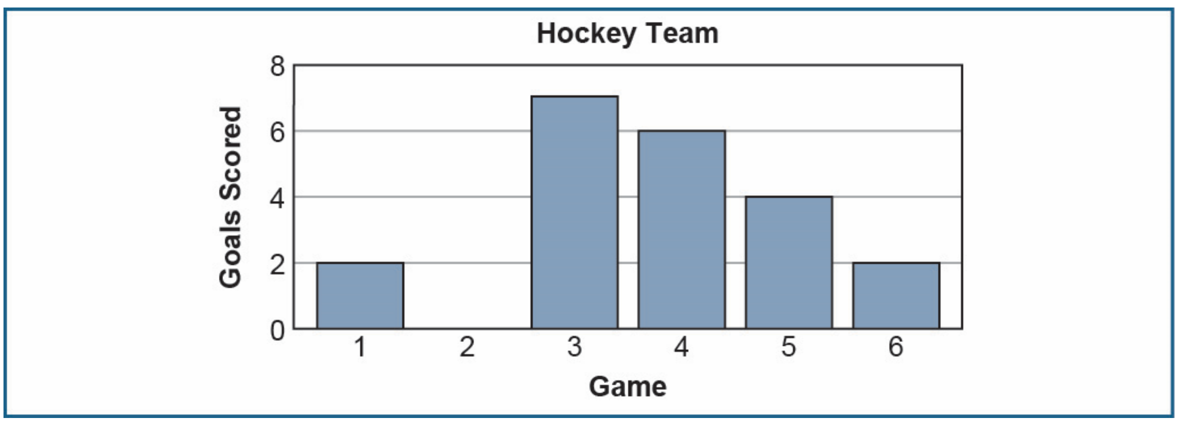
$$\text{median} = \frac{12 + 17}{2} = 14.5$$

Students should conclude that the median value for the data set is 14.5 seconds.

Students should determine the mode by identifying the data entry that is most common. In this data set, 4 seconds is the only data entry that repeats, so it is the mode.

- Ask students to make predictions utilizing the mean, median, or mode of a data set displayed in a table, line graph, or bar graph. For example, give students the following situation.

The numbers of goals a hockey team has scored in the first 6 games of the season are shown in the bar graph. Use the data to predict the number of goals the hockey team will score in the 7th game and justify your answer.



Students may make several different, but valid, predictions for this data set based on the measure of central tendency they choose to focus on. Justifications should include the choice of summary data they select along with their reason for considering that measure.

- Students may predict the team will score 3 or 4 goals (because 3.5 does not make sense as a prediction) in the next game based on the mean of the data, which they can calculate as

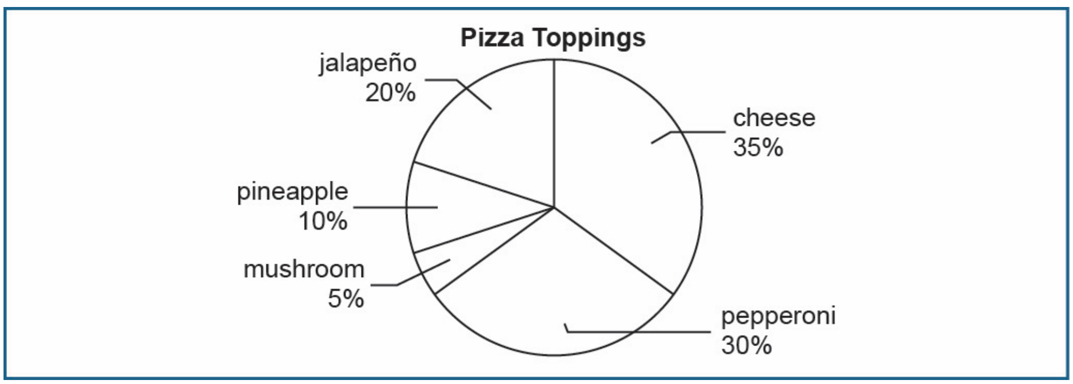
$\frac{2+0+7+6+4+2}{6}=3.5$ goals. Students can justify this prediction because it provides a pure average, which best interprets the number of goals scored to all of the games.

- Students may predict the team will score 3 goals in the next game based on the median of the data, which they can calculate as the average of the two value closest to the middle in the data set, 2 and 4. Students might justify using the median as their predictive value because it is less obscured by possible outlier results.

Students may predict the team will score 2 goals in the next game based on the mode of the data, which is 2. Students might justify the mode as their predictive value because it is the most common result of goals scored in a game: so this value has the best individual expected chance or probability of occurring.

Ask students to use a circle graph or pie chart to draw conclusions about a data set. For example, give students the following situation.

The circle graph shows the distribution of pizza topping preferences from a survey of 40 students at a high school.



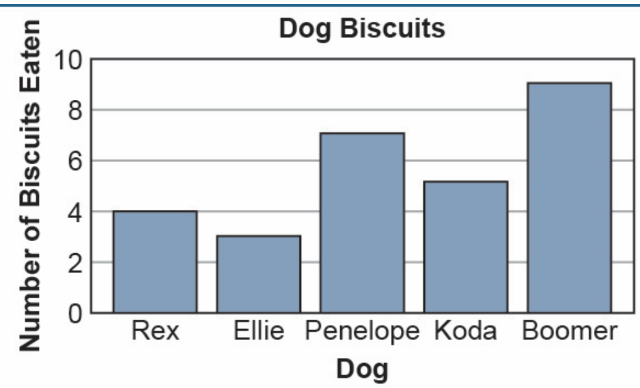
Students can answer the following questions.

1) How many of the 40 students prefer a topping of pineapple?
   - Students can discover how many students prefer pineapple by multiplying 40 (the total number of students) by 0.10 (because 10% prefer pineapple) to get (40)(0.10) = 4 students.
2) The principal is hosting a pizza party for the whole school and is deciding on the number of different types of pizza to order. Based on the circle graph results, how many of the 1,500 students at the school can be expected to prefer cheese or jalapeño toppings?

- Students can determine the number of students initially surveyed that prefer pineapple by multiplying the total number of students surveyed, 40, by the percentage of students that prefer pineapple, 10%, to determine a total of $(40)(0.10) = 4$ students.
- Students can then add the percentages of surveyed students who prefer cheese or jalapeño to get 35% + 20% = 55%. They can then determine 55% of 1,500 students: (0.55)(1,500) = 825 students.

- Ask students to use bar graphs to make predictions or solve problems about a data set. For example, give students the following situation.

The bar graph shows the number of biscuits each dog at a dog daycare ate on Wednesday.

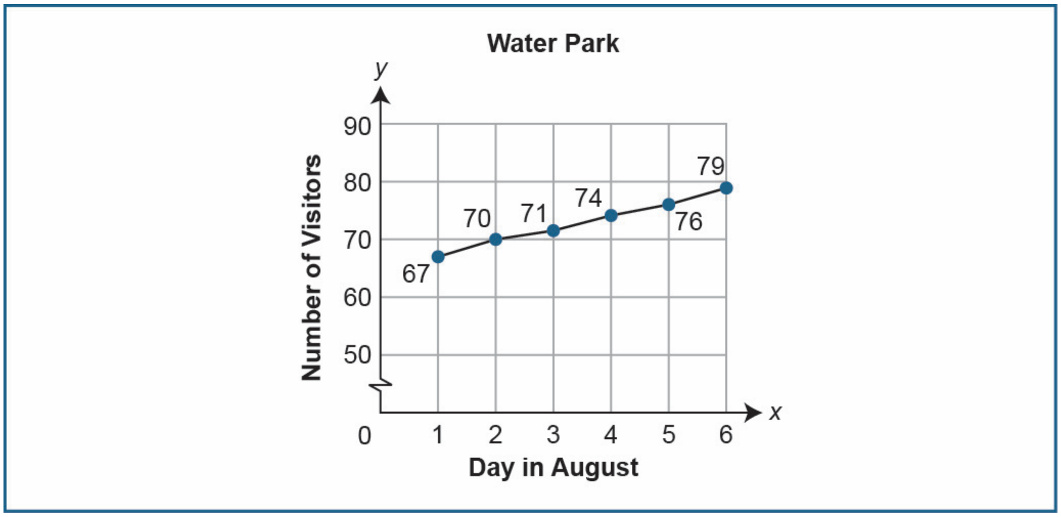


Students can answer the following questions.

1) What is the mean of this data set?
    - To calculate the mean, students should divide the sum of the data points by the total number of entries: $\frac{4+3+7+5+9}{5} = 5.6$. This means that each dog ate an average of 5.6 biscuits on Wednesday.
2) Another dog, Shadow, had 8 biscuits on Wednesday, but that data point is not recorded in the bar graph. How will including this data point affect the mean of the data set?
    - Students should conclude that the mean will increase, since 8 biscuits is greater than the mean of 5.6 biscuits. The updated mean is $\frac{4+3+7+5+9+8}{6} = 6$ biscuits.
3) The daycare anticipates caring for 30 dogs on Friday. Based on the adjusted mean in question 2, how many biscuits will the daycare need?

- Students should use the average of 6 biscuits per dog to predict that they will need $(6)(30) = 180$ biscuits for 30 dogs.

- Ask students to make a prediction based on data presented in a line graph. For example, give students the following situation.
  The line graph shows the number of visitors to a water park during the first 6 days of August.



Students can answer the following questions. Note: Students may arrive at a variety of reasonable answers depending on how they choose to interpret the data. It is important to analyze the responses through correct interpretation of measures of central tendency and data trends in addition to constraints on real-world contexts; responses should not necessarily be judged through a universally "correct" or "incorrect" prediction.

1) Based on the line graph, what would you predict the number of visitors to be on day 7? Explain your reasoning.
   - Students may predict that on day 7 the water park will have 82 visitors. They might justify this prediction by noting that on each of the first 6 days, there was an increase of between 1 and 3 visitors, with both the median and the mode of the daily increase being 3 visitors. Adding 3 visitors to the most recent day's total of 79 visitors results in 82 visitors. Students may acknowledge that the average increase is slightly less than 3, so they may predict more strongly 81 or 82 visitors but likely no more than that.
2) If the data continue to follow the current trend, on what day would you anticipate the water park will have 90 visitors?

- Students should use the average expected increase chosen in their prediction on day 7 to iteratively, or through multiplication, determine an anticipated number of days until a total of 90 visitors is reached.

Water Park Predictions

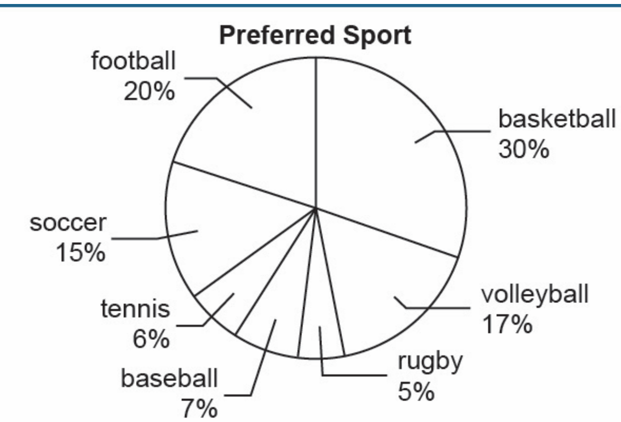
Day	7	8	9	10	11
Number of Visitors	82	85	88	91	94

By continuing to use a median increase of 3, students may predict that the number of visitors will reach 90 by day 10. They may also conclude that, since the initial set of data often increased by 3 or less and occasionally by 1 or 2 visitors, a better prediction may indicate 90 visitors by day 11.

▶ Identifying and Addressing Common Misconceptions

- Students may multiply by a percentage without converting it to a decimal when solving problems using circle graphs. Give students the following situation.

  The following circle graph shows the favorite sport of students in a community.

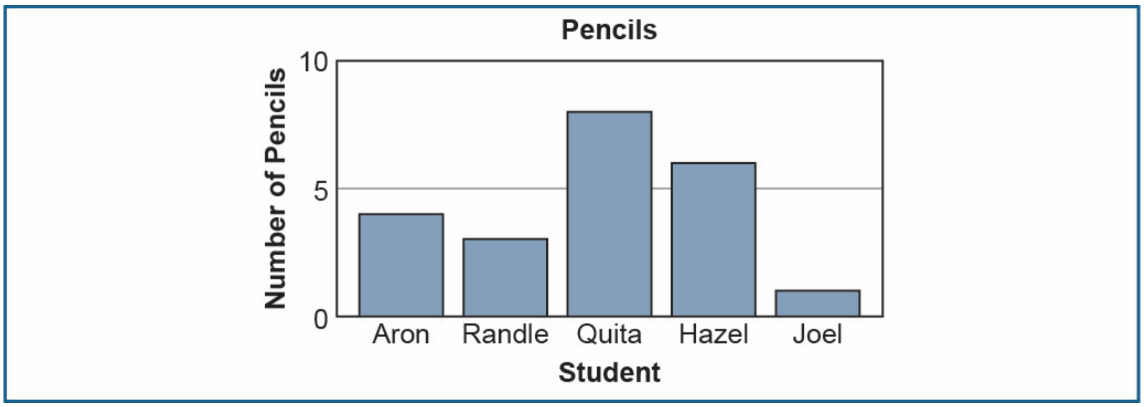


If there are 320 students in the community, how many prefer the sport of rugby?

Students who mistakenly multiply by a percentage without converting it to a decimal may multiply 320 students by 5 percent to get a result of $(320)(5) = 1{,}600$ students preferring rugby. Remind students that a percentage is equivalent to a hundredth fraction or decimal and when multiplying it should be converted to a decimal value. This will lead to a correct interpretation that 5% is 0.05, so $(320)(0.05) = 16$ students who prefer rugby.

- Students may forget to average the middle values when calculating the median of a data set with an even number of data points. Ask students to determine the median of the following data set: $\{10, 20, 30, 40, 50, 60\}$. Students who forget to average middle values may count the data points and assume that 30 and 40 are both middle values, so either or both can be the median. Remind students that when there is an even number of data entries, they must take the average of the two middle values, which means adding them and dividing by 2. Students should correctly identify the median in this data set to be $\frac{30 + 40}{2} = 35$.

- Students may misinterpret data points in a line or in a bar graph by "eyeballing" or guessing unlabeled values. Ask students to determine the mean of the following data set.



Students may misread this data set by identifying Quita as having 7 pencils or Joel having 2 pencils. This would lead to a miscalculated mean of 4.2 pencils per student. Remind students about precision when reading data points in a bar graph and to draw incremental horizontal lines to help ensure accuracy. Students can correctly identify the mean in this example to be $\frac{4 + 3 + 8 + 6 + 1}{5} = 4.4$ pencils per student.

- Students may need to be reminded that a data set can have more than one mode or no mode. Also, remind students not to forget counting a data point of 0 when calculating mean, median, or mode. Remind the students about understanding the benefits of one measure of central tendency over another when they make a prediction.

▶ Enrichment

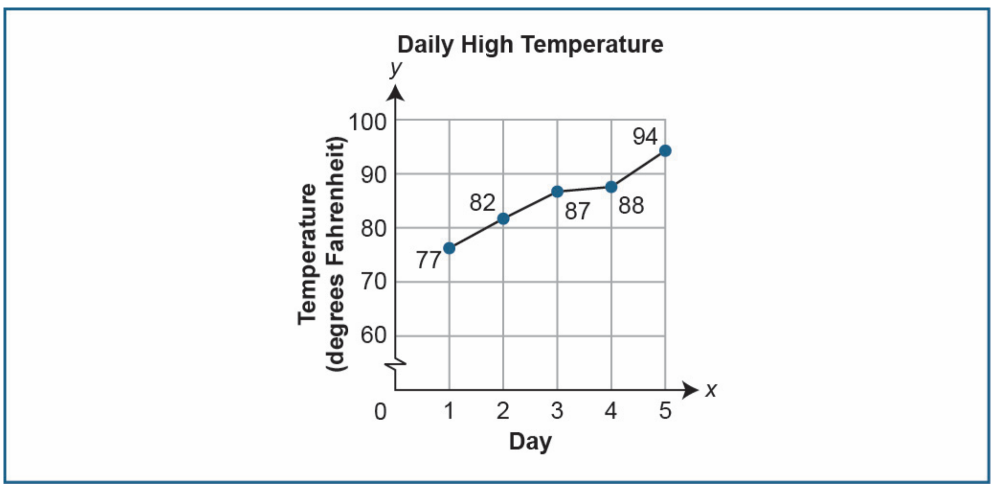
- Ask students to solve problems about unknown data points using the mean, median, or mode. For example, give students the following situation.

A training program for long-distance swimming categorizes the longest swim distance in each workout. Raul knows that the program includes 10 workouts and has a median longest swim distance of 400 meters. So far, he has completed workouts with the following longest swim distances: $\{200, 150, 400, 350, 200, 550, 500, 650, 600\}$. What can Raul conclude about the longest swim distance in the final workout?

- Ask students to combine data sets to make predictions with summary statistics. For example, give students the following situation.

  Mrs. Vasquez gave a quiz to each of her two history classes. Her first period class had a mean score of 88%, while her second period class had a mean score of 81%. If there are 18 students in her first period class and 25 students in her second period class, what is the mean score for both history classes combined?

- Ask students to identify constraints of real-world contexts and use them when interpreting and making predictions about a data set. Give students the following situation.
  The line graph shows the daily high temperature of a Midwestern town over the last 5 days.



  Based on the line graphed, what can you predict the daily high temperature will be on day 15? Explain your reasoning.

  Students should recognize that there is a clearly increasing trend in the data set and that making a prediction based solely on the data would result in a temperature of over 130°F. However, they should understand that a daily temperature of 130°F is likely an unrealistically high temperature and that daily temperatures often rise, fall, and level off.

- Ask students to continue to explore real-world boundaries by creating realistic hypothetical data sets that have a distinct relationship or trend but are otherwise constrained by real-world circumstances.

► Key Terms

bar graph, central tendency, circle graph (pie graph), line graph, mean (average), median, mode, outlier, prediction, trend

A1.2.3.2.2

▶ Module, Assessment Anchor, Anchor Descriptor

Linear Functions and Data Organizations

Data Analysis

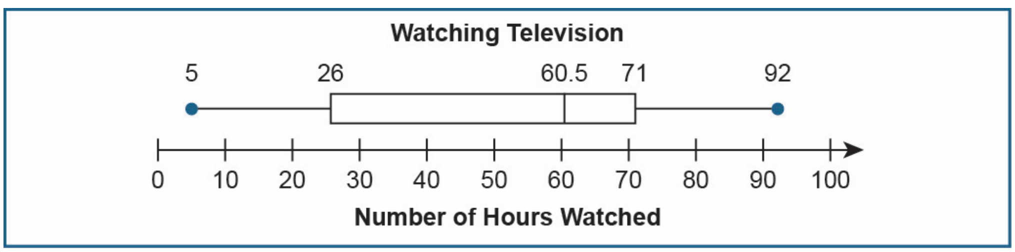
Use data displays in problem-solving settings and/or make predictions.

▶ Eligible Content

A1.2.3.2.2 Analyze data, make predictions, and/or answer questions based on displayed data (box-and-whisker plots, stem-and-leaf plots, scatter plots, measures of central tendency, or other representations).

▶ Instructional Supports

- Ask students to interpret a box-and-whisker plot (box plot) and answer questions about the data it presents. For example, give students the following situation and series of questions.
  A box-and-whisker plot (box plot) displays the number of hours that 40 people surveyed said they watched television over the duration of a month.



1) What percentage of the survey population watched between 60.5 and 92 hours of television during the month?
2) How many of the people surveyed watched 26 or fewer hours of television during the month?
   - Students can use the box plot to create a five-number summary:
     - Min = 5 hours
     - Q1 = 26 hours
     - Median = 60.5 hours
     - Q3 = 71 hours

    - Max = 92 hours
  - Based on this summary, students can identify that 50% of the data lies between 60.5 and 92 as the median splits the data set into two parts with an equal number of entries.
  - Students should recognize that Q1= 26 hours, which indicates that 25% of the data lies between the minimum value and Q1. Since there were 40 survey respondents, multiplying $(40)(0.25) = 10$ respondents who watched 26 or fewer hours.

- Ask students to interpret a stem-and-leaf plot and answer questions about the data it represents. For example, give students the following situation and series of questions.
  The given stem-and-leaf plot shows the number of pieces of candy that each child at a party picked up from a piñata.

Pieces of Candy

1	2 2 3 5
2	1 7 8
3	1 2 2 3 3 4 5
4	2

Key
3 | 1 = 31

1) What is the median number of pieces of candy each child picked up?

2) How many children picked up fewer than 20 pieces of candy?

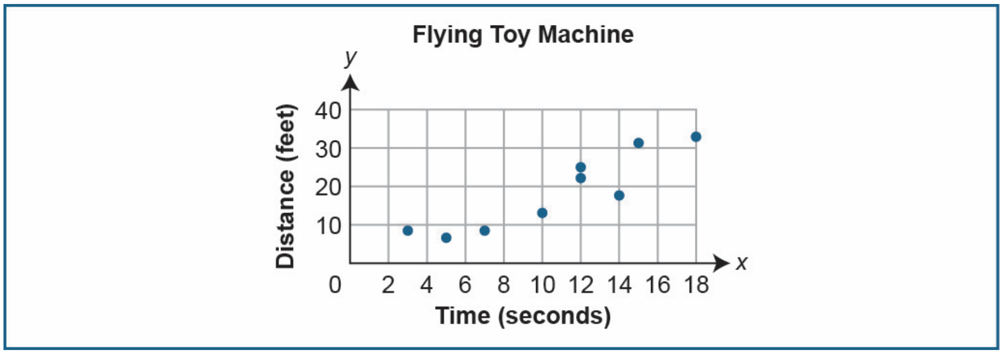
  - Students should first count the number of data entries, 15, so that they can determine the median, or middle value, of the data set. Since 15 is odd, students should count to the 8th data entry to determine a median value of 31.
  - Students may find it difficult to read the stem-and-leaf plot, so provide a scaffold by prompting them to rewrite the data set as a list of discrete data entries and then have them interpret the data in that list.

    {12, 12, 13, 15, 21, 27, 28, 31, 32, 32, 33, 33, 34, 35, 42}

  - Students should identify that the number of children who picked up fewer than 20 pieces of candy can be quickly identified as the only data entries with a stem of 1. They can conclude that 4 children picked up fewer than 20 pieces of candy.

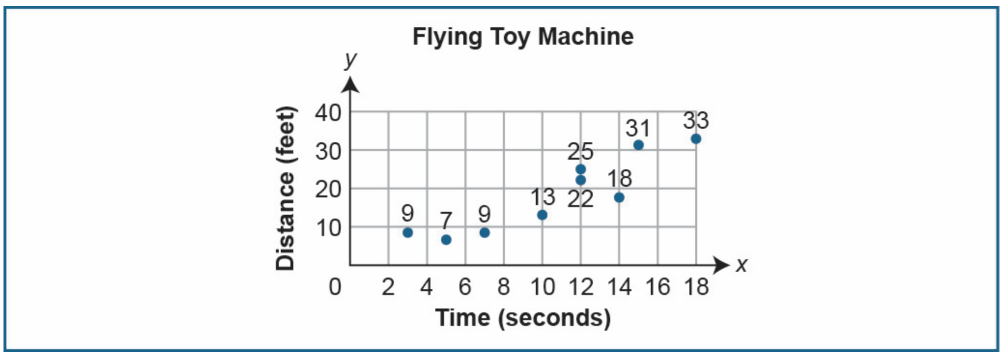
- Ask students to interpret a scatter plot and answer questions about the data it represents. For example, give students the following situation and series of questions.

Jaron is an inventor creating a flying toy machine. He has conducted several trial flights with the machine and measured the distance the toy flew and the total time for each flight. The results are recorded in the scatter plot shown.



1) What is the approximate difference between the longest and shortest flight distances among the trials?

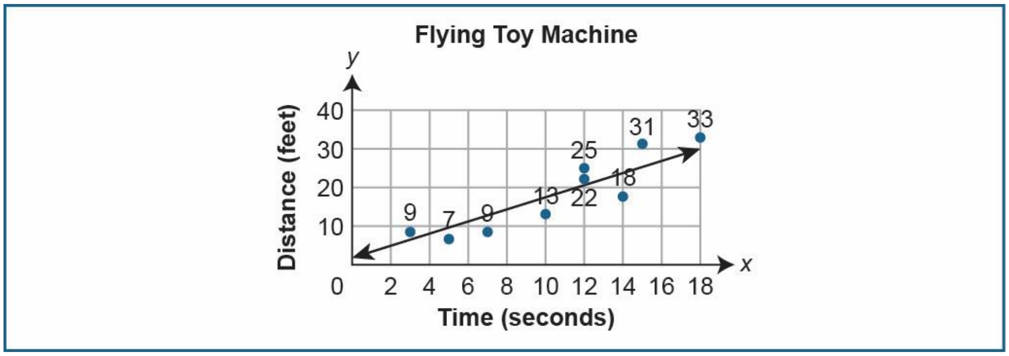
To determine the approximate distance between the longest and shortest flights, or the range of the distances in the scatter plot, students should identify the points with the highest and lowest *y*-values on the plot and estimate their values based on the given scale.



Students should subtract the minimum value, 7, from the maximum value, 33, to determine a range of flight distances of 26 feet.

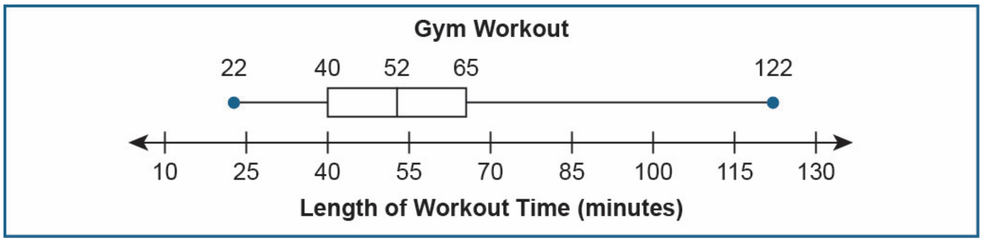
2) Do the data suggest a positive correlation, a negative correlation, or no correlation? Explain your reasoning.

Students should interpret the scatter plot as having a positive correlation, because the distance values tend to increase as the time of flight values increase. Student should recognize that the data follow a general linear trend with a positive slope.



- Ask students to make predictions in context based on a box plot. For example, give students the following situation and series of questions.

  The box plot shown summarizes the length of time members spent at a local gym this week.



  As the last member of the day arrives at the gym, what is the probability (or range of probabilities) this member will work out for more than an hour, based on the box plot? Explain your reasoning.

  - Students should identify that 60 minutes lies in the third quartile of the data set, which is below the Q3 value of 65 minutes and above the median value of 52 minutes. Students should make the following interpretations based on the data.

    1) There is at least a 25% chance the member will work out for more than 60 minutes, because this value is less than the value for Q3.

    2) There is a greater than 50% chance the member will work out for less than 60 minutes, because this value is greater than the median value.

    Students should conclude there is a chance between 25% and 50% that the member will work out for longer than an hour and that value is likely closer to 25%, but this cannot be stated with absolute certainty without further knowledge of the data.

- Ask students to make predictions in context based on a stem-and-leaf plot. For example, give students the following situation.

  The stem-and-leaf plot shown organizes the length of commute, in minutes, for teachers at a school.

Commute Times for Teachers

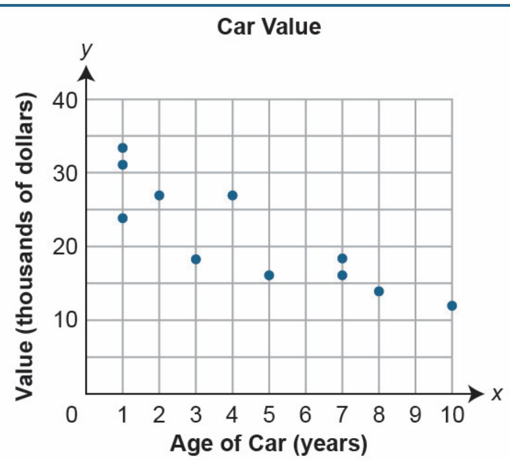
1	1 1 3 5 7
2	1 2 5 5 8 8
3	3 5 7
4	2 4
5	1 1 7

Key
3 | 1 = 31 minutes

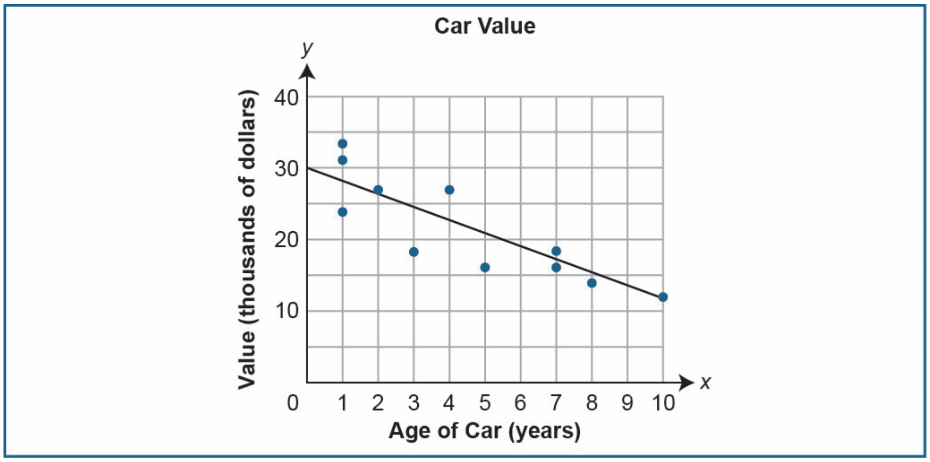
Based on the data, predict whether a teacher selected at random will have a commute of less than 30 minutes.

Students should count the entries in the stem-and-leaf plot to identify that there are 11 teachers who have a stem of 1 or 2, which means they have a commute less than 30 minutes. Similarly, there are 8 teachers who have a commute greater than 30 minutes. Students can conclude that there is a greater than 50% chance that a teacher selected at random will have a commute of less than 30 minutes.

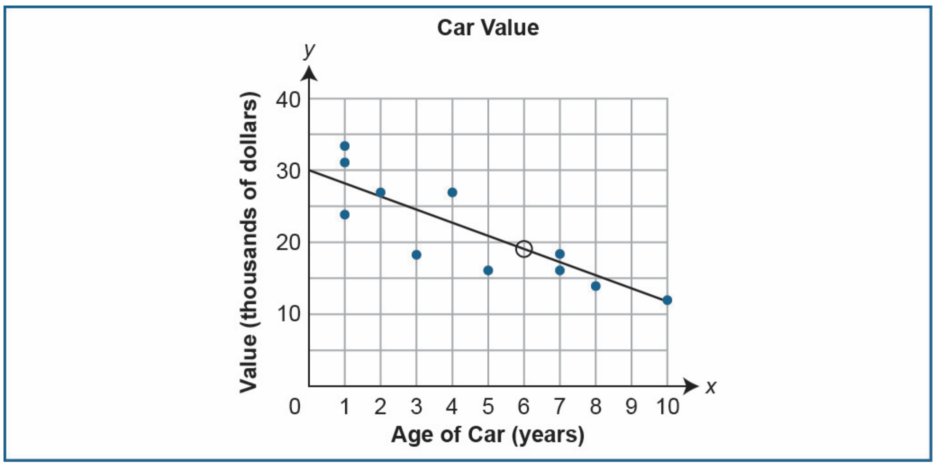
- Ask students to make predictions in context based on a scatter plot. For example, provide students with this scatter plot depicting the current value of cars in a used car lot and the age of those cars.



Ask students to predict the price of a car in the lot that is 6 years old and justify their response. Students should note that there is a general negative trend in the data and may sketch a line of best fit to interpret that trend.

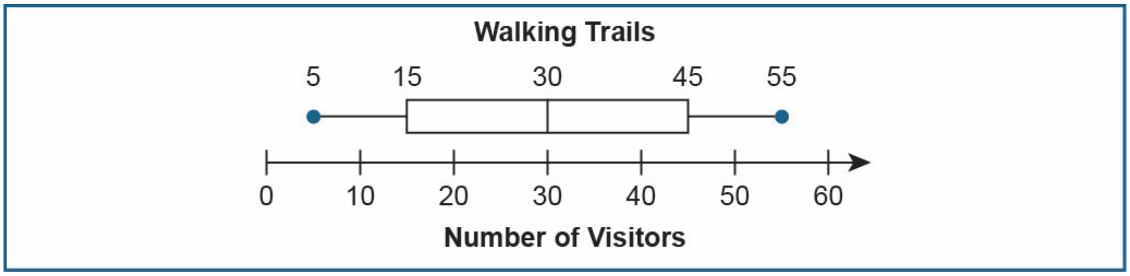


While drawing, students should try to create a line that allows half of the data entries to lie above and below the line while minimizing the average distance from the data points to the line. After sketching a line of best fit, students should predict from the trend that a 6-year-old car in the lot will likely have a cost close to $18,500.

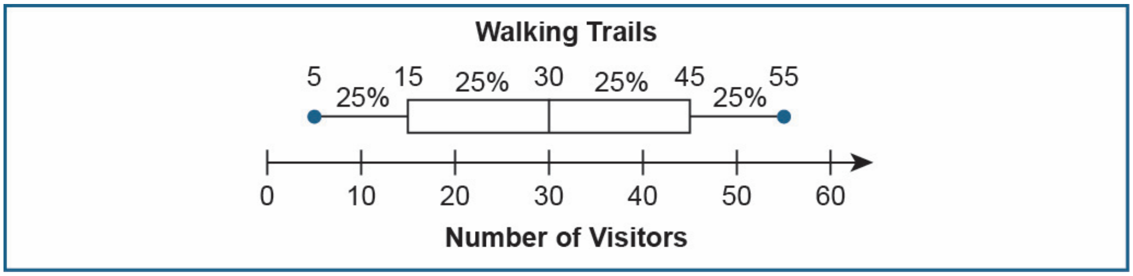


▶ Identifying and Addressing Common Misconceptions

- Students may misinterpret a box plot by assuming that all data values reside in the box portion. Provide students with the following box plot and ask them to identify the greatest data value that the plot represents.

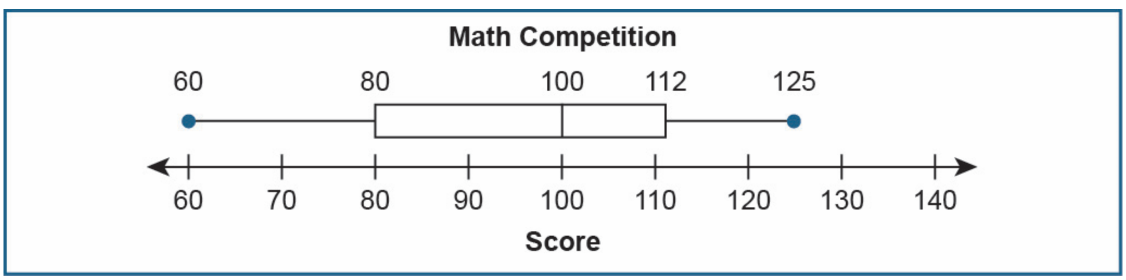


Students who misinterpret a box plot may interpret 45 as the absolute maximum. Remind students that each section in a box plot, including the whiskers, represents 25% of the data and that the whiskers extend to the absolute maximum and absolute minimum of the data.



Students should identify that the greatest data value in this set is 55.

- Students may believe that data must be evenly spread within a quartile when given a five-number summary or box plot. Provide students with the following box plot and questions.



The box plot shows the results of round 1 of a math competition. After the first round, the participants with the lowest 40% of the scores are dropped and do not continue to the next round. Mohraeel claims that every participant with a score less than 92 will not be able to move on to the next round. Is her claim valid? Why or why not?

Students who misinterpret the spread of data may view 92 as the valid landing spot of the 40%-marker of the data because it is three-fifths of the distance across the second quartile; this may lead students to believe that Mohraeel has come to a valid conclusion. Remind students that while quartile values can mark the percentage of data above and below specific values, the data may not be evenly spread. For example, the math competition box plot could represent this data set:

{60, 65, 80, 99, 100, 100, 105, 112, 112, 125}

In this data set, the cutoff, or the 40% marker in the data, would be at a score of 99. Students should thus conclude that Mohraeel's claim is not valid.

- Students may misread stem-and-leaf plots, leading to a miscount of the number of data entries or an incorrect interpretation of the data values. Ask students to rewrite this stem-and-leaf plot as a list of data values:

5	1 1 6
6	2 9
8	1 1

Key
3 | 1 = 3.1

There are several common ways to misread this stem-and-leaf plot.

  - Students may misread this data set as explicit entries for every number listed:

    {1, 1, 1, 1, 2, 5, 6, 6, 8}

  - Students may misread the place value by inappropriately incorporating the key:

    {51, 51, 56, 62, 69, 81, 81}

  - Students may flip stems and leaves:

    {15, 15, 65, 26, 96, 18, 18}

Remind students that a stem-and-leaf plot organizes data into visual groupings that are typically categorized by place value and that they should carefully apply the key when interpreting data entries. Students should correctly read this plot as the data set {5.1, 5.1, 5.6, 6.2, 6.9, 8.1, 8.1}

- Students may mix up the median and mean when interpreting measures of central tendency in a data set. Give students the following situation.

Sasha has taken 10 environmental science quizzes this semester and has earned a median score of 87% on those assessments. What information does this provide about her assessment scores? Students who confuse the median and mean may believe that 87% is the average of all her test scores or that adding all her test scores up and dividing the total by 10 will result in 87%. Remind students that the median represents a "middle value" in a data set, which divides the data set into two groups where half of the data is above and below that data point. The mean, on the other hand, represents the raw average of a data set. Students should correctly conclude from the data that Sasha must have 5 assessments that scored at 87% or lower and 5 assessments that scored at 87% or higher.

▶ Enrichment

- Ask students to compare real-world data sets based on measures of central tendency. For example, give students the following situation.

  Kealia has a lawnmowing service and is updating her fleet of riding lawnmowers. She is researching the reliability and lifespan of two brands, Cutterz and Sundriven. In her research, she determines that Cutterz riding lawnmowers have a median lifespan of 146 weeks of heavy use and a mean of 108 weeks of heavy use. Sundriven mowers have a median lifespan of 140 weeks of heavy use and a mean lifespan of 145 weeks of heavy use. Which company should Kealia purchase lawnmowers from? Provide detailed support for your recommendation by using the provided data.

- Ask students to determine which measure of central tendency is most appropriate to describe a real-world data set. For example, provide students with the number of points a basketball player scored over several games: {5, 12, 13, 41, 8, 15, 29}. Ask students to use the data to predict the number of points the player will score in the next game. Have the students identify the measure of central tendency, if any, that they used to make their prediction. Be sure they explain their reasoning.

- Ask students to predict how a data set may change based on the addition or removal of data points. For example, give students the following situation and series of questions.
  A marine biologist is recording the mass of sand crabs along the coast of San Diego. So far, he has weighed 28 crabs and found a mean mass of 225 grams and a median mass of 240 grams.

  1) After reviewing his data, the biologist decides to remove the masses of 3 young crabs that each had a mass below 100 grams. Predict how this may affect his data, specifically his median and mean values.

  2) The biologist decides to combine his data with those of another biologist who is also studying sand crabs. He shares his original data set without removing the young crabs while the other biologist contributes the mass of 14 crabs that have a total range in mass from 235 to 242 grams. Use this information to predict the new mean and median of the combined data set. Justify your response.

▶ Key Terms

box-and-whisker plot (box plot), central tendency, interquartile range (IQR), line of best fit, median, mean (average), mode, quartile, range, scatter plot, stem-and-leaf plot, trend

A1.2.3.2.3

▶ Module, Assessment Anchor, Anchor Descriptor

Linear Functions and Data Organizations

Data Analysis

Use data displays in problem-solving settings and/or to make predictions.

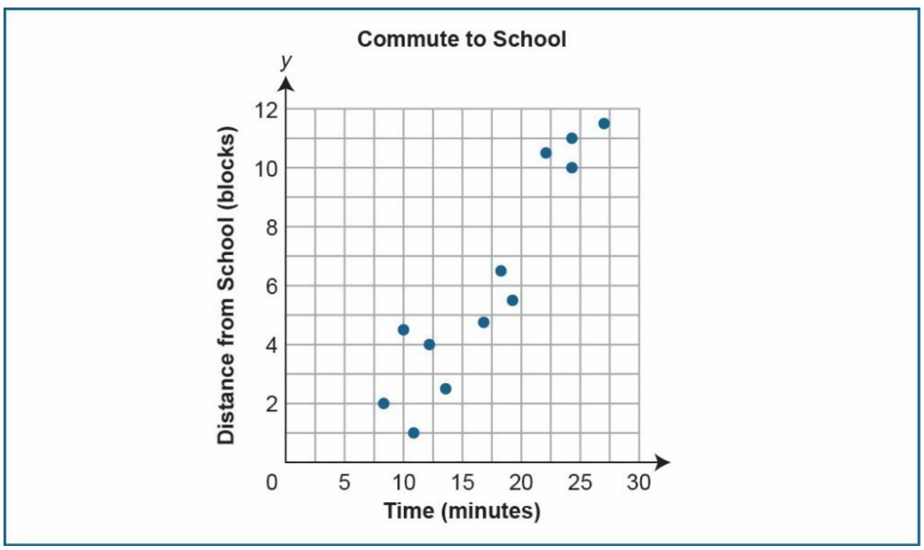
▶ Eligible Content

A1.2.3.2.3 Make predictions using the equations or graphs of best-fit lines of scatter plots.

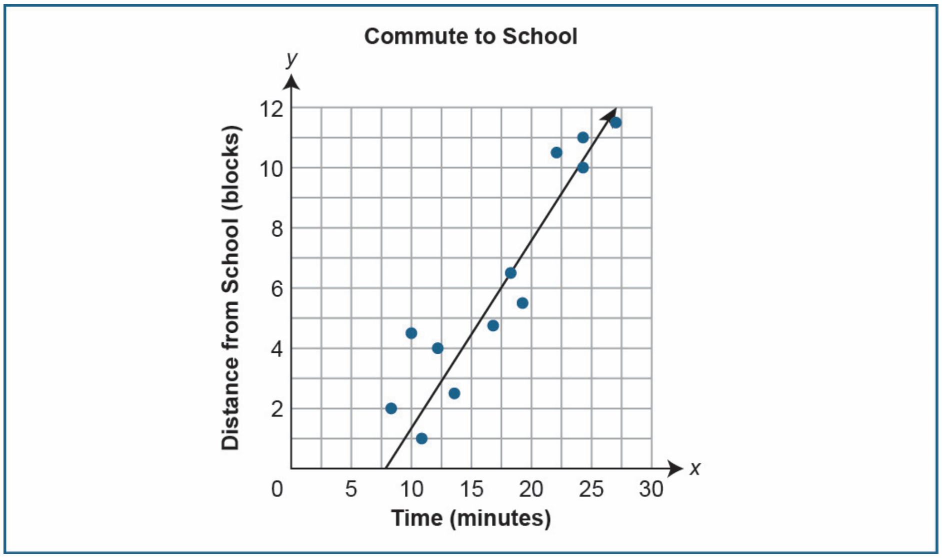
▶ Instructional Supports

- Ask students to draw a line of best fit for a scatter plot and then use it to make a prediction. For example, give students the following situation.

  The scatter plot shows the distance each student in Mara's class lives from school and the time it takes those students to commute to school in the morning.

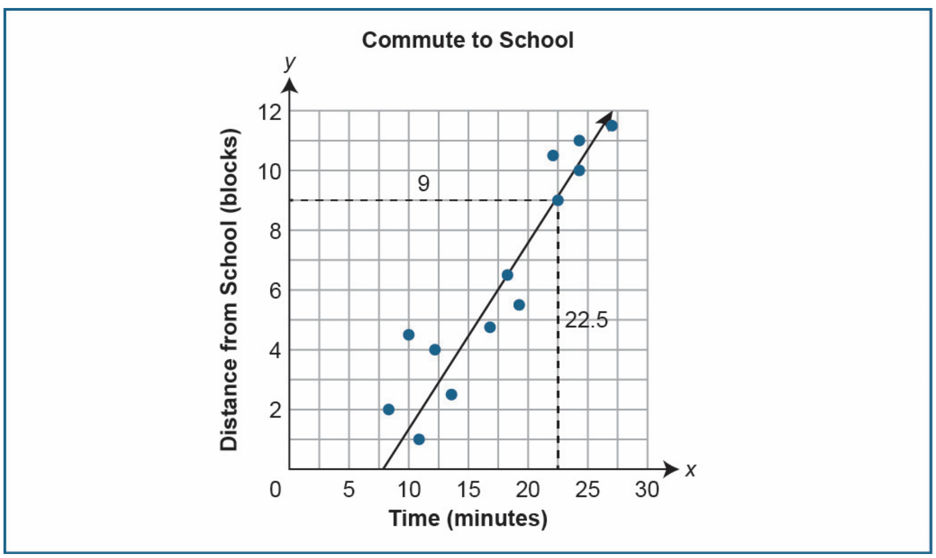


  Mara wants to better understand the data and make predictions for students who were not included in the data, so she drew the following line of best fit.



Based on the line of best fit, what could Mara expect the commute time to be for a student who lives 9 blocks from school?

Students should recognize that the line of best fit shows a good approximation for values that follow the trend of the scatter plot. To determine the best estimate for a commute time for someone living 9 blocks from school, students should identify the point on the trend line where $y = 9$ and determine the input value.



Students should conclude from visual estimation that the expected commute time will be about 22.5 minutes.

- Ask students to make a prediction using the equation of a line of best fit for a scatter plot. For example, give students the following situation.

Kieran is an electrician who works on newly built homes. He wants to determine the amount of wire he will likely need for each project based on the square footage of the house. He creates a data set that connects the input value, x, of the square footage of each of the last 12 houses he has worked on with an output value, y, of the amount of wire, in feet, he used for each project. Kieran calculates a linear regression, or a line of best fit, for his data set as the equation $y = 1.3x + 375$. Based on the equation for this line of best fit, how much wire should Kieran expect to need for a 2,150-square-foot house?

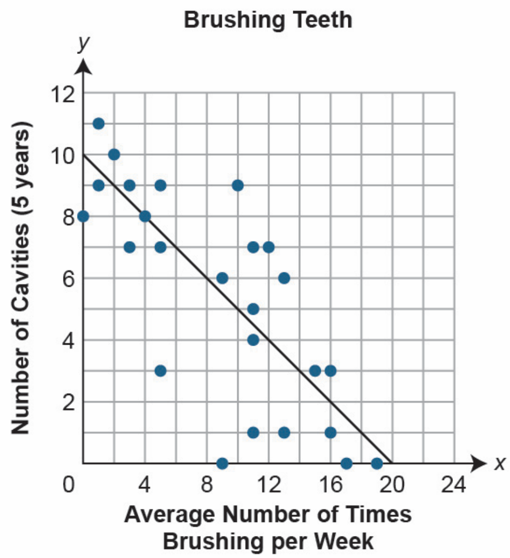
- Students should substitute 2,150 in for the x-value of the equation for the line of best fit to determine a y-value of $y = 1.3(2{,}150) + 375 = 2{,}795 + 375 = 3{,}170$. Students should conclude that Kieran will need 3,170 feet of wire based on the trend of his data set.

  Additionally, ask a modified form of this question that requires students to predict an input value, x, given the output, y. For example, in this context, give students the following problem.

  Kieran's assistant notices that he has bought 4,275 feet of wire to start a job at another house. Using the equation for the line of best fit, predict the square footage of this house.

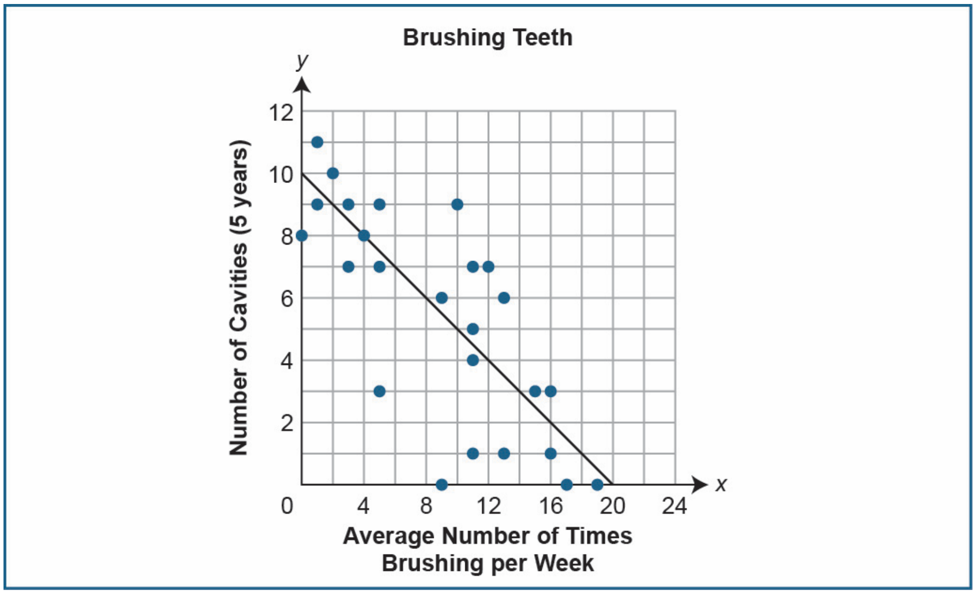
  In this modified problem, students should substitute 4,275 in for the y-value in the equation for the line of best fit: $y = 1.3x + 375 \longrightarrow 4{,}275 = 1.3x + 375$. Students may then isolate the x-variable to determine that Kieran is likely planning to work on a house that has a square footage of $x = 3{,}000$ square feet.

- Ask students to make a prediction about a scatter plot. For example, give students the following scatter plot displaying the data from a survey asking people the average number of times they brushed their teeth each week and the number of cavities they had over a 5-year span. A line of best fit is included on the scatter plot, as shown.



The equation for the line of best fit for the scatter plot is $y = -\frac{1}{2}x + 10$. Ask students to predict what the average number of cavities would be over a 5-year span for people who brush their teeth an average of 7 times per week and again for people who brush their teeth an average of 14 times per week.

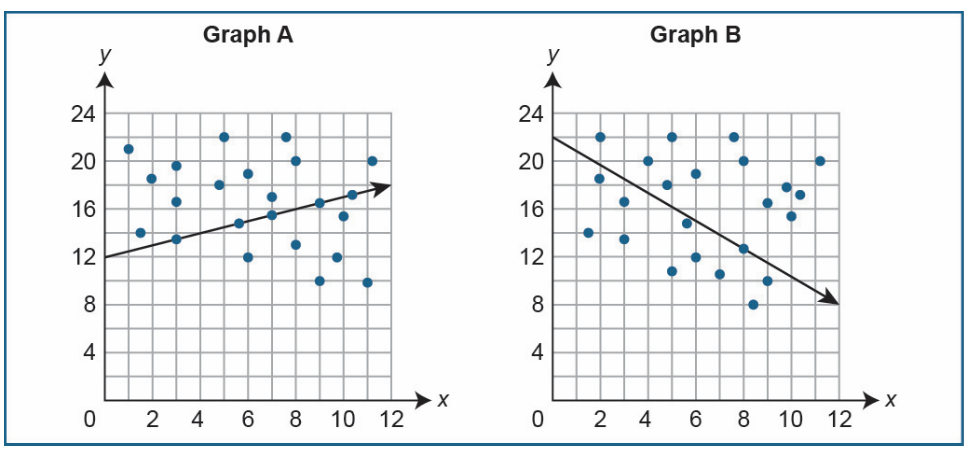
- Students may use this equation to predict the values for $x = 7$ times per week and $x = 14$ times per week by substituting these values into the equation for the line of best fit. The expected number of cavities for people brushing 7 times per week is $y = -\frac{1}{2}(7) + 10 = 6.5$, while the expected number of cavities for people brushing 14 times per week is $y = -\frac{1}{2}(14) + 10 = 3$.
- Students may also directly track the line of best fit on the graph to predict values as shown.



By identifying the approximate *y*-value when the input value is 7, students may predict that people who brush their teeth 7 times per week will have 6.5 cavities over a 5-year span. Similarly, they may predict that people who brush their teeth 14 times per week will have 3 cavities over a 5-year span.

▶ Identifying and Addressing Common Misconceptions

- Students may believe that a line of best fit is the line that "fits" the most points or that a line that touches the most points is the line of best fit. Ask students to determine which graph has a stronger correlation for the line of best fit for the same data set.



Students who misunderstand a line of best fit may believe that the line of Graph A has a stronger correlation because it touches more data points. Remind students that a line of best fit models the general trend of a data set, has approximately the same number of points above and below the line, and minimizes the average distances from each data point to the line. Students should correctly interpret the line of Graph B as having the line of best fit for the data set.

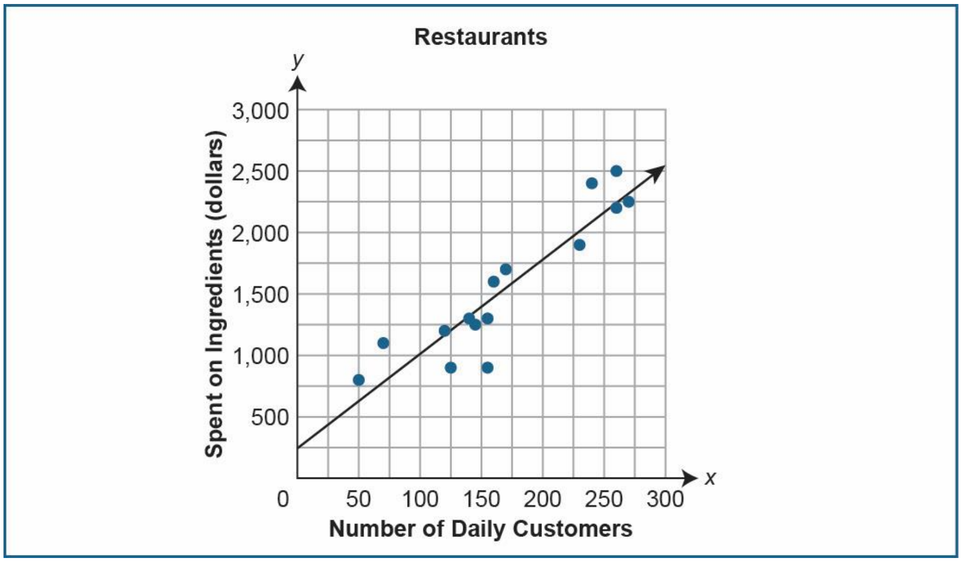
- Students may mix up independent and dependent variables when making a prediction using an equation for a line of best fit. Give students the following situation.

  Caleb created a scatter plot to show the relationship between the number of hours his band classmates spend practicing weekly, x, and their performance scores in a solo contest, y. After creating the scatter plot, he drew a line of best fit that had an equation of $y = 0.65x + 1.7$. Based on his data, how long should the musicians practice if they want to earn a score of 8 on the assessment?

  Students who mix up independent and dependent variables may enter the score of 8 into the equation for the line of best fit as x, leading to a solution of $y = 0.65(8) + 1.7 = 6.9$. Remind students to always carefully distinguish their input (independent) and output (dependent) variables when interpreting a two-variable equation in context. In this example, the problem states that the number of hours spent practicing is the independent, or x, variable, and the resultant score is the dependent, or y, variable. Students should correctly substitute 8 for y in the equation for the line of best fit, $8 = 0.65x + 1.7$. They should then solve for x to determine a predicted weekly practice time of about 9.7 hours.

► Enrichment

- Ask students to interpret the rate of change and initial value of lines of best fit in real-world contexts. For example, give students the following situation.
  A scatter plot shows the relationship between the number of daily customers at several restaurants and the average daily amount that each restaurant spends on ingredients.



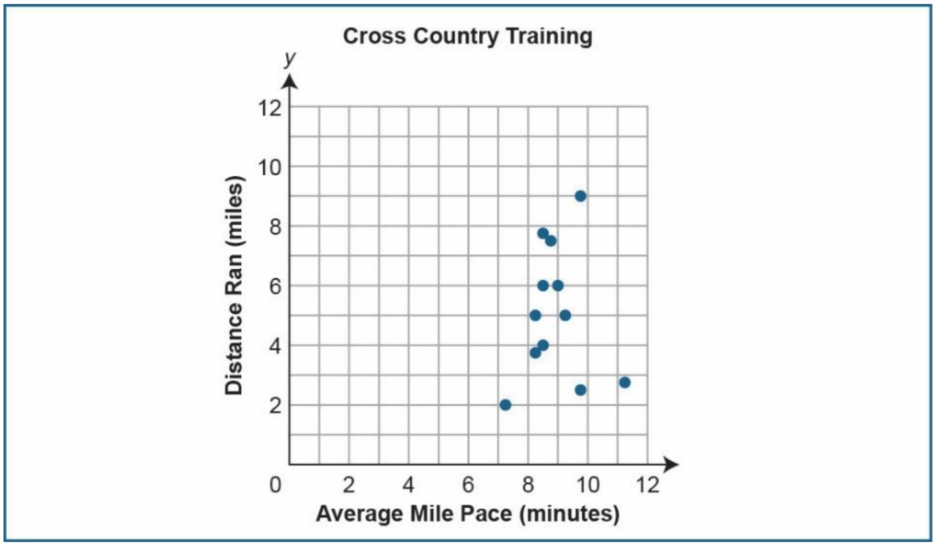
What does the y-intercept of the line of best fit represent in this context? What conclusion can be drawn from the rate of change of the line of best fit?

- Ask students to detect the limitations of real-world constraints on predictions based on the line of best fit for a data set. For example, give students the following situation.

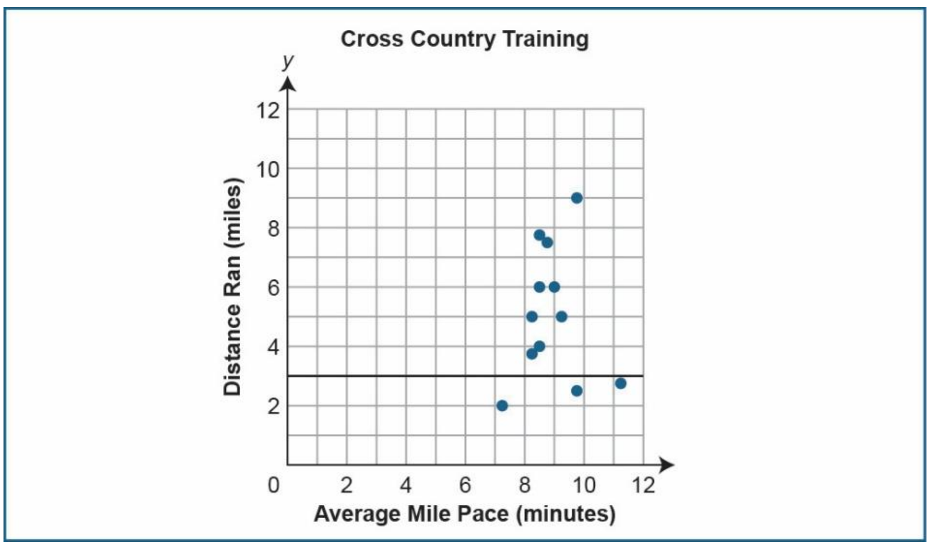
  A local veterinarian has collected data on the age of puppies her office has treated in months, x, and the weight, in pounds, of those puppies, y. The office manager creates a trend line with the data with an equation of $y = 4.5x + 2$. Ask students to use the line of best fit to determine the weight of a dog that is 7 years old (84 months). Ask students to consider why this may be an unrealistic prediction and to identify whether any assumptions can be made about the data.

- Ask students to interpret the effect of outliers on a scatter plot and trend line and make predictions on how removing or adding data points may affect a line of best fit. For example, give students the following situations.

  A high school cross country runner tracked her workouts over the course of three weeks and plotted the distance she ran, x, along with the average pace for each mile she ran, y, on the scatter plot shown.



After gathering her data, she drew a trend line to help approximate an average mile pace when running 3 miles based on her workouts.



1) What does the trend line suggest her 3-mile pace would be? Do you think this is justified based on the data?

2) The runner wasn't feeling well last weekend. She still went on a run each day but ran at a much slower pace than she usually averages. She decides to exclude these data points from her data set to get a better visualization of what her anticipated pace will be when she's feeling healthy. How will these data affect the trend line? What might be a better prediction for her pace at 3 miles?

► Key Terms

correlation, line of best fit, regression, scatter plot, slope (rate of change), slope-intercept form, trend, y-intercept (initial value)

A1.2.3.3.1

▶ Module, Assessment Anchor, Anchor Descriptor

Linear Functions and Data Organizations

Data Analysis

Apply probability to practical situations.

▶ Eligible Content

A1.2.3.3.1 Find probabilities for compound events (e.g., find probability of red and blue, find probability of red or blue) and represent as a fraction, decimal, or percent.

▶ Instructional Supports

- Ask students to determine the probabilities of simple events and communicate them in the form of a fraction, a decimal, and a percentage. For example, ask students to communicate the probability of rolling a 2 face-up on a fair six-sided number cube. Students should determine simple probabilities by dividing the number of favorable outcomes, or the desired results, by the total number of possible outcomes. Students often write this as the fraction $P = \frac{\text{\# of favorable outcomes}}{\text{total \# of possible outcomes}}$. Students should recognize that out of 6 total outcomes when rolling a fair six-sided number cube, there is only one side marked with a value of 2, so they should write this probability as the fraction $P = \frac{1}{6}$. Students may further divide 1 ÷ 6 to determine an equivalent decimal probability of $P = 0.1\overline{6}$, which in turn can be rewritten as a percent probability by multiplying the decimal by 100, $P = 16.\overline{6}\%$.
- Ask students to distinguish the difference between "and" and "or" probabilities for compound events. For example, ask students to determine the probabilities of selecting the following cards when randomly selecting a single card from a standard deck of playing cards.

  1) a card that is a jack and is black
  2) a card that is a jack or a card that is black

  Prompt students to initially determine each probability by counting the total number of favorable outcomes and dividing it by the total number of cards. After determining both probabilities, ask students to explain why the results are different.

- Students should first identify that there are exactly 2 jacks in a standard deck of playing cards that are black (the jack of clubs and the jack of spades). They should also note that there are 52 total possible cards that can be selected, which allows them to write a fractional probability of $P = \frac{2 \text{ favorable outcomes}}{52 \text{ total possible outcomes}} = \frac{1}{26}$ of selecting a card that is a jack and is black from a standard deck of playing cards.
- Students should then consider that there are 4 jacks in a standard deck of playing cards and 26 possible cards that are black. Also, there are 2 cards that fall in both categories (the jack of clubs and the jack of spades) and should be included only once in the number of possible favorable outcomes. This should lead students to create a list of possible favorable outcomes of 4 jacks and the remaining black cards in the deck, not including the jacks already counted $(26 - 2) = 24$. Students should then predict a probability of $P = \frac{4 + 24 \text{ favorable outcomes}}{52 \text{ total possible outcomes}} = \frac{28}{52} = \frac{7}{13}$ of selecting a card that is a jack or black.
- Students should recognize that when identifying compound probabilities that combine elements using an “and,” the probability of the event occurring is smaller (or remains the same) because it restricts the total number of favorable outcomes. Students should also recognize that when using “or,” the probability of the events becomes greater (or remains the same) because the number of favorable outcomes is increased.

- Ask students to determine the probability of independent compound events. For example, give students the following problem.

  A board game requires players to roll a fair six-sided number cube and a six-sided color cube. The color cube has two sides that are red, two sides that are green, and two sides that are blue. What is the probability of a player landing a 6 face-up and the color blue face-up on a given turn?

  - Students should first identify that the probability of landing a 6 is $\frac{1}{6}$ and the probability of landing the color blue is $\frac{2}{6}$. They should therefore conclude that the two events do not affect each other’s outcomes, so they can be described as independent. Students should recall the generalized formula for determining the probability of independent events $P(A \text{ and } B) = P(A) \bullet P(B)$ and multiply the probability of rolling a 6 and the probability of rolling the color blue together. In an equation, that may look like $P(6) \bullet P(\text{blue}) = \frac{1}{6} \bullet \frac{2}{6} = \frac{2}{36} = \frac{1}{18}$.

- Ask students to determine the probability of compound dependent events. For example, give students the following problem.

  A bag has 3 red marbles and 6 blue marbles in it. All the marbles are the same size and shape. What is the probability of randomly selecting a red marble and then another red marble without replacing the first marble?

  - Students should identify the probability of first randomly selecting a red marble as $P(\text{red}_1) = \frac{3\text{ red}}{9\text{ total}} = \frac{1}{3}$. They should conclude that once the first red marble is removed, the probability for the second event changes, because there are now only 2 red marbles remaining inside the bag and 8 total remaining marbles. Students should then identify the probability of randomly selecting the second red marble as $P(\text{red}_2) = \frac{2\text{ red}}{8\text{ total}} = \frac{1}{4}$. Students should finally determine the combined probability of both results by multiplying $P(\text{red}_1) \bullet P(\text{red}_2) = \frac{1}{3} \bullet \frac{1}{4} = \frac{1}{12}$.
    - Note: Ask students to interchangeably provide responses as a fraction, decimal, or percentage in similar situations.

- Ask students to determine the probabilities of mutually exclusive compound events. For example, give students the following problem.

  A gem collector has a bag containing gems. There are 3 diamonds, 4 rubies, and 5 sapphires inside the bag. All the gems are the same size and shape. What is the probability that one gem that the collector randomly selects from the bag will be a ruby or a sapphire?

  - Students can determine the probability by identifying the total number of favorable outcomes and dividing by the number of total possible outcomes. In this case, there are 4 rubies and 5 sapphires, so the number of favorable outcomes is 4 + 5 = 9. There are 12 total gems inside the bag, so $P = \frac{9}{12} = \frac{3}{4}$.
  - Students may also use the formula for mutually exclusive events $P(A \text{ or } B) = P(A) + P(B)$. In this situation, the probability of selecting a ruby is $\frac{4}{12}$ and the probability of selecting a sapphire is $\frac{5}{12}$, so the probability of selecting either a ruby or a sapphire is $P = \frac{4}{12} + \frac{5}{12} = \frac{9}{12} = \frac{3}{4}$.

- Ask students to determine the probability of a compound event where one or both of two favorable outcomes may occur. For example, give students the following situation.

  50 middle school students were surveyed about their favorite types of art to create. They were given a list of several options and were able to select up to three choices. In the results, 22 students selected digital photography, 15 selected watercolor painting, and 4 selected both digital photography and watercolor painting. If a surveyed student is chosen at random, what is the probability that the student selected digital photography or watercolor painting?

  - Students should identify the probability by directly counting the number of favorable outcomes and dividing by the total number of possible outcomes. In this problem, students should note that there are 22 students who selected digital photography and 15 who selected watercolor painting. However, 4 of the students who selected watercolor painting would have already been identified and counted as a favorable outcome because they also selected digital photography. This leaves $22 + 15 - 4 = 33$ possible favorable outcomes. Students should conclude that the probability of a random student selecting digital photography or watercolor painting to be $P = \frac{33}{50}$.
  - Students may also use the formula for mutually inclusive events $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$. In this problem the probability of a student selecting digital photography is $\frac{22}{50}$, the probability of a student selecting watercolor painting is $\frac{15}{50}$, and the probability of a student selecting both is $\frac{4}{50}$. Substituting this into the formula results in a total probability of $P = \frac{22}{50} + \frac{15}{50} - \frac{4}{50} = \frac{33}{50}$.

► Identifying and Addressing Common Misconceptions

- Students may add probabilities of independent events when they should multiply instead. Ask students to determine the probability of rolling 2 fair six-sided cubes simultaneously and having a result of 1 face-up on both cubes. Students who mistakenly add probabilities may add the probability of each cube rolling a 1 as $P = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$. For students who make this error, ask them to create an experimental probability by rolling two number cubes repeatedly and tracking the frequency of rolling 1 face-up on both cubes. Students will quickly see the resulting probability is much lower than $\frac{1}{3}$. Remind students that when combining the probability of independent events, they should multiply the probability of each event. Students should correctly identify the probability of rolling 1 face-up on both cubes simultaneously to be $P = \frac{1}{6} \bullet \frac{1}{6} = \frac{1}{36}$.

- Students may mistake an "and" compound probability for an "or" compound probability and vice versa. Ask students to determine the probability of rolling a 2 or a 5 on a fair six-sided cube. Students who confuse compound probabilities may multiply the probability of rolling a 2, $P = \frac{1}{6}$, and the probability of rolling a 5, $P = \frac{1}{6}$, to determine a combined probability of $P = \frac{1}{6} \bullet \frac{1}{6} = \frac{1}{36}$. Remind students that when using formulas to calculate probability they should think about the probabilities conceptually in terms of possible favorable outcomes against total possible outcomes. In this problem there are two possible favorable outcomes for a single roll, 2 and 5, and six total outcomes, leading to a probability of $P = \frac{2}{6} = \frac{1}{3}$. Students should interpret this through the formula for mutually exclusive events $P(A \text{ or } B) = P(A) + P(B) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$.

▶ Enrichment

- Ask students to use an understanding of complementary probabilities to determine real-world compound probabilities. For example, give students the following problem.

  The weather forecast in Cincinnati, Ohio, currently has a 15% chance of rain on Saturday and a 35% chance of rain on Sunday. Based on the forecast, what is the probability that there will be no rain over the weekend?

- Ask students to create probability models from real-world gathered data. For example, ask students to collect a survey of 20 classmates answering two "yes" or "no" questions, such as "Do you play on a sports team?" and "Do you play a musical instrument?" Once students collect the data, ask them to calculate
  - the probability a respondent plays on a sports team,
  - the probability a respondent plays a musical instrument
  - the probability a respondent does both,
  - the probability a respondent does neither, and
  - the probability a respondent either plays on a sports team or plays a musical instrument.

  Ask students to compare the responses of each result and specifically identify the relationship between the last two probabilities. Additionally, ask students whether or not the questions are more likely to have overlapping responses and have them explain why in context.

- Ask students to create a board game that requires the use of compound probabilities for progression. For example, have students create a board game and a set of rules where two independent simple probability events are used for each turn, like flipping a coin, using a spinner, rolling dice, or drawing cards randomly. In addition, direct students to create a set of rules in which certain combinations lead to negative or backward progress while maintaining that those results have a lower probability of occurring than results leading to a positive progression in the game.

► Key Terms

complement, compound event, dependent event, event, independent event, mutually exclusive event, mutually inclusive event, overlap, probability, simple event